2012

# White Noise Based Stochastic Calculus Associated With a Class of Gaussian Processes

Daniel Alpay
*Chapman University*, alpay@chapman.edu

Haim Attia
*Sami Shamoon College of Engineering*

David Levanony
*Ben Gurion University of the Negev*



Recommended Citation

D. Alpay, H. Attia and D. Levanony. White noise based stochastic calculus associated with a class of Gaussian processes. Opuscula Mathematica, vol. 32/3 (2012), pp. 401-422.

Opuscula Mathematica • Vol. 32 • No. 3 • 2012

http://dx.doi.org/10.7494/OpMath.2012.32.3.401

# WHITE NOISE BASED STOCHASTIC CALCULUS ASSOCIATED WITH A CLASS OF GAUSSIAN PROCESSES

Daniel Alpay, Haim Attia, and David Levanony

**Abstract.** Using the white noise space setting, we define and study stochastic integrals with respect to a class of stationary increment Gaussian processes. We focus mainly on continuous functions with values in the Kondratiev space of stochastic distributions, where use is made of the topology of nuclear spaces. We also prove an associated Ito formula.

**Keywords:** white noise space, Wick product, stochastic integral.

**Mathematics Subject Classification:** 60H40, 60H05, 60G15, 60G22, 46A12.

## 1. INTRODUCTION

In this paper we study stochastic integration with respect to stationary increment Gaussian processes $\{X_m(t), t \in \mathbb{R}\}$ with covariance functions of the form

$$\begin{aligned} C_m(t,s) \overset{\text{def.}}{=} E[X_m(t)\overline{X_m}(s)] &= \int_{\mathbb{R}} \frac{(e^{iut}-1)(e^{-ius}-1)}{u^2} m(u)du = \\ &= r(t) + \overline{r(s)} - r(t-s) - r(0), \end{aligned} \tag{1.1}$$

where $m$ is a measurable positive function subject to

$$\int_{\mathbb{R}} \frac{m(u)du}{1+u^2} < \infty, \tag{1.2}$$

and

$$m(u) \leq \begin{cases} K\,|u|^{-b} & \text{if} \quad |u| \leq 1, \\ K|u|^{2N} & \text{if} \quad |u| > 1, \end{cases} \tag{1.3}$$

with $b < 2$, $N \in \mathbb{N}_0$, and $0 < K < \infty$, where

$$r(t) = -\int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2+1} \right\} \frac{m(u)}{u^2} du.$$

This class includes in particular fractional Brownian motion; see (1.6) below. An associated Ito formula is subsequently derived. We use methods from infinite dimensional analysis, and the paper should be of interest to readers in this field, as well as to readers in stochastic processes, operator theory and reproducing kernel spaces.

We work in the white noise space setting as developed by T. Hida, and in particular the Gelfand triple $(S_1, \mathcal{W}, S_{-1})$ consisting of the Kondratiev space $S_1$ of stochastic test functions, of the white noise space $\mathcal{W}$, and the Kondratiev space $S_{-1}$ of stochastic distributions; see [21–23]. Various notions pertaining to these works, which are used in the introduction, are recalled in Sections 2 and 4. In infinite dimensional analysis usually a number of Gelfand triples may be used to study a given problem. The reason of using this Gelfand triple is the existence of an inequality, called Våge inequality, associated with the Wick product, (see (4.13)), which allows us to use locally Hilbert space methods and offers a powerful framework for stochastic integration. The *a priori* estimate (4.13) for the Wick product offers new insights into convergence questions in the representing formulas for the stochastic processes under consideration, and improvements and extensions of earlier versions of Ito integration and formulas. For other recent applications of this inequality to problems in infinite dimensional analysis, see for instance [2] and [7].

Explicit constructions of $X_m$ and its derivative, which we use below, are detailed in [4] utilizing this setting. A key role in the arguments of [4] is played by the operator $T_m$, defined via

$$\widehat{T_m f}(u) \stackrel{\text{def.}}{=} \sqrt{m(u)}\widehat{f}(u), \tag{1.4}$$

where $\widehat{f}$ denotes the Fourier transform of $f$:

$$\widehat{f}(u) = \int_{\mathbb{R}} e^{-iux} f(x) dx.$$

Since $m$ satisfies (1.3), the domain of $T_m$ contains in particular the Schwartz space $\mathscr{S}(\mathbb{R})$. We note that the operator $T_m$ will not be local in general: The support of $T_m f$ need not be included in the support of $f$. The example

$$m(u) = u^4 e^{-2u^2}, \tag{1.5}$$

given in [4] illustrates this point. The choice

$$m(u) = \frac{1}{2\pi}|u|^{1-2H}, \quad H \in (0,1), \tag{1.6}$$

corresponds to the fractional Brownian motion $B_H$ with Hurst parameter $H$, such that

$$E(B_H(t)B_H(s)) = V_H \left\{ |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right\},$$

where

$$V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi(1-2H)H}, \tag{1.7}$$

with $\Gamma$ denoting the Gamma function. For this choice of $m$, the operator $T_m$ has been introduced in [15, (2.10), p. 304] and in [11, Definition 3.1, p. 354].

It is easy to see that, when $m$ is even,

$$\overline{T_m f} = T_m \overline{f}. \tag{1.8}$$

In this paper we focus on the real-valued case, and therefore will restrict ourselves to even functions $m$.

The new results in the present paper are as follows: In Section 3 we prove a new result on continuous functions with values in the dual of a countably normed Hilbert space; see Theorem 3.1. This result, together with Våge inequality, allows us to define the stochastic integral in Section 5: Let $Y$ be an $S_{-1}$-valued continuous function defined for $t \in [a, b]$. Our main result, see Theorem 5.1 below, states that the integral

$$\int_{\mathbb{R}} Y(t) \lozenge W_m(t) dt, \tag{1.9}$$

(with $\lozenge$ being the Wick product to be defined below) usually understood in the sense of Pettis, is a limit of Riemann sums, with convergence in a Hilbert space norm sense. In the case of the fractional Brownian motion, a related characterization was given in [13, (3.16), p. 591]. Still, our methods and the methods of [13] are quite different. Finally in Section 6 we prove an associated Ito formula.

The paper consists of six sections besides the introduction. With the proof of Theorem 5.1 in mind, we begin Section 2 with a short review of the topology of countably normed spaces and of their duals. In the next section, we prove Theorem 3.1 on continuous functions with valued in the dual of a perfect space. The main features of Hida's theory of the white noise space are then reviewed in Section 4. Various notions, such as the Wick product and the Kondratiev spaces, appearing in this introduction, are defined there. In Section 5 we state and prove the stochastic integration Theorem 5.1. An Ito-type formula is proved in Section 6. The last section is devoted to a number of concluding observations. In particular, our results are compared with other stochastic calculus results. In this regard, we note that Gaussian processes with singular measures and their associated stochastic calculus are studied in [5]. While the framework of this work is set within the white noise space, the infinite dimensional tools utilized in [5] and here are different.

Notation is standard. In particular we set

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{and} \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

## 2. COUNTABLY NORMED SPACES

Nuclear spaces are an indispensable part of the foundation upon which white noise theory, to be utilized below, is built. In this section we review part of the theory of nuclear spaces, as developed in [19] and [20]. We use the notation of these books.

Let $\Phi$ be a vector space (on $\mathbb{R}$ or $\mathbb{C}$) endowed with a sequence of norms $(\|\cdot\|_p)_{p\in\mathbb{N}}$. Assume that the norms are defined by inner products and that the sequence is increasing:

$$p \le q \Longrightarrow \|h\|_p \le \|h\|_q, \quad \forall h \in \Phi.$$

Denote by $\mathcal{H}_p$ the closure of $\Phi$ with respect to the norm $\|\cdot\|_p$. For $p \le q$, a Cauchy sequence in $\mathcal{H}_q$ is a Cauchy sequence in $\mathcal{H}_p$, and this defines a natural map from $\mathcal{H}_q$ into $\mathcal{H}_p$. In general, this map need not be one-to-one. A counterexample is presented in [19, p. 13]. This phenomenon will not occur in the case of reproducing kernel Hilbert spaces, as is shown in the following proposition. In this statement, recall that the positive kernel $K_q$ is said to be smaller than the positive kernel $K_p$ if the difference $K_p - K_q$ is positive.

**Proposition 2.1.** *Given the notation above, let $p \le q$, and assume that $\mathcal{H}_p$ and $\mathcal{H}_q$ are reproducing kernel Hilbert spaces of functions defined on a common set $\Omega$ with respective reproducing kernels $K_p$ and $K_q$. Assume that*

$$K_q(z,w) \le K_p(z,w)$$

*in the sense of reproducing kernels. Then $\mathcal{H}_q$ is a subset of $\mathcal{H}_p$ and the inclusion is contractive.*

*Proof.* This follows from the decomposition

$$K_p(z,w) = K_q(z,w) + (K_p(z,w) - K_q(z,w))$$

of $K_p$ into a sum of two positive kernels, and of the characterization of the reproducing kernel Hilbert space associated with a sum of positive kernels. See [8, §6] for the latter. □

In the sequel we assume that $\mathcal{H}_q \subset \mathcal{H}_p$ when $p \le q$. The inclusion will not be in general an isometry. The space $\Phi$ is the projective limit of the spaces $\mathcal{H}_p$. It will be complete if and only if

$$\Phi = \bigcap_{n=1}^{\infty} \mathcal{H}_n.$$

See [19, Théorème 1, p. 17].

Denote by $\Phi'$ the topological dual of $\Phi$. Then

$$\Phi' = \bigcup_{n=1}^{\infty} \mathcal{H}_n',$$

where $\mathcal{H}_n'$ denotes the topological dual of $\mathcal{H}_n$. Furthermore, denote by

$$\langle v, u\rangle, \quad v \in \Phi', \quad u \in \Phi,$$

the duality between $\Phi$ and $\Phi'$. By definition, for $u \in \mathcal{H}_r$ and $v \in \mathcal{H}_r'$ one has

$$\langle v, u\rangle = \langle v, u\rangle_r,$$

where $\langle v, u\rangle_r$ denotes the duality between $\mathcal{H}_r$ and $\mathcal{H}_r'$, and

$$\|v\|_{\mathcal{H}_r'} = \sup_{\substack{u\in\mathcal{H}_r,\\ \|u\|_{\mathcal{H}_r}=1}} \langle v, u\rangle_r, \quad \text{and} \quad |\langle v, u\rangle_r| \le \|v\|_{\mathcal{H}_r'}\|u\|_{\mathcal{H}_r}. \tag{2.1}$$

Moreover, note that, for $p \ge r$ and $v \in \mathcal{H}_p'$ and $u \in \mathcal{H}_r \subset \mathcal{H}_p$, one has

$$\langle v, u\rangle = \langle v, u\rangle_r = \langle v, u\rangle_p. \tag{2.2}$$

Indeed, (2.2) expresses the values of the linear functional $v$ on $u$. See [20, p. 56] for a discussion of this point.

We refer the reader to [19, §5.1, pp. 41-44] for the definition of the strong topology on $\Phi'$. To ease the reading of Gelfand-Shilov [19], we make the following remark: In verifying that a topological vector space $V$ is Hausdorff, it is necessary and sufficient to check the following: For every $v \in V$ there exists a neighborhood of 0, say $\mathcal{N}$, such that $v \notin \mathcal{N}$. See for instance [17, Proposition 9, p. 70]. This is the condition which is used and verified in [19, §5.1].

The complete, countably normed space $\Phi$ is called perfect, or a *Montel space*, when any subset of $\Phi$ is bounded and closed if and only if it is compact. A sufficient condition for $\Phi$ to be perfect is that, for every $r \in \mathbb{N}$ there exists a $p > r$ such that the inclusion from $\mathcal{H}_p$ into $\mathcal{H}_r$ is compact. See [19, Théorème 1, p. 55]. The condition is not necessary, and insures in fact that we have a Schwartz space. See the papers [27] and [16]. It will be called nuclear if the above inclusion can be chosen to be of trace class (as an operator between Hilbert spaces).

## 3. CONTINUOUS FUNCTIONS WITH VALUES IN THE DUAL OF A PERFECT SPACE

The following theorem is the key to our construction of the stochastic integral as a limit of Riemann sums. See also [25, Proposition 2.1, p. 990] for a related discussion.

**Theorem 3.1.** *Let $(E, d)$ be a compact metric space, and let $f$ be a continuous function from $E$ into the dual of a countably normed perfect space $\Phi = \bigcap_{n=1}^{\infty} \mathcal{H}_n$, endowed with the strong topology. Then there exists a $p \in \mathbb{N}$ such that $f(E) \subset \mathcal{H}_p'$, and $f$ is uniformly continuous from $E$ into $\mathcal{H}_p'$, the latter being endowed with its norm induced topology.*

*Proof.* We divide the proof into a number of steps.

*Step 1.* $f(E)$ is compact.

Indeed, the dual space $\Phi'$ endowed with the strong topology is a Hausdorff space; see [19, §5.1, pp. 41-42]. Since $E$ is compact and the function $f$ is continuous, it follows that $f(E) \subset \Phi'$ is compact.

*Step 2.* There exists an $r \in \mathbb{N}$ such that $f(E) \subset \mathcal{H}'_r$ and $f(E)$ is bounded in $\mathcal{H}'_r$.

Since $f(E)$ is compact, it is bounded in $\Phi'$ [19, Proposition 1, p. 54], and, see [19, Définition 5, p. 30] for the notion of a bounded set in a topological vector space. By the characterization of bounded sets in the strong dual of a perfect space, see [19, Théorème 2, p. 45], there exists an $r \in \mathbb{N}$ such that $f(t) \in \mathcal{H}'_r$ for all $t \in E$.

*Step 3.* Set $r$ as in the previous step. Let $t \in E$ and let $(s_m)_{m\in\mathbb{N}}$ be a sequence of elements of $E$ such that $\lim_{m\to\infty} d(t, s_m) = 0$. Then, for every $h \in \mathcal{H}_r$,

$$\lim_{m\to\infty} \langle f(s_m) - f(t), h\rangle = \lim_{m\to\infty} \langle f(s_m) - f(t), h\rangle_r = 0. \tag{3.1}$$

Indeed, the function $f$ is continuous in the strong topology of $\Phi'$, and therefore sequentially continuous in the strong topology, and hence weakly sequentially continuous.

*Step 4.* There exists a $p > r$ such that the inclusion map from $\mathcal{H}_p$ into $\mathcal{H}_r$ is compact.

Such a $p$ exists since the space $\Phi$ is assumed perfect.

Before turning to the fifth step, we observe the following: Since $E$ is a metric space, it is enough to verify continuity by using sequences; see for instance [17, Théorème 4, p. 58]. Furthermore, we note that in Step 3, we cannot say in general that $f(s_n)$ tends to $f(t)$ in the norm of $\mathcal{H}'_r$, yet we have:

*Step 5.* Set $p$ as in Step 3. The function $f$ is continuous from $E$ into $\mathcal{H}'_p$, the latter endowed with its norm topology.

Set $t \in E$ and let $(t_n)_{n\in\mathbb{N}}$ be a sequence of elements of $E$ such that $\lim_{n\to\infty} d(t_n, t) = 0$. Since $f$ is continuous, it follows that

$$f(t_n) \to f(t) \tag{3.2}$$

in the strong topology of $\Phi'$. Using [19, Théorème 4, p. 58], one has $f(t_n) \to f(t)$ in norm in $\mathcal{H}'_p$. In the proof of [19, Théorème 4, p. 58], the integers $r$ and $p$ depend *a priori* on the sequence. We repeat this argument and verify that the *same* $r$ and $p$ can be taken for all sequences $f(t_n)$.

The argument of [19] is as follows. Assume that (3.2) does not hold. Then, there exist an $\epsilon > 0$, a subsequence $(t_{n_m})_{m\in\mathbb{N}}$, and a sequence $(h_m)_{m\in\mathbb{N}}$ of elements in the closed unit ball of $\mathcal{H}_p$ such that

$$|\langle f(t_{n_m}) - f(t), h_m\rangle_p| \geq \epsilon. \tag{3.3}$$

Since the inclusion map from $\mathcal{H}_p$ into $\mathcal{H}_r$ is compact, the sequence $(h_m)_{m\in\mathbb{N}}$ has a convergent subsequence in $\mathcal{H}_r$. Denote this subsequence by $(h_m)_{m\in\mathbb{N}}$ as well, and set

$$h = \lim_{m\to\infty} h_m \in \mathcal{H}_r.$$

Writing (recall (2.2))

$$\langle f(t_{n_m}) - f(t), h_m\rangle_p = \langle f(t_{n_m}) - f(t), h_m - h\rangle_r + \langle f(t_{n_m}) - f(t), h\rangle_r,$$

we see that

$$\lim_{m\to\infty} \langle f(t_{n_m}) - f(t), h_m\rangle_p = 0.$$

Indeed, using (2.1), we have

$$\lim_{m\to\infty} \langle f(t_{n_m}) - f(t), h_m - h\rangle_r = 0,$$

since $f(E)$ is bounded in $\mathcal{H}'_r$, and from Step 3, with $s_m = t_{n_m}$,

$$\lim_{n\to\infty} \langle f(t_{n_m}) - f(t), h\rangle = 0.$$

A contradiction with (3.3) is hence obtained, thus verifying the Step 5 statement.

*Step 6.* $f$ is uniformly continuous from $E$ into $\mathcal{H}'_p$.
This stems from the fact that $E$ is compact and that $\mathcal{H}'_p$ is Hausdorff. □

We conclude this section with the definition of a Gelfand triple: Consider a complete countably normed space $\Phi$, and let $(\cdot,\cdot)$ denote an inner product on $\Phi$, which is separately continuous in each variable with respect to the topology of $\Phi$. Let $\mathcal{H}$ be the closure of $\Phi$ with respect to the norm defined by that inner product. The triple $(\Phi, \mathcal{H}, \Phi')$ is called a Gelfand triple. See [20, p. 101]. An important Gelfand triple consists of the Schwartz space $\mathscr{S}(\mathbb{R})$ of rapidly decreasing functions, of the Lebesgue space $\mathbf{L}_2$ on the real line and of the Schwartz space of tempered distributions. In the following section we recall a stochastic counterpart of this Gelfand triple, which is used below.

## 4. THE WHITE NOISE SPACE

Let $\mathscr{S}(\mathbb{R})$ denote the Schwartz space of real-valued, rapidly decreasing functions. It is a nuclear space, and by the Bochner-Minlos theorem (see [20, Théorème 2, p. 342]), there exists a probability measure $P$ on the Borel sets $\mathcal{F}$ of the dual space $\Omega = \mathscr{S}(\mathbb{R})'$ such that

$$\int_\Omega e^{i\langle\omega,s\rangle} dP(\omega) = e^{-\frac{\|s\|^2}{2}}, \quad \forall s \in \mathscr{S}(\mathbb{R}). \tag{4.1}$$

The real-valued space

$$\mathcal{W} = \mathbf{L}_2(\Omega, \mathcal{F}, P)$$

is called the white noise space. For $s \in \mathscr{S}(\mathbb{R})$, let $Q_s$ denote the random variable

$$Q_s(\omega) = \langle \omega, s\rangle.$$

It follows from (4.1) that

$$\|s\|_{\mathbf{L}_2(\mathbb{R})} = \|Q_s\|_{\mathcal{W}}.$$

Therefore, $Q_s$ extends continuously to an isometry from $\mathbf{L}_2(\mathbb{R})$ into $\mathcal{W}$, which we will still denote by $Q$. In [4] we define

$$X_m(t) = Q_{T_m(1_{[0,t]})}. \tag{4.2}$$

It follows form the construction in [4] that $X_m(t)$ is real-valued when $m$ is even. See formula (4.7) below.

In the presentation of the Gelfand triple associated with the white noise space we follow [23]. Let $\ell$ to be the set of sequences

$$(\alpha_1, \alpha_2, \ldots), \tag{4.3}$$

indexed by $\mathbb{N}$ with values in $\mathbb{N}_0$, for which only a finite number of elements $\alpha_j \neq 0$. The white noise space $\mathcal{W}$, being a space of $\mathbf{L}_2$ random variables on the probability space $(\Omega, \mathcal{F}, P)$ specified above, admits a special orthogonal basis $(H_\alpha)_{\alpha\in\ell}$, indexed by the set $\ell$ and built in terms of the Hermite functions $\widetilde{h_k}$ and of the Hermite polynomials $h_k$ as

$$H_\alpha(\omega) = \prod_{k=1}^\infty h_{\alpha_k}(Q_{\widetilde{h_k}}(\omega)).$$

We refer the reader to [23, Definition 2.2.1, p. 19] for more information. In terms of this basis, any element $F \in \mathcal{W}$ can be written as

$$F = \sum_{\alpha\in\ell} f_\alpha H_\alpha, \quad f_\alpha \in \mathbb{R}, \tag{4.4}$$

with

$$\|F\|^2_{\mathcal{W}} = \sum_{\alpha\in\mathcal{W}} f_\alpha^2 \alpha! < \infty.$$

There are quite a number of Gelfand triples associated with $\mathcal{W}$. In [3,6], and here, we focus on $(S_1, \mathcal{W}, S_{-1})$, namely the Kondratiev space $S_1$ of stochastic test functions, $\mathcal{W}$ defined above, and the Kondratiev space $S_{-1}$ of stochastic distributions. To define these spaces we first introduce, for $k \in \mathbb{N}$, the Hilbert space $\mathcal{H}_k$ which consists of series of the form (4.4) such that

$$\|F\|_k \stackrel{\text{def.}}{=} \left(\sum_{\alpha\in\ell} (\alpha!)^2 f_\alpha^2 (2\mathbb{N})^{k\alpha}\right)^{1/2} < \infty, \tag{4.5}$$

where

$$(2\mathbb{N})^{\pm k\alpha} = (2\cdot 1)^{\pm k\alpha_1}(2\cdot 2)^{\pm k\alpha_2}(2\cdot 3)^{\pm k\alpha_3}\cdots,$$

and the Hilbert space $\mathcal{H}'_k$ consisting of sequences $G = (g_\alpha)_{\alpha\in\ell}$ such that

$$\|G\|'_k \stackrel{\text{def.}}{=} \left(\sum_{\alpha\in\ell} g_\alpha^2 (2\mathbb{N})^{-k\alpha}\right)^{1/2} < \infty,$$

and the duality between an element $F = \sum_{\alpha\in\ell} f_\alpha H_\alpha \in \mathcal{H}_k$ and a sequence $G = (g_\alpha)_{\alpha\in\ell} \in \mathcal{H}'_k$ is given by

$$\langle G, F\rangle_{S_{-1},S_1} = \sum_{\alpha\in\ell} \alpha! f_\alpha g_\alpha.$$

The map which to $F \in \mathcal{W}$ associates its sequence of coefficients with respect to the basis $(H_\alpha)_{\alpha\in\ell}$ allows to identify $\mathcal{W}$ as a subspace of $\mathcal{H}'_k$ for every $k \in \mathbb{N}_0$, and it is important to note that

$$\|F\|'_k \leq \|F\|_{\mathcal{W}}, \quad \forall F \in \mathcal{W}. \tag{4.6}$$

The spaces $S_1$ and $S_{-1}$ are defined by

$$S_1 = \bigcap_{k=1}^{\infty} \mathcal{H}_k \quad \text{and} \quad S_{-1} = \bigcup_{k=1}^{\infty} \mathcal{H}'_k.$$

The space $S_1$ is nuclear, see [23].

The process $\{X_m(t)\,,\, t \in \mathbb{R}\}$ defined in (4.2), is written in the series form

$$X_m(t) = \sum_{k=1}^{\infty} \int_0^t T_m \widetilde{h}_k(u) du H_{\epsilon^{(k)}}, \tag{4.7}$$

where the series converges in the norm of $\mathcal{W}$, and has an $S_{-1}$-valued derivative given by the obvious formula

$$W_m(t) = \sum_{k=1}^{\infty} (T_m \widetilde{h_k})(t) H_{\epsilon^{(k)}}, \tag{4.8}$$

where $\epsilon^{(k)}$ is the sequence in $\ell$ with all entries equal to 0, with the exception of the $k$-th, which is equal to 1. Furthermore, the series (4.8) converges in the norm of $\mathcal{H}'_{N+3}$, where $N$ is as in (1.3). See [4, Theorem 7.2].

**Remark 4.1.** Obviously, as $H_{\epsilon^{(k)}} = H_{\epsilon^{(k)}}(\omega)$, it follows that $X_m(t) = X_m(t,\omega)$, $W_m(t) = W_m(t,\omega)$. To simplify the notation, we however omit the $\omega$-dependence throughout, unless specifically required.

**Proposition 4.2.** *We claim that:*

(a) $W_m(t) \in \mathcal{H}'_{N+3}$ *for all* $t \in \mathbb{R}$,

(b) *there exists a constant* $C_N$ *such that*

$$\|W_m(t) - W_m(s)\|_{\mathcal{H}'_{N+3}} \leq C_N |t-s|, \quad \forall t, s \in \mathbb{R}, \tag{4.9}$$

(c) *it holds that*

$$X'_m(t) = W_m(t), \quad t \in \mathbb{R}, \tag{4.10}$$

*in the norm of* $\mathcal{H}'_{N+3}$, *and more generally, in the norm of any* $\mathcal{H}'_p$ *with* $p \geq N+3$.

*Proof.* Claim (a) is proved in [4, Proof of Theorem 3.2, p. 1098]. It is also shown there, see [4, Lemma 3.8, p. 1089], that there exist constants $C_1, C_2$ such that

$$\forall t,s \in \mathbb{R}\colon |T_m\widetilde{h}_k(t) - T_m\widetilde{h}_k(s)| \le |t-s| \cdot (C_1 k^{\frac{N+2}{2}} + C_2). \tag{4.11}$$

Since

$$\|Q_{\widetilde{h_k}}\|_{\mathcal{H}'_{N+3}} = (2k)^{-N-3},$$

we can write for all $t,s \in \mathbb{R}$ then

$$\begin{aligned}\|W_m(t) - W_m(s)\|_{\mathcal{H}'_{N+3}} &\le \sum_{k=1}^{\infty} |T_m\widetilde{h}_k(t) - T_m\widetilde{h}_k(s)| \|Q_{\widetilde{h_k}}\|_{\mathcal{H}'_{N+3}} \le \\ &\le |t-s| \left\{ \sum_{k=1}^{\infty} (C_1 k^{\frac{N+2}{2}} + C_2)(2k)^{-N-3} \right\} = \\ &= C_N |t-s|\end{aligned}$$

with

$$C_N = \sum_{k=1}^{\infty} (C_1 k^{\frac{N+2}{2}} + C_2)(2k)^{-N-3}, \tag{4.12}$$

which proves (b). We now prove (c). For $t, s \in \mathbb{R}$, with $t \neq s$, and $C_N$ as in (4.12), we have

$$\begin{aligned}\left\| \frac{X_m(t) - X_m(s)}{t-s} - W_m(t) \right\|_{\mathcal{H}'_{N+3}} &= \left\| \frac{\sum_{k=1}^{\infty} \int_s^t (T_m\widetilde{h}_k(u) - T_m\widetilde{h}_k(t)) du H_{\epsilon^{(k)}}}{t-s} \right\|_{\mathcal{H}'_{N+3}} \le \\ &\le C_N \frac{\int_s^t |u-t| du}{|t-s|} \le \\ &\le \frac{C_N |t-s|}{2} \quad \longrightarrow 0 \quad \text{as} \quad s \to t.\end{aligned}$$

The last claim follows from the fact that the spaces $\mathcal{H}'_n$ are increasing with decreasing norms. □

The Wick product is defined with respect to the basis $(H_\alpha)_{\alpha \in \ell}$ by

$$H_\alpha \lozenge H_\beta = H_{\alpha+\beta}.$$

It extends to a continuous map from $S_1 \times S_1$ into itself and from $S_{-1} \times S_{-1}$ into itself. Let $l > 0$, and let $k > l+1$. Consider $h \in \mathcal{H}'_l$ and $u \in \mathcal{H}'_k$. Then, Våge's inequality holds:

$$\|h \lozenge u\|_k \le A(k-l)\|h\|_l \|u\|_k, \tag{4.13}$$

where

$$A(k-l) = \left( \sum_{\alpha \in \ell} (2\mathbb{N})^{(l-k)\alpha} \right)^{1/2} < \infty. \tag{4.14}$$

See [23, Proposition 3.3.2, p. 118].

To conclude this section, we specify the conditions under which the process $X_m$ has $P$-a.s. continuous sample paths. This property will be utilized in the last step of the proof of (6.1) below, the Ito formula associated with $X_m$.

By [4, Lemma 6.1], (1.3) with $N = 0$ leads to

$$\begin{aligned} E[|X_m(t) - X_m(s)|^2] &= 2Re\{r(t-s)\} \leq \\ &\leq 2(C_1|t-s|^2 + C_2|t-s\ ) \leq \\ &\leq 2(C_1 + C_2)(|t-s|^2 \vee |t-s|), \end{aligned} \tag{4.15}$$

where the first equality is due to item (2) of aforementioned lemma, while the following inequality is due to item (3), with $C_1$, $C_2$ some finite, positive constants.

Recall that $X_m$ is a Gaussian process. Then, for all $t, s \in \mathbb{R}$, $|t-s| \leq 1$, it follows from (4.15) that

$$E[|X_m(t) - X_m(s)|^4] \leq 12(C_1 + C_2)^2|t-s|^2. \tag{4.16}$$

By Kolmogorov's continuity criterion, see e.g. [26, Theorem I-1.8], it follows that there exists a $P$-a.s. continuous modification of $X_m$, a modification we consider here.

**Remark 4.3.** Obviously, there are functions $m$, e.g. $m$ corresponding to the fractional Brownian motion with Hurst parameter $H < 1/2$, that do not meet (1.3) with $N = 0$, hence the continuity of the associated sample paths should be verfied by other means.

## 5. THE WICK-ITO INTEGRAL

The main result of this section is the following theorem.

**Theorem 5.1.** *Let $Y(t)$, $t \in [a,b]$ be an $S_{-1}$-valued function, continuous in the strong topology of $S_{-1}$. Then, there exists a $p \in \mathbb{N}$ such that the function $t \mapsto Y(t)\lozenge W_m(t)$ is $\mathcal{H}'_p$-valued, and*

$$\int_a^b Y(t,\omega)\lozenge W_m(t)dt = \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} Y(t_k,\omega)\lozenge\left(X_m(t_{k+1}) - X_m(t_k)\right),$$

*where the limit is in the $\mathcal{H}'_p$ norm, with $\Delta : a = t_0 < t_1 < \cdots < t_n = b$ a partition of the interval $[a,b]$ and $|\Delta| = \max_{0\leq k\leq n-1}(t_{k+1} - t_k)$.*

*Proof.* We proceed in a number of steps.
*Step 1.* $W_m(t) \in \mathcal{H}'_{N+3}$ for $t \in \mathbb{R}$, and satisfies (4.9):

$$\|W_m(t) - W_m(s)\|_{\mathcal{H}'_{N+3}} \leq C_N|t-s|, \quad \forall t, s \in \mathbb{R}$$

for some constant $C_N$.

See Proposition 4.2.

*Step 2.* There exists a $p \in \mathbb{N}$, $p > N+3$, such that $Y(t) \in \mathcal{H}'_p$ for all $t \in [a,b]$, being uniformly continuous from $[a,b]$ into $\mathcal{H}'_p$.

Theorem 3.1 with $E = [a,b]$ ensures that a $p$ (not necessarily larger than $N+3$) exists with the stated properties. Since the norms $\|\cdot\|_{\mathcal{H}'_p}$ are decreasing, we may assume that $p > N+3$.

Using Våge's inequality (4.13), it follows that, for $p > N+3$,

$$
\begin{aligned}
&\left\|Y(t)\Diamond W_m(t) - Y(s)\Diamond W_m(s)\right\|_p \leq \\
&\qquad \leq \left\|(Y(t)-Y(s))\Diamond W_m(t)\right\|_p + \left\|Y(s)\Diamond(W_m(t)-W_m(s))\right\|_p \leq \\
&\qquad \leq A(p-N-3)\|Y(t)-Y(s)\|_p\|W_m(s)\|_{N+3}+ \\
&\qquad\quad + A(p-N-3)\|Y(s)\|_p\|W_m(t)-W_m(s)\|_{N+3},
\end{aligned}
$$

where $A(p-N-3)$ is defined by (4.14), with $\|\cdot\|_p \overset{\text{def.}}{=} \|\cdot\|_{\mathcal{H}'_p}$ used to simplify the notation.

In view of Step 2, the integral $\int_a^b Y(t)\Diamond W_m(t)dt$ makes sense as a Riemann integral of a Hilbert space valued continuous function.

*Step 3.* Let $\Delta$ be a partition of the interval $[a,b]$. We now compute an estimate for

$$
\begin{aligned}
&\int_a^b Y(t)\Diamond W_m(t)dt - \sum_{k=0}^{n-1} Y(t_k)\Diamond\left(X_m(t_{k+1}) - X_m(t_k)\right) = \\
&\qquad\qquad = \sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}} (Y(t)-Y(t_k))\Diamond W_m(t)dt\right).
\end{aligned}
$$

Let $p$ be as in Step 2, and let $\epsilon > 0$. Since $Y$ is uniformly continuous on $[a,b]$ there exists an $\eta > 0$ such that

$$
|t-s| < \eta \Longrightarrow \|Y(t)-Y(s)\|_p < \epsilon.
$$

Set

$$
\tilde{C} = \max_{s\in[a,b]} \|W_m(s)\|_{N+3} \quad \text{and} \quad A = A(p-N-3).
$$

Let $\Delta$ be a partition of $[a,b]$ with

$$
|\Delta| = \max\left\{|t_{k+1}-t_k|\right\} < \eta.
$$

We then have

$$
\begin{aligned}
&\left\| \sum_{k=0}^{n-1} \left( \int_{t_k}^{t_{k+1}} (Y(t) - Y(t_k)) \Diamond W_m(t) dt \right) \right\|_p \leq \\
&\leq \sum_{k=0}^{n-1} \left( \int_{t_k}^{t_{k+1}} \left\| (Y(t) - Y(t_k)) \Diamond W_m(t) \right\|_p dt \right) \leq \\
&\leq A \sum_{k=0}^{n-1} \left( \int_{t_k}^{t_{k+1}} \left\| (Y(t) - Y(t_k)) \right\|_p \left\| W_m(t) \right\|_{N+3} dt \right) \leq \\
&\leq \tilde{C} A \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| (Y(t) - Y(t_k)) \right\|_p dt \leq \\
&\leq \epsilon \tilde{C} A (b - a).
\end{aligned}
$$

□

## 6. AN ITO FORMULA

We extend the classical Ito's formula to the present setting. We need the extra assumption that the function

$$r(t) = \|T_m 1_{[0,t]}\|_{\mathbf{L}_2(\mathbb{R})}$$

is absolutely continuous with respect to the Lebesgue measure. This is in particular the case for the fractional Brownian motion. This is also the case e.g. for the function $m$ defined in (1.5), where, for that $m$,

$$r(t) = \frac{\sqrt{2\pi}}{8} \left\{ 1 - e^{-\frac{t^2}{8}} (1 + t^2) \right\}.$$

**Theorem 6.1.** *Suppose that $r(t)$ is absolutely continuous with respect to the Lebesgue measure, and that the process $X_m$ has a.s. continuous sample paths. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^2(\mathbb{R})$ function. Then*

$$
\begin{aligned}
f(X_m(t)) = f(X_m(t_0)) &+ \int_{t_0}^{t} f'(X_m(s)) \Diamond W_m(s) ds + \\
&+ \frac{1}{2} \int_{t_0}^{t} f''(X_m(s)) r'(s) ds, \quad t_0 < t \in \mathbb{R},
\end{aligned}
\tag{6.1}
$$

*where the equality holds in the $P$-almost sure sense.*

*Proof.* We prove for $t > t_0 = 0$. The proof for any other interval in $\mathbb{R}$ is essentially the same. We divide the proof into a number of steps. Step 1–Step 8 are constructed so as to show that (6.1) holds, for all $t > 0$, for $C^2$ functions with compact support, with the equality holding in the $\mathcal{H}'_p$ sense. This implies its validity in the $P$-a.s. sense (actually, holding for all $\omega \in \Omega$), hence, setting the ground for the concluding Step 9, in which the result is extended to hold for all $C^2$ functions $f$.

*Step 1.* For every $(u,t) \in \mathbb{R}^2$, it holds that

$$e^{iuX_m(t)} \in \mathcal{W},$$

and

$$e^{iuX_m(t)} \lozenge W_m(t) \in \mathcal{H}'_{N+3}. \tag{6.2}$$

Indeed, since $X_m$ is real, we have

$$|e^{iuX_m(t)}| \leq 1, \quad \forall u, t \in \mathbb{R},$$

and hence $e^{iuX_m(t)} \in \mathcal{W}$. Since $\mathcal{W} \subset \mathcal{H}'_{N+1}$ and since $W_m(t) \in \mathcal{H}'_{N+3}$ for all $t \in \mathbb{R}$, it follows from Våge's inequality (4.13) that (6.2) holds.

In the following two steps, we prove formula (6.1) for exponential functions. For $\alpha \in \mathbb{R}$, we set

$$g(x) = \exp(i\alpha x).$$

*Step 2.* It holds that

$$g'(X_m(t)) = i\alpha g(X_m(t)) \lozenge W_m(t) + \frac{1}{2}(i\alpha)^2 g(X_m(t)) r'(t). \tag{6.3}$$

Indeed, $g(X_m(t))$ belongs to $S_{-1}$, see [23, p. 65], and we have from [23, Lemma 2.6.16, p. 66]:

$$\begin{aligned} g(X_m(t)) = \exp(i\alpha X_m(t)) &= \exp^{\lozenge}\left(i\alpha X_m(t) + \frac{1}{2}(i\alpha)^2 \|T_m I_t\|^2_{\mathbf{L}_2(\mathbb{R})}\right) = \\ &= \exp^{\lozenge}\left(i\alpha X_m(t) + \frac{1}{2}(i\alpha)^2 r(t)\right). \end{aligned}$$

The hypothesis that $r$ is absolutely continuous with respect to Lebesgue measure comes now into play. Since the function $t \mapsto W_m(t)$ is continuous in $S_{-1}$ and since $X'_m = W_m$, see Proposition 4.2, an application of [23, Theorem 3.1.2] with

$$h(t) = i\alpha W_m(t) - \frac{\alpha^2}{2} r'(t)$$

leads to

$$\begin{aligned} g'(X_m(t)) &= g(X_m(t)) \lozenge (i\alpha W_m(t) + \frac{1}{2}(i\alpha)^2 r(t)') = \\ &= g(X_m(t)) \lozenge (i\alpha W_m(t)) + \frac{1}{2}(i\alpha)^2 g(X_m(t)) r'(t). \end{aligned}$$

We thus obtain (6.3).

*Step 3.* Equation (6.1) holds for exponentials.

Indeed, it follows from (6.3) that

$$g(X_m(t)) = g(X_m(0)) + \int_0^t i\alpha g(X_m(s))\lozenge W_m(s)ds + \\ + \frac{1}{2}\int_0^t (i\alpha)^2 g(X_m(s))r'(s)ds.$$

This can be written

$$g(X_m(t)) = g(0) + \int_0^t g'(X_m(s))\lozenge W_m(s)ds + \\ + \frac{1}{2}\int_0^t g''(X_m(s))r'(s)ds,$$

that is

$$e^{iuX_m(t)} = 1 + \int_0^t iue^{iuX_m(s)}\lozenge W_m(s)ds + \\ + \frac{1}{2}\int_0^t (iu)^2 e^{iuX_m(s)}r'(s)ds. \tag{6.4}$$

In the following two steps, we prove (6.1) to hold for Schwartz functions.

*Step 4.* The function $(u,t) \mapsto e^{iuX_m(t)}\lozenge W_m(t)$ is continuous from $\mathbb{R}^2$ into $\mathcal{H}'_{N+3}$.

We first recall that the function $t \mapsto X_m(t)$ is continuous, and even uniformly continuous, from $\mathbb{R}$ into $\mathcal{W}$, and hence from $\mathbb{R}$ into $\mathcal{H}'_{N+5}$ since

$$\|u\|_{\mathcal{H}'_{N+3}} \le \|u\|_{\mathcal{W}} \quad \text{for} \quad u \in \mathcal{W}.$$

Therefore, the function $(u,t) \mapsto e^{iuX_m(t)}$ is continuous from $\mathbb{R}^2$ into $\mathcal{H}'_{N+3}$. Furthermore,

$$\begin{aligned}
&\|e^{iu_1X_m(t_1)}\lozenge W_m(t_1) - e^{iu_2X_m(t_2)}\lozenge W_m(t_2)\|_{\mathcal{H}'_{N+3}} \le \\
&\le \|(e^{iu_1X_m(t_1)} - e^{iu_2X_m(t_2)})\lozenge W_m(t_1)\|_{\mathcal{H}'_{N+1}} + \\
&\quad + \|e^{iu_1X_m(t_1)}\lozenge(W_m(t_2) - W_m(t_1))\|_{\mathcal{H}'_{N+3}} \le \\
&\le A(2)\|(e^{iu_1X_m(t_1)} - e^{iu_2X_m(t_2)})\|_{\mathcal{H}'_{N+1}} \cdot \|W_m(t_1)\|_{\mathcal{H}'_{N+3}} + \\
&\quad + A(2)\|e^{iu_1X_m(t_1)}\|_{\mathcal{H}'_{N+1}} \cdot \|(W_m(t_2) - W_m(t_1))\|_{\mathcal{H}'_{N+3}},
\end{aligned}$$

where $A(2)$ is defined by (4.14). This completes the proof of Step 4, since $t \mapsto W_m(t)$ is continuous in the norm of $\mathcal{H}'_{N+3}$ and $(u,t) \mapsto e^{iuX_m(t)}$ is continuous in the norm of $\mathcal{H}'_{N+1}$.

*Step 5.* (6.1) holds for $f$ in the Schwartz space.

Let $s$ be in the Schwartz space. Replace $u$ by $-u$ in (6.4), and multiply both sides of this equation by $s(u)$. Integrating with respect to $u$, and interchanging order of integration, we obtain

$$\int_{\mathbb{R}} s(u)e^{-iuX_m(t)}du = \int_{\mathbb{R}} s(u)du + \int_0^t \left(\int_{\mathbb{R}} (-iu)s(u)e^{-iuX_m(s)}du\right) \lozenge W_m(s)ds + \\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \left((-iu)^2 s(u)e^{-iuX_m(s)}\right) dur(s)'ds.$$

Continuity of the function in the previous step allows the use Fubini's theorem for functions with values in a Hilbert space (see [14, Theorem 2.6.14, p. 65], [12, Proposition 9, p. 97]), and to interchange the order of integration.

Since

$$\widehat{s}\,'(x) = \int_{\mathbb{R}} (-iu)s(u)e^{-iux}du \quad \text{and} \quad \widehat{s}\,''(x) = \int_{\mathbb{R}} (-iu)^2 s(u)e^{-iux}du,$$

we obtain

$$\widehat{s}(X_m(t)) = \widehat{s}(0) + \int_0^t (\widehat{s})'(X_m(s))\lozenge W_m(s)ds + \\ + \frac{1}{2}\int_0^t (\widehat{s})''(X_m(s))r(s)'ds.$$

This completes the proof of Step 5, since the Fourier transform maps the Schwartz space onto itself.

To show that (6.1) holds for $f$ of class $C^2$ with compact support, we use the concept of approximate identity. For $\epsilon > 0$, define

$$k_\epsilon(x) = \frac{1}{\sqrt{2\pi}\epsilon}\exp\left(-\frac{x^2}{2\epsilon^2}\right).$$

*Step 6.* It holds that

$$\int_{\mathbb{R}} k_\epsilon(x)dx = 1, \tag{6.5}$$

and for every $r > 0$

$$\lim_{\epsilon \to 0} \int_{|x|>r} k_\epsilon(x)dx = 0. \tag{6.6}$$

Indeed, (6.5) follows directly from the fact that $k_\epsilon$ is an $\mathcal{N}(0,\epsilon^2)$ density. Furthermore, for $|x| > r > 0$,

$$\frac{1}{\epsilon\sqrt{2\pi}}\int_r^\infty e^{-\frac{x^2}{2\epsilon^2}}dx = \frac{1}{\epsilon\sqrt{2\pi}}\int_r^\infty \frac{x}{\epsilon^2}e^{-\frac{x^2}{2\epsilon^2}}\frac{\epsilon^2}{x}dx \leq$$
$$\leq \frac{\epsilon}{r\sqrt{2\pi}}\int_r^\infty \frac{x}{\epsilon^2}e^{-\frac{x^2}{2\epsilon^2}}dx = \frac{\epsilon}{r\sqrt{2\pi}}e^{-\frac{r^2}{2\epsilon^2}} \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0.$$

The properties in Step 6 express the fact that $k_\epsilon$ is an approximate identity. Therefore, it follows from [18, Theorem 1.2.19, p. 25] that, for every continuous function with compact support,

$$\lim_{\epsilon\to 0}\|k_\epsilon * f - f\|_\infty = 0.$$

*Step 7.* The functions

$$(k_\epsilon * f)(x) = \frac{1}{\sqrt{2\pi}\epsilon}\int_{\mathbb{R}} \exp\Big(-\frac{(u-x)^2}{\epsilon^2}\Big)f(u)du$$

are in the Schwartz space.

One proves by induction on $n$ that the $n$-th derivative

$$(k_\epsilon * f)^{(n)}(x)$$

is a finite sum of terms of the form

$$\frac{1}{\sqrt{2\pi}\epsilon}\int_{\mathbb{R}} \exp\Big(-\frac{(u-x)^2}{\epsilon^2}\Big)p(x-u)f(u)du,$$

where $p$ is a polynomial. That all limits,

$$\lim_{|x|\to\infty} x^m(k_\epsilon * f)^{(n)}(x) = 0,$$

is then shown using the dominated convergence theorem.

*Step 8.* (6.1) holds for $f$ of class $C^2$ and with compact support.

A function $f$ of class $C^2$ with compact support can be approximated, together with first two derivatives, in the supremum norm by Schwartz functions. This is done as follows. Take for simplicity $\epsilon = \frac{1}{n}$, $n = 1, 2, \ldots$. We apply [18, Theorem 1.2.19, p. 25] to $f$, $f'$ and $f''$. Set

$$a_n = k_{1/n} * f, \quad b_n = k_{1/n} * f', \quad \text{and} \quad c_n = k_{1/n} * f''.$$

Integration by parts shows that

$$\begin{aligned} a_n' &= b_n \\ b_n' &= c_n. \end{aligned}$$

Furthermore,

$$\begin{aligned} \lim_{n\to\infty} \|a_n - f\|_\infty &= 0, \\ \lim_{n\to\infty} \|b_n - f'\|_\infty &= 0, \\ \lim_{n\to\infty} \|c_n - f''\|_\infty &= 0. \end{aligned}$$

For every $n$, we have

$$a_n(X_m(t)) = a_n(0) + \int_0^t a_n'(X_m(s))\lozenge W_m(s)ds + \frac{1}{2}\int_0^t a_n''(X_m(s))r'(s)ds.$$

We claim that, for a given $t$, the sequence $(a_n(X_m(t)))_{n\in\mathbb{N}}$ is a Cauchy sequence in any $\mathcal{H}_p'$, since

$$\|a_n(X_m(t)) - a_m(X_m(t))\|_{\mathcal{H}_p'} \leq \|a_n(X_m(t)) - a_n(X_m(s))\|_{\mathcal{W}} \leq \|a_n - a_m\|_\infty,$$

and denote by $f(X_m(t))$ the corresponding limit.

Similarly, the sequence

$$\Big(\int_0^t b_n(X_m(u))\lozenge W_m(u)du\Big)_{n\in\mathbb{N}}$$

is a Cauchy sequence in $\mathcal{H}_p'$, since

$$\begin{aligned} &\left\| \int_0^t b_n(X_m(u))\lozenge W_m(u)du - \int_0^t b_m(X_m(u))\lozenge W_m(u)du \right\|_{\mathcal{H}_p'} \leq \\ &\leq \int_0^t \|(b_n - b_m)(X_m(u))\lozenge W_m(u)\|_{\mathcal{H}_p'} du \leq \\ &\leq A(2)\int_0^t \|(b_n - b_m)(X_m(u))\|_{\mathcal{H}_p'} \|W_m(u)\|_{\mathcal{H}_{N+3}'} du \leq \\ &\leq A(2)\|b_n - b_m\|_\infty \int_0^t \|W_m(u)\|_{\mathcal{H}_{N+3}'} du. \end{aligned}$$

Denote

$$\int_0^t f'(X_m(u))\Diamond W_m(u)du$$

to be its limit. A similar argument holds for

$$\int_0^t c_n(X_m(u))r'(u)du.$$

Details are omitted.

Note that we have actually shown (6.1) to hold for $C^2$ functions with compact support with the equality understood in the $\mathcal{H}'_p$ sense. This implies that it also holds in the $P$-a.s. sense.

*Step 9.* (6.1) holds with probability 1 for all $f \in C^2(\mathbb{R})$.

Here, we follow key arguments of the corresponding proofs of the standard Ito rule, given in e.g., [26, Theorem IV-3.3, p. 138], [24, Theorem 3.3, p. 149]. Specifically, the following standard localization argument is utilized. Let $\tau_N$ be a stopping time defined by

$$\tau_N = \inf\left\{s > 0 \,:\, |X_m(s)| > N\right\}.$$

Set

$$X_m^N(s) \overset{\text{def.}}{=} X_m(s \wedge \tau_N).$$

Then, by Step 8, (6.1) holds for $\left\{X_m^N(s)\,,\, s \geq 0\right\}$, a.s., for all $f \in C^2(\mathbb{R})$. Fix an arbitrary $\epsilon > 0$ and let $N = N(\epsilon) < \infty$ be such that

$$P\Big(\sup_{0\leq s\leq t} |X_m(s)| > N\Big) < \epsilon.$$

As (6.1) holds for $X_m^N$ a.s., it then follows that (6.1) holds for $X_m$ with probability greater that $1-\epsilon$, for all $f \in C^2(\mathbb{R})$. The arbitrariness of $\epsilon$ completes the proof of the fact that (6.1) holds, $P$-a.s. for all $t \in \mathbb{R}$. This suggests that both sides of (6.1) are *modifications* of one another. Since both are $t$-continuous (see the discussion at the end of Section 4 specifying sufficient conditions on $m$ for the continuity of the LHS), they are in fact *indistinguishable* processes, which is to say that (6.1) holds for all $t \in \mathbb{R}$, $P$-a.s. □

## 7. CONCLUDING REMARKS

**1.** Note that no adaptability of the integrand with respect to an underlying filtration is assumed. In this sense, one may regard the integral defined here in fact as a Wick-Skorohod integral.

**2.** Due to the fact that $\|F\|'_p \leq \|F\|_{\mathcal{W}}$ for all $F \in \mathcal{W}$ (see (4.6)), it follows that the integral defined in Theorem 5.1, being an $\mathcal{H}'_p$ limit of Riemann sums, is defined in a

weaker sense than the standard Ito and Skorohod integrals, which are defined in an $\mathbf{L}_2$ sense. This is a reasonable price to pay so as to integrate with respect to a larger class of non-$\mathbf{L}_2$ integrands. This places the proposed integral well within existing stochastic integration theory, as a non-trivial extension is formulated, at the (expected) expense of a somewhat weaker sense of convergence. Specifically, one may consider the integral proposed in [1], written as a limit of Riemann sums, see [1, Proposition 4, Section 7], making it compatible with the integral presented here in Theorem 5.1. Then, while the limit in [1] is understood in the $\mathbf{L}_2$ sense, here, the limit of the Riemann sums is taken in the weaker $\mathcal{H}'_p$ sense.

**3.** We note the reduction of the calculus derived here to the standard Ito calculus when $r(t) = |t|$. This case corresponds to setting $H = 1/2$ in (1.6) and (1.7), so that $V_{1/2} = 1$ and $m(u) = \frac{1}{2\pi}$. For example, for $Y(t) = B(t)$, both stochastic integrals give

$$\int_0^t B(t)dB(t) = \frac{B^2(t) - t}{2}.$$

See [3]. Furthermore, for the fractional Brownian motion with Hurst parameter $H \in (0, 1)$, i.e. (up to a multiplicative constant) $r(t) = |t|^{2H}$, our integral coincides with that proposed in e.g. [13] and [9, 10].

**4.** Note that our Ito formula for $X_m$ being the fractional Brownian motion with Hurst parameter $H \in (0, 1)$, hence $m(t) = \frac{1}{2\pi}|t|^{1-2H}$, coincides with that of Bender [9] specified for $C^2$ functions with values in a space of distributions. We note however the difference between the proofs. In [9] Bender shows that the $S$-transforms of both sides of equation (6.1) agree. The conclusion that (6.1) holds in fact for all $\omega \in \Omega$ follows from the fact that the $S$-transform is injective. Here one can use Bender's approach for $C^2$ distributions (with derivatives understood in the sense of distributions), replacing Bender's $r'(t) = t^{2H-1}$ $(t > 0)$ with the derivative of a general $r$. We omit the computations which are essentially the same. An apparent advantage of our (and Bender's [9,10]) white noise space based approach to the construction of the associated Ito formula, lies in the conditions assumed: While [1] requires the integrand $u$ to belong (in the notation of that paper) to $\mathrm{Dom}\,\delta^B$, effectively to a subclass of $\mathbf{L}_2$, no restriction is required here. Moreover, while a specific exponential growth condition cited in [1, (16)] is assumed in [1, Theorem 2], no growth conditions are imposed on the $C^2(\mathbb{R})$ functions dealt with in Theorem 6.1.

**Acknowledgements**
*D. Alpay thanks the Earl Katz family for endowing the chair which supported his research, as well as the Israel Science Foundation grant 1023/07 and the Binational Science Foundation Grant number 2010117.*

REFERENCES

[1] E. Alos, O. Mazet, D. Nualart, *Stochastic calculus with respect to gaussian processes*, Ann. Probab. **29** (2001) 2, 766–801.

[2] D. Alpay, H. Attia, *An interpolation problem for functions with values in a commutative ring*, Oper. Theory Adv. Appl. **218** (2011), 1–17.

[3] D. Alpay, H. Attia, D. Levanony, *Une généralisation de l'intégrale stochastique de Wick-Itô*, C. R. Math. Acad. Sci. Paris **346** (2008) 5–6, 261–265.

[4] D. Alpay, H. Attia, D. Levanony, *On the characteristics of a class of Gaussian processes within the white noise space setting*, Stochastic Process. Appl. **120** (2010), 1074–1104.

[5] D. Alpay, P. Jorgensen, D. Levanony, *A class of Gaussian processes with fractional spectral measures*, J. Funct. Anal. **261** (2011), 507–541.

[6] D. Alpay, D. Levanony, *Linear stochastic systems: a white noise approach*, Acta Appl. Math. **110** (2010), 545–572.

[7] D. Alpay, G. Salomon, *A New Family of Topological Rings with Applications in Linear System Theory*, ArXiv e-prints, eprint number 1106.5746. 2011.

[8] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[9] C. Bender, *An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter*, Stochastic Process. Appl. **104** (2003), 181–106.

[10] C. Bender, *An S-transform approach to integration with respect to a fractional Brownian motion*, Bernoulli **9** (2003) 6, 955–983.

[11] F. Biagini, B. Øksendal, A. Sulem, N. Wallner, *An introduction to white-noise theory and Malliavin calculus for fractional Brownian motion, stochastic analysis with applications to mathematical finance*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **460** (2004) 2041, 347–372.

[12] N. Bourbaki, *Éléments de mathématique. Fasc. XXI. Livre VI: Intégration. Chapitre V: Intégration des mesures*, Springer, Paris, 2007.

[13] T.E. Duncan, Y. Hu, B. Pasik-Duncan, *Stochastic calculus for fractional Brownian motion. I. Theory*, SIAM J. Control Optim. **38** (2000) 2, 582–612 (electronic).

[14] N. Dunford, J. Schwartz, *Linear operators*, vol. 1. Interscience, 1957.

[15] R.J. Elliott, J. van der Hoek, *A general fractional white noise theory and applications to finance*, Math. Finance **13** (2003) 2, 301–330.

[16] K. Floret, *Fréchet-Montel-spaces which are not Schwartz-spaces*, Port. Math. 42(1):1–4 (1985), 1983/84.

[17] J. Garsoux, *Espaces vectoriels topologiques et distributions*, Avec la collaboration de Daniel Ribbens; Préface de Pierre Houzeau de Lehaie. Dunod, Paris, 1963.

[18] L. Grafakos, *Modern Fourier analysis*, vol. 250 of Graduate Texts in Mathematics, Springer, New York, 2nd ed., 2009.

[19] I.M. Guelfand, G.E. Shilov, *Les distributions. Tome 2*, Collection Universitaire de Mathématiques, No. 15. Dunod, Paris, 1964.

[20] I.M. Guelfand, N.Y. Vilenkin, *Les distributions. Tome 4: Applications de l'analyse harmonique*, Collection Universitaire de Mathématiques, No. 23. Dunod, Paris, 1967.

[21] T. Hida, H. Kuo, J. Potthoff, L. Streit, *White noise*, volume 253 of Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1993. An infinite-dimensional calculus.

[22] T. Hida, Si Si, *Lectures on white noise functionals*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[23] H. Holden, B. Øksendal, J. Ubøe, T. Zhang, *Stochastic partial differential equations*, Probability and Its Applications. Birkhäuser Boston Inc., Boston, MA, 1996.

[24] I. Karatzas, S.E. Shreve, *Brownian motion and stochastic calculus*, volume 113 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1988.

[25] I. Mitoma, *Tightness of probabilities on $C([0,1];\mathcal{S}')$ and $D([0,1];\mathcal{S}')$*, Ann. Probab. **11** (1983) 4, 989–999.

[26] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 1991.

[27] V. Wrobel, *Generating Fréchet-Montel spaces that are not Schwartz by closed linear operators*, Arch. Math. (Basel) **46** (1986) 3, 257–260.

Daniel Alpay
dany@math.bgu.ac.il

Ben Gurion University of the Negev
Department of Mathematics
P.O.B. 653, Be'er Sheva 84105, Israel

Haim Attia
haima@sce.ac.il

Sami Shamoon College of Engineering
Department of Mathematics
Be'er Sheva 84100, Israel

David Levanony
levanony@ee.bgu.ac.il

Ben Gurion University of the Negev
Department of Electrical Engineering
P.O.B. 653, Be'er Sheva 84105, Israel

*Received: November 30, 2011.*
*Revised: December 4, 2011.*
*Accepted: December 11, 2011.*