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RATIONAL HYPERHOLOMORPHIC FUNCTIONS IN \mathbb{R}^4

DANIEL ALPAY, MICHAEL SHAPIRO, AND DAN VOLOK

ABSTRACT. We introduce the notion of rationality for hyperholomorphic functions (functions in the kernel of the Cauchy-Fueter operator). Following the case of one complex variable, we give three equivalent definitions: the first in terms of Cauchy-Kovalevskaya quotients of polynomials, the second in terms of realizations and the third in terms of backward-shift invariance. Also introduced and studied are the counterparts of the Arveson space and Blaschke factors.

1. INTRODUCTION

It is well known that functions holomorphic in a domain $\Omega \subset \mathbb{C}$ are exactly the elements of the kernel of the Cauchy-Riemann differential operator

$$\bar{\partial} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$$

restricted to Ω . A polynomial in x and y is holomorphic if, and only if, it is a polynomial in the complex variable $z = x + iy$, and rational holomorphic functions are quotients of polynomials.

Holomorphic functions of one complex variable have a natural generalization to the quaternionic setting when one replaces the Cauchy-Riemann operator by the Cauchy-Fueter operator

$$D = \frac{\partial}{\partial x_0} + \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}.$$

In this expression the x_j are real variables and the \mathbf{e}_j are imaginary units of the skew-field \mathbb{H} of quaternions (see Section 2 below for more details). Solutions of the equation $Df = 0$ are called left-hyperholomorphic functions (they are also called left-hyperanalytic, or left-monogenic, or regular, functions, see [18], [13], [21]). Right-hyperholomorphic functions are the solutions of the equation

$$fD = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} \mathbf{e}_1 + \frac{\partial f}{\partial x_2} \mathbf{e}_2 + \frac{\partial f}{\partial x_3} \mathbf{e}_3 = 0.$$

When trying to generalize the notions of polynomial and rational functions to the hyperholomorphic setting, one encounters several obstructions. For instance,

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the quaternionic variable

$$x = x_0 + x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3$$

is not hyperholomorphic. Moreover, the point-wise product of two hyperholomorphic functions is not hyperholomorphic in general and the point-wise inverse of a non-vanishing hyperholomorphic function need not be hyperholomorphic.

For the polynomials these difficulties were overcome by Fueter, who introduced in [16] the symmetrized multi-powers of the three elementary functions

$$\zeta_1(x) = x_1 - \mathbf{e}_1 x_0, \quad \zeta_2(x) = x_2 - \mathbf{e}_2 x_0, \quad \text{and} \quad \zeta_3(x) = x_3 - \mathbf{e}_3 x_0.$$

The polynomials thus obtained are known today as the Fueter polynomials. They are (both right and left) hyperholomorphic and appear in power series expansions of hyperholomorphic functions. In particular, a hyperholomorphic polynomial is a linear combination of the Fueter polynomials.

In this paper we introduce the notion of rational hyperholomorphic function. We obtain three equivalent characterizations: the first one in terms of quotients and products of polynomials, the second one in terms of realization and the last one in terms of backward-shift-invariance. These various notions need to be suitably defined in the hyperholomorphic setting. A key tool here is the Cauchy-Kovalevskaya product of hyperholomorphic functions.

We also introduce a reproducing kernel Hilbert space of left-hyperholomorphic functions which seems to be the counterpart of the Arveson space of the ball – the reproducing kernel Hilbert space of functions holomorphic in the open unit ball of \mathbb{C}^N with the *rational* reproducing kernel $\frac{1}{1 - \sum z_j \bar{w}_j}$. When $N = 1$, this is just the Hardy space of the open unit disk. It was first introduced by S. Drury in [14] and proved in recent years to be a better extension of the Hardy space than the classical Hardy space of the unit ball of \mathbb{C}^N , at least for problems in operator theory (see for instance the papers [1], [2], [9], [10], [12] for a sample of examples and applications). In particular, it is invariant under the operators M_{z_j} of multiplication by the variables z_j , $j = 1, \dots, N$, and it holds that

$$(1.1) \quad I - \sum_{j=1}^N M_{z_j} M_{z_j}^* = C^* C$$

where C is the point evaluation at the origin.

To explain our approach let us consider briefly first the case of holomorphic functions of one complex variable. Let f and g be two functions holomorphic in a neighborhood of the origin, with the power series expansions

$$(1.2) \quad f(z) = \sum_{n=0}^{\infty} z^n a_n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} z^n b_n$$

at the origin. Then the point-wise product $(fg)(z) = f(z)g(z)$ has at the origin the expansion

$$(1.3) \quad (fg)(z) = \sum_{n=0}^{\infty} z^n c_n,$$

where the sequence $\{c_n\}$, given by

$$(1.4) \quad c_n := \sum_{m=0}^n a_m b_{n-m},$$

is called the convolution of the sequences $\{a_n\}$ and $\{b_n\}$. It appears that the substitute for pointwise product in the hyperholomorphic setting (the Cauchy-Kovalevskaya product) is also a convolution.

In the sixties of the previous century, the state space theory of linear systems gave rise to a representation of a rational function called *realization* (see [20], [11]). Still assuming analyticity in a neighborhood of the origin, this representation is of the form

$$(1.5) \quad r(z) = D + zC(I - zA)^{-1}B$$

where A, B, C, D are matrices of appropriate dimensions. It is particularly suitable for the study of matrix-valued rational functions.

Realization theory has various extensions in the setting of several complex variables; see e.g. [17], [26]. One approach, related to functions holomorphic in the unit ball, exploits the so-called Gleason problem (see [4], [3]). A solution of the Gleason problem, due to Leibenson (see [19], [24, §15.8, p.151]), was adapted to the setting of hyperholomorphic functions in [8] and [7]. It leads naturally to the analogues of (1.2) – (1.4); these are the expansions in terms of Fueter polynomials and the Cauchy-Kovalevskaya product, mentioned above. Moreover, in this way we obtain the analogue of the realization (1.5) and other equivalent descriptions of the class of rational hyperholomorphic functions, as well as the reproducing kernel of the counterpart of the Arveson space (quite different from the quaternionic Cauchy kernel).

This paper is organized as follows. In Section 2 we review facts from the quaternionic analysis and present some preliminary results, concerning backward-shift operators in the hyperholomorphic setting. In Section 3 we give three definitions of a rational function in the hyperholomorphic case and prove their equivalence. In Section 4 we define and study the counterparts of the Arveson space of the unit ball and the Blaschke factors.

Some of the results presented here were announced in [5]. In forthcoming papers we will consider the theory of linear systems in the quaternionic case and Beurling-Lax-type theorems for the Arveson space in the present setting.

2. QUATERNIONS AND HYPERHOLOMORPHIC FUNCTIONS

2.1. The skew-field of quaternions. In this section, we provide some background on quaternionic analysis needed in this paper. For more information, we refer the reader to [25] and to [6]. The Hamilton skew-field of quaternions \mathbb{H} is the real four-dimensional linear space \mathbb{R}^4 equipped with the product, defined as follows.

For the elements of the standard basis $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the rules of multiplication form the Cayley table:

$$(2.1) \quad \begin{array}{c|cccc} & \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \hline \mathbf{e}_0 & \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \mathbf{e}_1 & \mathbf{e}_1 & -\mathbf{e}_0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ \mathbf{e}_2 & \mathbf{e}_2 & -\mathbf{e}_3 & -\mathbf{e}_0 & \mathbf{e}_1 \\ \mathbf{e}_3 & \mathbf{e}_3 & \mathbf{e}_2 & -\mathbf{e}_1 & -\mathbf{e}_0 \end{array}$$

Given two elements

$$x = \sum_{i=0}^3 x_i \mathbf{e}_i, \quad x_i \in \mathbb{R},$$

$$y = \sum_{j=0}^3 y_j \mathbf{e}_j, \quad y_j \in \mathbb{R},$$

of \mathbb{H} , their product is defined by

$$xy := \sum_{i,j=0}^3 x_i y_j \mathbf{e}_i \mathbf{e}_j,$$

where $\mathbf{e}_i \mathbf{e}_j$ are calculated according to (2.1). Note that \mathbf{e}_0 is the identity element of \mathbb{H} (for convenience, we identify it with the real unit: $\mathbf{e}_0 = 1$).

The quaternionic modulus $|\cdot|$ coincides with the Euclidean norm in \mathbb{R}^4 :

$$|x| = \|x\|_{\mathbb{R}^4} = \sqrt{\sum_{k=0}^3 x_k^2},$$

and it holds that

$$|xy| = |x||y| \quad \forall x, y \in \mathbb{H}.$$

The conjugation in \mathbb{H} is defined by

$$\bar{x} = x_0 - \sum_{i=1}^3 x_i \mathbf{e}_i.$$

It holds that

$$\bar{x}x = x\bar{x} = |x|^2$$

and hence

$$\forall x \in \mathbb{H} \setminus \{0\} : x^{-1} = \bar{x}|x|^{-2}.$$

2.2. Hyperholomorphic functions and the Cauchy-Kovalevskaya product. We have already mentioned in Section 1 that an \mathbb{H} -valued function f , \mathbb{R} -differentiable in an open connected set $\Omega \subset \mathbb{H}$, is said to be left-hyperholomorphic in Ω if it satisfies in Ω the following differential equation:

$$(2.2) \quad \sum_{i=0}^3 \mathbf{e}_i \frac{\partial f}{\partial x_i} = 0.$$

Analogously, an \mathbb{H} -valued function f , \mathbb{R} -differentiable in an open connected set $\Omega \subset \mathbb{H}$, is said to be right-hyperholomorphic in Ω if it satisfies in Ω the differential equation

$$(2.3) \quad \sum_{i=0}^3 \frac{\partial f}{\partial x_i} \mathbf{e}_i = 0.$$

The differential operator

$$D = \sum_{i=0}^3 \mathbf{e}_i \frac{\partial}{\partial x_i}$$

is called the Cauchy-Fueter operator. It satisfies the identity

$$D\overline{D} = \overline{D}D = \Delta_4,$$

where

$$\overline{D} = \frac{\partial}{\partial x_0} - \sum_{j=1}^3 \mathbf{e}_j \frac{\partial}{\partial x_j} \quad \text{and} \quad \Delta_4 = \sum_{i=0}^3 \frac{\partial^2}{\partial x_i^2}.$$

Thus hyperholomorphic functions are, in particular, harmonic.

In the sequel we shall restrict ourselves to the case of left-hyperholomorphic functions. One can, of course, obtain analogous results for right-hyperholomorphic functions, as well.

Let us denote the right- \mathbb{H} -module of functions, left-hyperholomorphic in Ω , by $\mathcal{O}_{\mathbb{H}}(\Omega)$. Assume that Ω is a ball, centered at the origin. Then, as was proved in [7], any element $f \in \mathcal{O}_{\mathbb{H}}(\Omega)$ can be written in the form

$$(2.4) \quad f(x) = f(0) + \sum_{n=1}^3 \zeta_n(x) \mathcal{R}_n f(x),$$

where

$$(2.5) \quad \zeta_n(x) := x_n - x_0 \mathbf{e}_n$$

are entire (both right and left) hyperholomorphic functions, and the operators

$$\mathcal{R}_n : \mathcal{O}_{\mathbb{H}}(\Omega) \mapsto \mathcal{O}_{\mathbb{H}}(\Omega)$$

are defined by

$$(2.6) \quad \mathcal{R}_n f(x) = \int_0^1 \frac{\partial f}{\partial x_n}(tx) dt.$$

(see [22, p. 118], [24, §15.8 p.151] and [3] for these operators in the setting of the unit ball of \mathbb{C}^N). Note that it follows from the hyperholomorphic Cauchy integral formula that $\mathcal{O}_{\mathbb{H}}(\Omega) \subset C^\infty(\Omega)$, hence the operators \mathcal{R}_n commute:

$$\mathcal{R}_m \mathcal{R}_n f(x) = \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_n \partial x_m}(utx) t dt du = \mathcal{R}_n \mathcal{R}_m f(x),$$

and that

$$\mathcal{R}_n f(0) = \frac{\partial f}{\partial x_n}(0).$$

Hence, applying the formula (2.4) for $\mathcal{R}_n f$, we get

$$f(x) = f(0) + \sum_{n=1}^3 \zeta_n(x) \frac{\partial f}{\partial x_n}(0) + \sum_{0 \leq n \leq m \leq 3} (\zeta_n(x) \zeta_m(x) + \zeta_m(x) \zeta_n(x)) \mathcal{R}_m \mathcal{R}_n f(x).$$

Iterating this process, one obtains an expansion of f in terms of symmetrized products of ζ_n , analogous to the classical Taylor power series expansion.

To be more precise, let us introduce the multi-index notation we shall use throughout this paper. The symmetrized product of $a_1, \dots, a_n \in \mathbb{H}$ is defined by

$$a_1 \times a_2 \times \dots \times a_n = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(n)},$$

where S_n is the set of all permutations of the set $\{1, \dots, n\}$. Furthermore, for $\nu, \mu \in \mathbb{Z}_+^3$ we use the usual notation

$$|\nu| = \nu_1 + \nu_2 + \nu_3, \quad \nu! = \nu_1! \nu_2! \nu_3!, \quad \nu \geq \mu \text{ if } \nu_j \geq \mu_j \ \forall j,$$

$$\partial^\nu = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \partial x_3^{\nu_3}},$$

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

Using the above notation, we can formally write

$$(2.7) \quad f(x) = \sum_{n=0}^{\infty} \sum_{|\nu|=n} \zeta^\nu(x) f_\nu,$$

where

$$(2.8) \quad \zeta^\nu(x) := \zeta_1(x)^{\times \nu_1} \times \zeta_2(x)^{\times \nu_2} \times \zeta_3(x)^{\times \nu_3},$$

$$(2.9) \quad f_\nu := \frac{1}{\nu!} (\partial^\nu f)(0).$$

The polynomials ζ^ν , defined by (2.8), are called the Fueter polynomials. It can be proved that the Fueter polynomials are entire (both left and right) hyperholomorphic and that the series (2.7) is normally convergent. Thus one can characterize the right- \mathbb{H} -module $\mathcal{O}_{\mathbb{H}}$ of functions, left-hyperholomorphic in a neighborhood of the origin, as follows (see [13]):

Theorem 2.1. *An \mathbb{H} -valued function f , defined in a neighborhood of the origin, belongs to the space $\mathcal{O}_{\mathbb{H}}$ if, and only if, it can be represented in the form (2.7), where*

$$(2.10) \quad \rho(f) = \limsup_{n \rightarrow \infty} \left(\sum_{|\nu|=n} |f_\nu| \right)^{\frac{1}{n}} < \infty.$$

In this case the series (2.7) converges uniformly on compact subsets of the ball

$$\{x \in \mathbb{H} : |x| \cdot \rho(f) < 1\}.$$

Corollary 2.2. *An \mathbb{H} -valued polynomial p of real variables x_0, x_1, x_2, x_3 is left-hyperholomorphic if, and only if, it is a finite linear combination of Fueter polynomials:*

$$p(x) = \sum_{n=0}^m \sum_{|\nu|=n} \zeta^\nu p_\nu, \quad p_\nu \in \mathbb{H}.$$

Remark 2.3. In view of Theorem 2.1, in the quaternionic analysis the elementary functions ζ_n play role, similar in a sense to that of z_n in several complex variables. Thus ζ_n are sometimes called the hyperholomorphic variables. The term "total variables" is used also referring to the fact that both ζ_n and all its powers are hyperholomorphic, see [13], [18]. We note, however, that ζ_n are neither independent, nor \mathbb{H} -linear. Moreover, the choice of *left*-hyperholomorphic variables is not unique: e.g., $\zeta_n(\mathbf{e}_1 x)$ are also suitable for this role, but are not right-hyperholomorphic.

It is useful to calculate the expressions for the operators \mathcal{R}_n , defined by (2.6), in terms of expansions (2.7).

Lemma 2.4. *Let $f \in \mathcal{O}_{\mathbb{H}}(\Omega)$ be given by (2.7). Then*

$$(2.11) \quad \mathcal{R}_n f(x) = \sum_{\nu \geq e_n} \frac{\nu_n}{|\nu|} \zeta^{\nu-e_n}(x) f_\nu.$$

Proof. Without loss of generality, f is a Fueter polynomial. But

$$\frac{\partial \zeta^\nu}{\partial x_n}(x) = \nu_n \zeta^{\nu-e_n}(x),$$

hence

$$\mathcal{R}_n \zeta^\nu(x) = \int_0^1 \frac{\partial \zeta^\nu}{\partial x_n}(tx) dt = \int_0^1 \nu_n t^{|\nu|-1} \zeta^{\nu-e_n}(x) dt = \frac{\nu_n}{|\nu|} \zeta^{\nu-e_n}(x).$$

□

In view of Lemma 2.4, we propose the following

Definition 2.5. The operators $\mathcal{R}_n : \mathcal{O}_{\mathbb{H}} \mapsto \mathcal{O}_{\mathbb{H}}$, defined by (2.11), are called the backward-shift operators.

Following the analogy with the complex case, we would like to impose on $\mathcal{O}_{\mathbb{H}}$ the structure of a ring. However, the point-wise product is not suitable here. For instance, the function $\zeta_1 \zeta_2$ is not hyperholomorphic. Instead, one can use (see [13, Section 14] and compare with (1.2) – (1.4) in Section 1) the following

Definition 2.6. The Cauchy-Kovalevskaya product (below: C-K-product) $f \odot g$ of the functions

$$f = \sum \zeta^\nu f_\nu, \quad g = \sum \zeta^\nu g_\nu,$$

left hyperholomorphic in a neighborhood of the origin, is defined by

$$(2.12) \quad f \odot g = \sum_{n=0}^{\infty} \sum_{|\eta|=n} \zeta^\eta \sum_{0 \leq \nu \leq \eta} f_\nu g_{\eta-\nu}.$$

Remark 2.7. In certain special cases the C-K-product coincides with the point-wise one. For instance, if $g(x) \equiv \text{const}$ then $f \odot g = fg$, but not necessarily $g \odot f = gf$! Another special case is discussed in Section 3.4.

Proposition 2.8. *The space $\mathcal{O}_{\mathbb{H}}$, equipped with the C-K-product, is a ring. Moreover,*

$$\rho(f \odot g) \leq \max\{\rho(f), \rho(g)\}.$$

Proof. Without loss of generality, we take $\rho(f) = \rho(g) = \rho$. Then $\forall \epsilon > 0, \exists C(\epsilon) > 0, \forall k :$

$$\begin{aligned} \sum_{|\nu|=k} |a_{\nu}| &\leq C(\epsilon)(\rho + \epsilon)^k, \\ \sum_{|\nu|=k} |b_{\nu}| &\leq C(\epsilon)(\rho + \epsilon)^k. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{|\eta|=n} \left| \sum_{0 \leq \nu \leq \eta} a_{\nu} b_{\eta-\nu} \right| &\leq \sum_{|\eta|=n} \sum_{0 \leq \nu \leq \eta} |a_{\nu}| |b_{\eta-\nu}| \leq \sum_{k=0}^n \sum_{|\nu|=k} |a_{\nu}| \sum_{|\mu|=n-k} |b_{\mu}| \\ &\leq (n+1)C(\epsilon)^2(\rho + \epsilon)^n \end{aligned}$$

and so

$$\rho(f \odot g) \leq \rho.$$

□

The C-K-product can be generalized to spaces of matrix-valued left-hyperholomorphic functions in the usual way: for

$$F = (f_{\alpha,\beta}) \in \mathcal{O}_{\mathbb{H}}^{m \times n}, \quad G = (g_{\beta,\gamma}) \in \mathcal{O}_{\mathbb{H}}^{n \times p}$$

we define

$$F \odot G := \left(\sum_{\beta} f_{\alpha,\beta} \odot g_{\beta,\gamma} \right)_{\alpha,\gamma}.$$

The question arises, when an element $F \in \mathcal{O}_{\mathbb{H}}^{n \times n}$ is C-K-invertible. In view of (2.12), a necessary condition is that the value $F(0)$ must be invertible in $\mathbb{H}^{n \times n}$. This turns out to be also sufficient:

Proposition 2.9. *Let $F \in \mathcal{O}_{\mathbb{H}}^{n \times n}$. If $F(0)$ is invertible in \mathbb{H} then F is C-K-invertible in $\mathcal{O}_{\mathbb{H}}^{n \times n}$ and its C-K-inverse $F^{-\odot} \in \mathcal{O}_{\mathbb{H}}^{n \times n}$ is given by the series*

$$(2.13) \quad F^{-\odot} = (F(0))^{-1} \odot (I_n - G)^{-\odot} = (F(0))^{-1} \odot \sum_{k=0}^{\infty} G^{\odot k},$$

where

$$G = I_n - F(F(0))^{-1}.$$

Proof. It suffices to show the normal convergence of the series

$$(2.14) \quad (I_n - G)^{-\odot} = \sum_{k=0}^{\infty} G^{\odot k}$$

in a neighborhood of the origin for arbitrary $G \in \mathcal{O}_{\mathbb{H}}^{n \times n}$, satisfying $G(0) = 0$. According to Theorem 2.1,

$$G = \sum_{p=1}^{\infty} \sum_{|\nu|=p} \zeta^{\nu} A_{\nu},$$

and there exists $A \in \mathbb{R}^+$ such that

$$\forall p > 0 : \sum_{|\nu|=p} \|A_\nu\| \leq A^p,$$

where $\|\cdot\|$ denotes the operator norm. Then

$$G^{\odot k} = \sum_{p=k}^{\infty} \sum_{|\nu|=p} \zeta^\nu \sum_{\substack{\mu_1+\dots+\mu_k=\nu \\ \mu_1, \dots, \mu_k \neq 0}} A_{\mu_1} \cdots A_{\mu_k}.$$

But for $p \geq k \geq 1$

$$\begin{aligned} \sum_{|\nu|=p} \left\| \sum_{\substack{\mu_1+\dots+\mu_k=\nu \\ \mu_1, \dots, \mu_k \neq 0}} A_{\mu_1} \cdots A_{\mu_k} \right\| &\leq \sum_{\substack{p_1+\dots+p_k=p \\ p_1, \dots, p_k > 0}} \sum_{|\mu_1|=p_1} \|A_{\mu_1}\| \cdots \sum_{|\mu_k|=p_k} \|A_{\mu_k}\| \\ &\leq \binom{p-1}{k-1} A^p < (2A)^p. \end{aligned}$$

Therefore,

$$|x| < \frac{1}{4A} \implies \|G^{\odot k}(x)\| \leq \frac{1}{2^{k-1}},$$

and the normal convergence of the series (2.14) in the ball $\{x : |x| < 1/4A\}$ follows. \square

2.3. The Gleason problem in the hyperholomorphic case. In view of Remark 2.3, the formula (2.4) may be considered as a solution for a Gleason problem with respect to the hyperholomorphic variables ζ_n (see [8], [7] for details and references). However, there is a disadvantage in that the point-wise product appears. In particular, the individual terms $\zeta_n(x)\mathcal{R}_n f(x)$ in the sum (2.4) need not be left-hyperholomorphic, in general. The goal of the present section is to consider the Gleason problem with the point-wise product being replaced by the C-K product.

Definition 2.10. Let $f \in \mathcal{O}_{\mathbb{H}}$. The Gleason problem for f is to find a triple of functions $g_1, g_2, g_3 \in \mathcal{O}_{\mathbb{H}}$, such that

$$f - f(0) = \sum_{n=1}^3 \zeta_n \odot g_n.$$

It turns out that the backward-shift operators \mathcal{R}_n provide a solution for this new Gleason problem, as well.

Theorem 2.11. Let $f \in \mathcal{O}_{\mathbb{H}}$. Then it holds that

$$(2.15) \quad f - f(0) = \sum_{n=1}^3 \zeta_n \odot \mathcal{R}_n f.$$

Proof. According to (2.11), we have

$$\sum_{n=1}^3 \zeta_n \odot \mathcal{R}_n f = \sum_{n=1}^3 \sum_{\nu \geq e_n} \frac{\nu_n}{|\nu|} \zeta^\nu f_\nu = \sum_{|\nu| > 0} \zeta^\nu f_\nu = f - f(0).$$

\square

In general, the solution for the Gleason problem, provided by the backward-shift operators, is not the only possible one. To illustrate this observation, let us consider the subspaces of $\mathcal{O}_{\mathbb{H}}^m$, in which the problem is solvable.

Definition 2.12. A subspace \mathcal{W} of $\mathcal{O}_{\mathbb{H}}^m$ is said to be resolvent-invariant if

$$\forall f \in \mathcal{W} \exists g_1, g_2, g_3 \in \mathcal{W} : f - f(0) = \sum_{n=1}^3 \zeta_n \odot g_n.$$

If, moreover, the space \mathcal{W} is \mathcal{R}_n -invariant for $n = 1, 2, 3$, it is said to be backward-shift-invariant.

Theorem 2.13. A finite-dimensional subspace \mathcal{W} of $\mathcal{O}_{\mathbb{H}}^m$ is resolvent-invariant (respectively, backward-shift-invariant) if, and only if, it is spanned by the columns of a matrix-valued function of the form

$$(2.16) \quad W = C \odot (I - \sum_{n=1}^3 \zeta_n A_n)^{-\odot},$$

where C and A_n are constant matrices with entries in \mathbb{H} (respectively, A_n commute).

In the proof of Theorem 2.13 we shall use the following

Lemma 2.14. Let A_1, A_2 and A_3 be in $\mathbb{H}^{\ell \times \ell}$. Then in a neighborhood of the origin it holds that

$$(2.17) \quad (I_{\ell} - \zeta_1 A_1 - \zeta_2 A_2 - \zeta_3 A_3)^{-\odot} = \sum_{\nu \in \mathbb{Z}_+^3} \zeta^{\nu} A^{\nu} \frac{|\nu|!}{\nu!},$$

where

$$(2.18) \quad A^{\nu} = A_1^{\times \nu_1} \times A_2^{\times \nu_2} \times A_3^{\times \nu_3}.$$

Proof. We have

$$(I_{\ell} - \zeta_1 A_1 - \zeta_2 A_2 - \zeta_3 A_3)^{-\odot} = \sum_{k=0}^{\infty} (\zeta_1 A_1 + \zeta_2 A_2 + \zeta_3 A_3)^{\odot k}.$$

Let us prove by induction on k that

$$(2.19) \quad (\zeta_1 A_1 + \zeta_2 A_2 + \zeta_3 A_3)^{\odot k} = \sum_{|\nu|=k} \zeta^{\nu} (A_1^{\times \nu_1} \times A_2^{\times \nu_2} \times A_3^{\times \nu_3}) \frac{|\nu|!}{\nu!}.$$

Indeed, (2.19) obviously holds for $k = 0$, and if it holds for some k then we have

$$\begin{aligned} (\zeta_1 A_1 + \zeta_2 A_2 + \zeta_3 A_3)^{\odot(k+1)} &= (\zeta_1 A_1 + \zeta_2 A_2 + \zeta_3 A_3) \odot \sum_{|\nu|=k} \zeta^{\nu} A^{\nu} \frac{|\nu|!}{\nu!} \\ &= \sum_{|\nu|=k+1} \zeta^{\nu} \frac{(|\nu|-1)!}{\nu!} (\nu_1 A_1 A^{\nu-e_1} + \nu_2 A_2 A^{\nu-e_2} + \nu_3 A_3 A^{\nu-e_3}) \\ &= \sum_{|\nu|=k+1} \zeta^{\nu} A^{\nu} \frac{|\nu|!}{\nu!}. \end{aligned}$$

□

Proof of Theorem 2.13. Let \mathcal{W} be a resolvent-invariant finite-dimensional subspace of $\mathcal{O}_{\mathbb{H}}^m$ and let W be a matrix-valued hyperholomorphic function whose columns

form a basis of \mathcal{W} . Then there exist constant matrices $A_n \in \mathbb{H}^{\ell \times \ell}$ (with $\ell = \dim \mathcal{W}$) and $C = W(0)$, such that

$$W = C + \sum_{n=1}^3 \zeta_n \odot W A_n = C + W \odot \sum_{n=1}^3 \zeta_n A_n,$$

hence W is of the form (2.16). If \mathcal{W} is, moreover, backward-shift-invariant then A_n can be chosen such that $\mathcal{R}_n W = W A_n$. Then A_n commute since the R_n do.

Conversely, let \mathcal{W} be the span of the columns of a matrix-valued function of the form (2.16). Then

$$\begin{aligned} W - W(0) &= C \odot \left(I - \sum_{n=1}^3 \zeta_n A_n \right)^{-\odot} - C \\ &= W \odot \left(I - \left(I - \sum_{n=1}^3 \zeta_n A_n \right) \right) = \sum_{n=1}^3 \zeta_n \odot W A_n, \end{aligned}$$

and hence \mathcal{W} is resolvent-invariant. If, moreover, the matrices A_n commute then, according to Lemma 2.14,

$$W = \sum_{\nu \in \mathbb{Z}_+^3} \zeta^\nu C A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} \frac{|\nu|!}{\nu!},$$

hence $\mathcal{R}_n W = W A_n$. This completes the proof. \square

3. RATIONAL HYPERHOLOMORPHIC FUNCTIONS

3.1. Definitions. In this section we give three definitions of a rational function, left-hyperholomorphic in a neighborhood of the origin. We prove that they are equivalent in Section 3.3.

The first definition parallels the classical definition in terms of quotients of polynomials in the complex case. Here polynomials are replaced by the Fueter polynomials, point-wise multiplication is replaced by the C-K-product, and inverses are replaced by the C-K-inverses.

Definition 3.1. An $\mathbb{H}^{m \times n}$ -valued function R , left-hyperholomorphic in a neighborhood of the origin, is said to be rational if all its entries belong to the minimal subring $\mathcal{Q}_{\mathbb{H}}$ of $\mathcal{O}_{\mathbb{H}}$, which contains hyperholomorphic polynomials and is closed under C-K-inversion:

$$r \in \mathcal{Q}_{\mathbb{H}}, r(0) \neq 0 \implies \exists r^{-\odot} \in \mathcal{Q}_{\mathbb{H}}.$$

Example 3.2. Let $j = 1, 2, 3$. The functions ζ_j, ζ_j^2 and more generally all the Fueter polynomials are rational.

Example 3.3. The function

$$\left(((1 - \zeta_1 \mathbf{e}_1)^{-\odot} + 2)^{-\odot} + \zeta_1 \odot \zeta_2^{\odot 3} \right)^{-\odot} + \zeta_3^{\odot 5} \mathbf{e}_3$$

is rational.

The next example will play an important role in the sequel.

Example 3.4. Let $a \in \mathbb{H}$. The function

$$(3.1) \quad x \mapsto (1 - \zeta_1 \overline{\zeta_1(a)} - \zeta_2 \overline{\zeta_2(a)} - \zeta_3 \overline{\zeta_3(a)})^{-\odot}$$

is rational.

The second definition parallels the realization (1.5) (see Section 1) in the complex case.

Definition 3.5. An $\mathbb{H}^{m \times n}$ -valued function R , left-hyperholomorphic in a neighborhood of the origin, is said to be rational if it can be represented in the form

$$(3.2) \quad R = D + C \odot (I - \zeta_1 A_1 - \zeta_2 A_2 - \zeta_3 A_3)^{-\odot} \odot (\zeta_1 B_1 + \zeta_2 B_2 + \zeta_3 B_3),$$

where A_i, B_i, C and D are constant matrices with entries in \mathbb{H} and of appropriate dimensions.

For brevity, from now on we shall use the notation

$$(3.3) \quad \zeta_{(\ell)} := (\zeta_1 I_\ell \quad \zeta_2 I_\ell \quad \zeta_3 I_\ell) \in \mathcal{O}_{\mathbb{H}}^{\ell \times 3\ell}.$$

The dimension ℓ will usually be understood from the context and omitted. Then (3.2) can be rewritten as:

$$(3.4) \quad R = D + C \odot (I - \zeta A)^{-\odot} \odot \zeta B$$

where

$$(3.5) \quad A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \text{ and } B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

The third definition is in terms of the resolvent-invariance.

Definition 3.6. An $\mathbb{H}^{m \times n}$ -valued function R , left-hyperholomorphic in a neighborhood of the origin, is said to be rational if there is a finite-dimensional resolvent-invariant space $\mathcal{W} \subset \mathcal{O}_{\mathbb{H}}^m$, such that for every $v \in \mathbb{H}^n$ the Gleason problem for Rv is solvable in \mathcal{W} .

The main result of the paper, presented in Section 3.3, is that all three definitions are equivalent. In view of Proposition 2.14, they are also equivalent to the following

Definition 3.7. An $\mathbb{H}^{m \times n}$ -valued function R , left-hyperholomorphic in a neighborhood of the origin, is said to be rational if it can be represented as

$$R = \sum_{n=0}^{\infty} \sum_{|\nu|=n} \zeta^\nu R_\nu,$$

where for $|\nu| \geq 1$

$$R_\nu = \frac{(|\nu| - 1)!}{\nu!} C \begin{pmatrix} \nu_1 A^{\nu-e_1} & \nu_2 A^{\nu-e_2} & \nu_3 A^{\nu-e_3} \end{pmatrix} B$$

with A, B, C being constant matrices of appropriate dimensions.

3.2. Preparatory lemmas. The proof of the equivalence of Definitions 3.1 – 3.6 is based on several technical lemmas.

Lemma 3.8. *Let $R \in \mathcal{O}_{\mathbb{H}}^{n \times n}$ admit the representation (3.4), where $D \in \mathbb{H}^{n \times n}$ is invertible. Then R is C - K -invertible and its C - K -inverse $R^{-\odot}$ admits the representation*

$$(3.6) \quad R^{-\odot} = D^{-1} - D^{-1}C \odot (I - Z\tilde{A})^{-\odot} \odot ZBD^{-1},$$

where $\tilde{A} = A - BD^{-1}C$.

Proof. We have:

$$\begin{aligned} & (D + C \odot (I - \zeta A)^{-\odot} \odot \zeta B) \odot (D^{-1} - D^{-1}C \odot (I - \zeta \tilde{A})^{-\odot} \odot \zeta BD^{-1}) \\ &= I - C \odot (I - \zeta \tilde{A})^{-\odot} \odot \zeta BD^{-1} + C \odot (I - \zeta A)^{-\odot} \odot \zeta BD^{-1} - \\ & \quad - C \odot (I - \zeta A)^{-\odot} \odot \zeta BD^{-1}C \odot (I - \zeta \tilde{A})^{-\odot} \odot \zeta BD^{-1} \\ &= I - C \odot \{ (I - \zeta \tilde{A})^{-\odot} - (I - \zeta A)^{-\odot} + \\ & \quad + (I - \zeta A)^{-\odot} \odot \zeta BD^{-1}C \odot (I - \zeta \tilde{A})^{-\odot} \} \odot \zeta BD^{-1}. \end{aligned}$$

But

$$\zeta BD^{-1}C = \zeta(A - \tilde{A}) = (I - \zeta \tilde{A}) - (I - \zeta A),$$

hence the expression in the curly brackets is equal to 0. \square

Lemma 3.9. *There exists a unitary matrix $U \in \mathbb{H}^{3(\ell+m) \times 3(\ell+m)}$ such that*

$$(3.7) \quad \text{diag}(\zeta_{(\ell)}, \zeta_{(m)}) = \zeta_{(\ell+m)} U.$$

Proof. It suffices to take

$$U = \begin{pmatrix} I_{\ell} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & I_{\ell} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & I_{\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m \end{pmatrix}.$$

\square

Lemma 3.10. *Let $R_i \in \mathcal{O}_{\mathbb{H}}^{m_i \times n_i}$, $i = 1, 2$, admit the representations*

$$R_i(x) = D^{(i)} + C^{(i)} \odot (I - \zeta A^{(i)})^{-\odot} \odot \zeta B^{(i)}.$$

If $n_2 = m_1$ then $R_1 \odot R_2$ admits the representation

$$\begin{aligned} R_1 \odot R_2 &= D^{(1)}D^{(2)} + \\ &+ (C^{(1)} \quad D^{(1)}C^{(2)}) \odot \left(I - \zeta U \begin{pmatrix} A^{(1)} & B^{(1)}C^{(2)} \\ 0 & A^{(2)} \end{pmatrix} \right)^{-\odot} \odot \zeta U \begin{pmatrix} B^{(1)}D^{(2)} \\ B^{(2)} \end{pmatrix}. \end{aligned}$$

If $m_1 = m_2$, $n_1 = n_2$ then $R_1 + R_2$ admits the representation

$$\begin{aligned} R_1 + R_2 &= D^{(1)} + D^{(2)} + \\ &+ (C^{(1)} \quad C^{(2)}) \odot \left(I - \zeta U \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \right)^{-\odot} \odot \zeta U \begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix}. \end{aligned}$$

In both formulas U is as in Lemma 3.9.

Proof. We have:

$$\begin{aligned}
R_1 \odot R_2 &= D^{(1)}D^{(2)} + D^{(1)}C^{(2)} \odot (I - \zeta A^{(2)})^{-\odot} \odot \zeta B^{(2)} + \\
&\quad + C^{(1)} \odot (I - \zeta A^{(1)})^{-\odot} \odot \zeta B^{(1)}D^{(2)} + \\
&\quad + C^{(1)} \odot (I - \zeta A^{(1)})^{-\odot} \odot \zeta B^{(1)}C^{(2)} \odot (I - \zeta A^{(2)})^{-\odot} \odot \zeta B^{(2)} \\
&= D^{(1)}D^{(2)} + (C^{(1)} \quad D^{(1)}C^{(2)}) \odot \begin{pmatrix} \alpha^{-\odot} & -\alpha^{-\odot} \odot \beta \odot \gamma^{-\odot} \\ 0 & \gamma^{-\odot} \end{pmatrix} \odot \begin{pmatrix} \zeta B^{(1)}D^{(2)} \\ \zeta B^{(2)} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= I - \zeta A^{(1)}, \\
\beta &= -\zeta B^{(1)}C^{(2)}, \\
\gamma &= I - \zeta A^{(2)}.
\end{aligned}$$

Using the formula

$$\begin{pmatrix} \alpha^{-\odot} & -\alpha^{-\odot} \odot \beta \odot \gamma^{-\odot} \\ 0 & \gamma^{-\odot} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}^{-\odot},$$

we have:

$$\begin{aligned}
R_1 \odot R_2 &= D^{(1)}D^{(2)} + \\
&\quad + (C^{(1)} \quad D^{(1)}C^{(2)}) \odot \begin{pmatrix} I - \zeta A^{(1)} & -\zeta B^{(1)}C^{(2)} \\ 0 & I - \zeta A^{(2)} \end{pmatrix}^{-\odot} \odot \begin{pmatrix} \zeta B^{(1)}D^{(2)} \\ \zeta B^{(2)} \end{pmatrix} \\
&\quad = D^{(1)}D^{(2)} + \\
&\quad + (C^{(1)} \quad D^{(1)}C^{(2)}) \odot \left(I - \zeta U \begin{pmatrix} A^{(1)} & B^{(1)}C^{(2)} \\ 0 & A^{(2)} \end{pmatrix} \right)^{-\odot} \odot \zeta U \begin{pmatrix} B^{(1)}D^{(2)} \\ B^{(2)} \end{pmatrix}.
\end{aligned}$$

In order to obtain the second formula, it is enough to apply the first one for

$$(R_1 \quad I) \odot \begin{pmatrix} I \\ R_2 \end{pmatrix}.$$

□

3.3. Equivalence between the various definitions.

Proposition 3.11. *Definitions 3.6 and 3.5 are equivalent.*

Proof. Indeed, if $R \in \mathcal{O}_{\mathbb{H}}^{m \times n}$ admits the representation (3.4), where $A_i \in \mathbb{H}^{p \times p}$, let us denote by \mathcal{W} the span of columns of the matrix-function $W = C \odot (I - \zeta A)^{-\odot}$. According to Theorem 2.13, the finite-dimensional space \mathcal{W} is resolvent-invariant, and $\forall v \in \mathbb{H}^n$ the functions

$$G_k = C \odot (I - \zeta A)^{-\odot} B_k v \in \mathcal{W}, \quad k = 1, 2, 3,$$

are a solution of the Gleason problem for Rv .

Conversely, assume that $\mathcal{W} \subset \mathcal{O}_{\mathbb{H}}^m$ is a finite-dimensional resolvent-invariant space, in which $\forall v \in \mathbb{H}^n$ the Gleason problem for Rv is solvable. According to

Theorem 2.13, there exists a matrix-function of the form $W = C \odot (I - \zeta A)^{-\odot}$, whose columns span \mathcal{W} . Hence there exist constant matrices B_k such that

$$R - R(0) = \sum_{k=1}^3 \zeta_k \odot W B_k.$$

Since the hyperholomorphic variables (and more, generally, all the Fueter polynomials) belong to the center of the ring $\mathcal{O}_{\mathbb{H}}$, we obtain for R the representation (3.2) with $D = R(0)$. \square

Proposition 3.12. *Definitions 3.1 and 3.5 are equivalent.*

Proof. First of all we note that, in view of Lemmas 3.8 and 3.10, the space of elements of $\mathcal{O}_{\mathbb{H}}$ which admit the representation (3.2) is a subring, which is closed under the C-K-inversion. Substituting in (3.2) $B_k = 0$ (respectively, $C = B_k = 1$, $D = A_k = 0$), we see that this subring contains constant functions (respectively, hyperholomorphic variables), hence it also contains $\mathcal{Q}_{\mathbb{H}}$. In other words, every function, rational in the sense of Definition 3.1, admits the representation (3.2).

In order to prove the converse implication, it suffices to show that every entry of

$$\left(I - \sum_{k=1}^3 \zeta_k A_k \right)^{-\odot}$$

belongs to $\mathcal{Q}_{\mathbb{H}}$. We proceed by induction on $\dim A_k$. Denote

$$A_k = \begin{pmatrix} \check{A}_k & \check{a}_k \\ \hat{a}_k & \hat{A}_k \end{pmatrix}.$$

Then

$$I - \sum_{k=1}^3 \zeta_k A_k = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= I - \sum_{k=1}^3 \zeta_k \check{A}_k, \quad \beta = - \sum_{k=1}^3 \zeta_k \check{a}_k, \\ \gamma &= - \sum_{k=1}^3 \zeta_k \hat{a}_k, \quad \delta = I - \sum_{k=1}^3 \zeta_k \hat{A}_k. \end{aligned}$$

Furthermore,

$$\begin{aligned} (I - \sum_{k=1}^3 \zeta_k A_k)^{-\odot} &= \\ &= \begin{pmatrix} (\tilde{\alpha})^{-\odot} & -(\tilde{\alpha})^{-\odot} \odot \beta \odot \delta^{-\odot} \\ -\delta^{-\odot} \odot \gamma \odot (\tilde{\alpha})^{-\odot} & \delta^{-\odot} + \delta^{-\odot} \odot \gamma \odot (\tilde{\alpha})^{-\odot} \odot \beta \odot \delta^{-\odot} \end{pmatrix}, \end{aligned}$$

where

$$\tilde{\alpha} = \alpha - \beta \odot \delta^{-\odot} \odot \gamma.$$

By definition and the induction assumption, all the entries of $(\tilde{\alpha})^{-\odot}, \beta, \gamma, \delta^{-\odot}$ belong to $\mathcal{Q}_{\mathbb{H}}$. \square

3.4. Rational functions of two complex variables.

Writing $x = z_1 + z_2 \mathbf{e}_2$, where $z_1 = x_0 + x_1 \mathbf{e}_1$, $z_2 = x_2 + x_3 \mathbf{e}_1$,

one can identify the skew field of quaternions \mathbb{H} with the two-dimensional complex space \mathbb{C}^2 , endowed with the special structure where, in particular,

$$z \mathbf{e}_2 = \mathbf{e}_2 \bar{z}.$$

The complex variables z_1 and z_2 have the following properties: z_1 is (both right and left) hyperholomorphic, z_2 is right-hyperholomorphic, \bar{z}_2 is left-hyperholomorphic. It holds that

$$z_1(x) = \zeta_1(x) \mathbf{e}_1, \quad \bar{z}_2(x) = \zeta_2(x) - \zeta_3(x) \mathbf{e}_1, \quad x \in \mathbb{R}^4.$$

Moreover, it follows from (2.9) that

$$z_1^m \bar{z}_2^n = z_1^{\odot m} \odot \bar{z}_2^{\odot n}.$$

From here we conclude that complex-valued functions of two complex variables z_1 and \bar{z}_2 , holomorphic in a neighborhood of the origin, are also left-hyperholomorphic, for which the C-K-product and the point-wise product coincide. It follows that rational functions of z_1 and \bar{z}_2 , holomorphic in a neighborhood of the origin, are also rational in the sense of our Definitions 3.1 – 3.6.

4. QUATERNIONIC ARVESON SPACE

4.1. Positive rational kernel. In this section we define and study what we believe to be the appropriate counterpart of the Arveson space of the unit ball (see Section 1) in the setting of left-hyperholomorphic functions.

To begin with, let us recall Definition 2.5 of the backward-shift operators \mathcal{R}_n and formulate the following

Proposition 4.1. *The common eigenvectors of the backward-shift operators \mathcal{R}_1 , \mathcal{R}_2 , \mathcal{R}_3 are functions of the form*

$$(1 - \zeta a)^{-\odot},$$

where $a \in \mathbb{H}^3$.

Proof. This is a special case of Theorem 2.13: we are looking for 1-dimensional backward-shift-invariant spaces. \square

Set

$$(4.1) \quad \Omega = \{x \in \mathbb{H} : 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1\},$$

$$(4.2) \quad k_y(x) = (1 - \zeta \zeta(y)^*)^{-\odot}(x), \quad y \in \Omega.$$

According to Definition 3.1, the left-hyperholomorphic function k_y is rational. In view of Lemma 2.14, we have

$$(4.3) \quad k_y(x) = \sum_{\nu \in \mathbb{Z}_+^3} \frac{|\nu|!}{\nu!} \zeta^\nu(x) \overline{\zeta^\nu(y)}.$$

The function $k_y(x)$ is therefore positive in Ω and there exists an associated right-linear reproducing kernel Hilbert space which is an extension of the classical Hardy space; see [6]. As a direct consequence of the power expansion (4.3) we obtain:

Theorem 4.2. *The reproducing kernel right-linear Hilbert space $\mathbf{H}(k)$ with reproducing kernel $k_y(x)$ (we shall call this space the (left) quaternionic Arveson space) is the set of functions of the form (2.7) endowed with the \mathbb{H} -valued inner product*

$$(4.4) \quad \langle f, g \rangle = \sum_{\nu \in \mathbb{Z}_+^3} \frac{\nu!}{|\nu|!} \overline{g_\nu} f_\nu.$$

Remark 4.3. We note that ζ^ν and ζ^μ are orthogonal in the Arveson space when $\nu \neq \mu$. In the quaternionic Hardy space this orthogonality condition holds only when moreover $|\nu| \neq |\mu|$.

Let us consider the C-K-multiplication operators

$$(4.5) \quad \mathcal{M}_{\zeta_n} f = \zeta_n \odot f, \quad n = 1, 2, 3.$$

Proposition 4.4. *For $n = 1, 2, 3$ the operator \mathcal{M}_{ζ_n} is a contraction from the space $\mathbf{H}(k)$ into itself.*

Proof. For $f \in \mathbf{H}(k)$ we have

$$\begin{aligned} \langle \mathcal{M}_{\zeta_n} f, \mathcal{M}_{\zeta_n} f \rangle &= \sum_{\nu \in \mathbb{Z}_+^3} \frac{(\nu + e_n)!}{|\nu + e_n|!} |f_\nu|^2 = \sum_{\nu \in \mathbb{Z}_+^3} \frac{\nu_n + 1}{|\nu| + 1} \frac{\nu!}{|\nu|!} |f_\nu|^2 \\ &\leq \sum_{\nu \in \mathbb{Z}_+^3} \frac{\nu!}{|\nu|!} |f_\nu|^2 = \langle f, f \rangle. \end{aligned}$$

□

Proposition 4.4 implies, in particular, that the C-K-multiplication operator \mathcal{M}_{ζ_n} is a bounded linear operator from $\mathbf{H}(k)$ into itself, hence, according to the quaternionic version of the Riesz theorem (see [13] and [23] for more details on quaternionic Hilbert spaces and quaternionic adjoint operators), it has the Hilbert adjoint $\mathcal{M}_{\zeta_n}^* : \mathbf{H}(k) \mapsto \mathbf{H}(k)$, defined by

$$\langle \mathcal{M}_{\zeta_n} f, g \rangle = \langle f, \mathcal{M}_{\zeta_n}^* g \rangle \quad \forall f, g \in \mathbf{H}(k).$$

The latter turns out to coincide with the backward-shift operator \mathcal{R}_n :

Proposition 4.5.

$$\mathcal{M}_{\zeta_n}^* = \mathcal{R}_n|_{\mathbf{H}(k)}.$$

Proof. We have $\forall \nu \in \mathbb{Z}_+^3, \forall \mu \geq e_n$:

$$\langle \mathcal{R}_n \zeta^\mu, \zeta^\nu \rangle = \langle \frac{\mu_n}{|\mu|} \zeta^{\mu - e_n}, \zeta^\nu \rangle = \frac{(\nu + e_n)!}{|\nu + e_n|!} \delta_\nu^{\mu - e_n} = \langle \zeta^\mu, \zeta^{\nu + e_n} \rangle = \langle \zeta^\mu, \mathcal{M}_{\zeta_n} \zeta^\nu \rangle.$$

Analogously, if $\mu_n = 0$ then

$$\langle \mathcal{R}_n \zeta^\mu, \zeta^\nu \rangle = 0 = \langle \zeta^\mu, \zeta^{\nu + e_n} \rangle = \langle \zeta^\mu, \mathcal{M}_{\zeta_n} \zeta^\nu \rangle.$$

□

Let us denote by $\mathcal{C} : \mathbf{H}(k) \mapsto \mathbb{H}$ the operator of evaluation at the origin: $\mathcal{C}f := f(0)$. Then, in view of Proposition 4.5 and Theorem 2.11 for the C-K-multiplication operator $\mathcal{M}_\zeta : \mathbf{H}(k)^3 \mapsto \mathbf{H}(k)$ the following operator identity holds true

$$(4.6) \quad I - \mathcal{M}_\zeta \mathcal{M}_\zeta^* = \mathcal{C}^* \mathcal{C}.$$

The identity (4.6) is the quaternionic counterpart of (1.1). In the next section we shall use it to obtain the counterpart of the Blaschke factors in the quaternionic Arveson space.

4.2. Blaschke factors.

Definition 4.6. Let $a \in \Omega$. We define the Blaschke factor $B_a \in \mathbf{H}(k)^{1 \times 3}$ by

$$(4.7) \quad B_a = (1 - \zeta(a)\zeta(a)^*)^{\frac{1}{2}}(1 - \zeta\zeta(a)^*)^{-\odot} \odot (\zeta - \zeta(a)) (I - \zeta(a)^*\zeta(a))^{-\frac{1}{2}}.$$

Theorem 4.7. *The C-K-multiplication operator*

$$\mathcal{B}_a = \mathcal{M}_{B_a} : \mathbf{H}(k)^3 \mapsto \mathbf{H}(k)$$

is a contraction, and the following operator identity holds:

$$(4.8) \quad I - \mathcal{B}_a \mathcal{B}_a^* = (1 - \zeta(a)^*\zeta(a)) \left(I - \mathcal{M}_\zeta \mathcal{M}_{\zeta(a)}^* \right)^{-1} \mathcal{C}^* \mathcal{C} \left(I - \mathcal{M}_\zeta \mathcal{M}_{\zeta(a)}^* \right)^{-*}.$$

Proof. The proof follows the arguments of [4]. We first note that the operators

$$I - \mathcal{M}_{\zeta(a)} \mathcal{M}_{\zeta(a)}^* \quad \text{and} \quad I - \mathcal{M}_{\zeta(a)}^* \mathcal{M}_{\zeta(a)}$$

are self-adjoint and strictly contractive and hence the operators

$$\left(I - \mathcal{M}_{\zeta(a)} \mathcal{M}_{\zeta(a)}^* \right)^{\pm 1/2} \quad \text{and} \quad \left(I - \mathcal{M}_{\zeta(a)}^* \mathcal{M}_{\zeta(a)} \right)^{\pm 1/2}$$

are well defined. We set

$$\mathcal{H} := \begin{pmatrix} \left(I - \mathcal{M}_{\zeta(a)} \mathcal{M}_{\zeta(a)}^* \right)^{-1/2} & -\mathcal{M}_{\zeta(a)} \left(I - \mathcal{M}_{\zeta(a)}^* \mathcal{M}_{\zeta(a)} \right)^{-1/2} \\ -\mathcal{M}_{\zeta(a)}^* \left(I - \mathcal{M}_{\zeta(a)} \mathcal{M}_{\zeta(a)}^* \right)^{-1/2} & \left(I - \mathcal{M}_{\zeta(a)}^* \mathcal{M}_{\zeta(a)} \right)^{-1/2} \end{pmatrix}.$$

Then it holds that

$$\mathcal{H} J \mathcal{H}^* = \mathcal{H}^* J \mathcal{H} = J,$$

where

$$J = \begin{pmatrix} I_{\mathbf{H}(k)} & 0 \\ 0 & -I_{\mathbf{H}(k)^3} \end{pmatrix}.$$

(\mathcal{H} is "the Halmos extension" of $-\mathcal{M}_{\zeta(a)}$; see [15].) Thus

$$(4.9) \quad \mathcal{C}^* \mathcal{C} = I - \mathcal{M}_\zeta \mathcal{M}_\zeta^* = \begin{pmatrix} I & \mathcal{M}_\zeta \end{pmatrix} J \begin{pmatrix} I \\ \mathcal{M}_\zeta^* \end{pmatrix} = \begin{pmatrix} I & \mathcal{M}_\zeta \end{pmatrix} \mathcal{H} J \mathcal{H}^* \begin{pmatrix} I \\ \mathcal{M}_\zeta^* \end{pmatrix} = \mathcal{X} J \mathcal{X}^*,$$

where

$$\begin{aligned} \mathcal{X} &= (\mathcal{X}_1 \quad \mathcal{X}_2), \\ \mathcal{X}_1 &= (I - \mathcal{M}_\zeta \mathcal{M}_{\zeta(a)}^*) \left(I - \mathcal{M}_{\zeta(a)} \mathcal{M}_{\zeta(a)}^* \right)^{-1/2}, \\ \mathcal{X}_2 &= (\mathcal{M}_\zeta - \mathcal{M}_{\zeta(a)}) \left(I - \mathcal{M}_{\zeta(a)}^* \mathcal{M}_{\zeta(a)} \right)^{-1/2}. \end{aligned}$$

To conclude we remark that the operator $I - \mathcal{M}_{\zeta(a)} \mathcal{M}_{\zeta(a)}^*$ is the operator of multiplication by the positive number $1 - \zeta(a)\zeta(a)^*$ and therefore commutes with all the other operators under consideration. Multiplying the first and the last expressions in the equality (4.9) by

$$\left(I - \mathcal{M}_{\zeta(a)} \mathcal{M}_{\zeta(a)}^* \right)^{1/2} (I - \mathcal{M}_\zeta \mathcal{M}_{\zeta(a)}^*)^{-1}$$

on the left and by its adjoint on the right we obtain (4.8). \square

Theorem 4.7 allows to get some preliminary results on interpolation in the Arveson space. Here we have:

Theorem 4.8. *Let $a \in \Omega$. Then*

$$(4.10) \quad \{f \in \mathcal{H}(k) : f(a) = 0\} \subset \text{ran } \mathcal{B}_a.$$

Proof. The identity (4.8) in Theorem 4.7 implies that

$$\text{ran}(I - \mathcal{B}_a \mathcal{B}_a^*) = \text{span}(k_a).$$

Hence

$$\ker(I - \mathcal{B}_a \mathcal{B}_a^*) = \{f \in \mathcal{H}(k) : \langle f, k_a \rangle = f(a) = 0\}.$$

On the other hand, $\ker(I - \mathcal{B}_a \mathcal{B}_a^*) \subset \text{ran } \mathcal{B}_a$. \square

We note that inequality is strict in (4.10). Indeed, the space $\text{ran } \mathcal{B}_a$ is invariant under the operators \mathcal{M}_{ζ_n} while the set of left-hyperholomorphic functions vanishing at a is not, if $a \neq 0$.

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