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Quaternionic Hardy spaces in the open unit ball and half space and Blaschke products

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Abstract. The Hardy spaces $H_2(\mathbb{B})$ and $H_2(\mathbb{H}_+)$, where $\mathbb{B}$ and $\mathbb{H}_+$ denote, respectively, the open unit ball of the quaternions and the half space of quaternions with positive real part, as well as Blaschke products, have been intensively studied in a series of papers where they are used as a tool to prove other results in Schur analysis. This paper gives an overview on the topic, collecting the various results available.

1. Introduction

In this paper we give an overview of the results available on the Hardy spaces $H_2(\Omega)$ where Ω is either the open unit ball of the quaternions or the half space of quaternions with positive real part. These spaces have been studied in a series of papers, see [1, 2, 3, 4, 5, 6] as a tool to prove other results in Schur analysis and the purpose of this survey is to collect them in one paper. For the proofs of the various results, we refer the reader to the original sources.

We will work in the framework of slice hyperholomorphic functions. Several functions spaces have been considered in this setting, e.g. the Bergman spaces, see [11, 12, 13], Besov, Bloch and Dirichlet spaces, see [9], the Fock space see [7] while other Hardy spaces $H_p(\mathbb{B})$ are in [14]. To define the class of slice hyperholomorphic functions, we need some terminology that is introduced below.

By $\mathbb{H}$ we denote the algebra of real quaternions, namely the set of elements of the form $p = x_0 + x_1 i + x_2 j + x_3 k$ where i, j, k is the standard basis of quaternions. Let $\mathbb{S}$ be the set of purely imaginary quaternions with norm 1. Any $I \in \mathbb{S}$ is such that $I^2 = -1$, so we can consider the complex plane of elements of the form $x + Iy$ where $x, y \in \mathbb{R}$. Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f : \Omega \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I$. We say that f is a (left) slice hyperholomorphic function in Ω if, for every $I \in \mathbb{S}$, f_I satisfies

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

An analogous definition can be given for right slice hyperholomorphic functions. The class of (left) slice hyperholomorphic functions includes, in particular, converging power series with quaternionic coefficients written on the right. For the basic notions and main properties of these functions we refer the reader to the books [10, 16]. The class of slice hyperholomorphic functions is not closed, in general, under the pointwise multiplication. However, slice hyperholomorphic

functions can be multiplied, at least on axially symmetric s-domains, in order to obtain a function of the same kind using the so-called $\star$-product. We recall that axially symmetric s-domains Ω are domains intersecting the real line, which remain connected when intersected with any complex plane $\mathbb{C}_I$ and such that whenever an element $x+Iy \in \Omega$ all the elements of the form $x+Jy \in \Omega$ when J varies in $\mathbb{S}$. In the special case in which Ω is a ball with center at the origin and if f, g are two slice hyperholomorphic functions they can be written as $f(p) = \sum_{k=0}^{\infty} p^k f_k$, $g(p) = \sum_{k=0}^{\infty} p^k g_k$. Their $*$-product is $(f \star g)(p) = \sum_{k=0}^{\infty} p^k (\sum_{r=0}^{k} f_r g_{k-r})$, thus it coincides with the product defined in [15].
Pointwise multiplication and slice multiplication can be related as in the following result, see e.g. [10, Proposition 4.3.22]:

Proposition 1.1 *Let $U \subseteq \mathbb{H}$ be an axially symmetric s-domain, $f, g : U \to \mathbb{H}$ be slice hyperholomorphic functions and let us assume that $f(p) \neq 0$. Then*

$$(f \star g)(p) = f(p) g(f(p)^{-1} p f(p)), \tag{1}$$

for all $p \in U$. If $f(p) = 0$ then $(f \star g)(p) = 0$.

Note that the transformation $p \to f(p)^{-1} p f(p)$ is clearly a rotation in $\mathbb{H}$, since $|p| = |f(p)^{-1} p f(p)|$.
Note also that if $(f \star g)(p) = 0$ then either $f(p) = 0$ or $g(f(p)^{-1} p f(p)) = 0$.
This latter fact is very well known for polynomials with quaternionic coefficients, see [17], for which it is also well known that they might have spheres of zeros. Given a nonreal quaternion α, it can be written as $\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k = a + Ib$ where $a = \alpha_0$, $I = (\alpha_1 i + \alpha_2 j + \alpha_3 k)/|\alpha_1 i + \alpha_2 j + \alpha_3 k| \in \mathbb{S}$, $b = |\alpha_1 i + \alpha_2 j + \alpha_3 k|$. The sphere defined by α is the set $[\alpha] = \{a + Jb \mid J \in \mathbb{S}\}$; it is the set of elements satisfying

$$p^2 - 2ap + (a^2 + b^2) = p^2 - 2\mathrm{Re}(\alpha)p + |\alpha|^2 = (p - \alpha) \star (p - \bar{\alpha}) = 0.$$

This discussion suggests that the definition of multiplicity of a zero has to be given in appropriate way:

Definition 1.2 *We say that the* multiplicity of the spherical zero $[\alpha]$ *of a function $Q(p)$ is m if m is the maximum of the integers r such that $(p^2 + 2\mathrm{Re}(\alpha)p + |\alpha|^2)^r$ divides $Q(p)$.*
Let $\alpha_j \in \mathbb{H} \setminus \mathbb{R}$ and let

$$Q(p) = (p - \alpha_1) \star \ldots \star (p - \alpha_n) \star g(p) \quad \alpha_{j+1} \neq \bar{\alpha}_j, \quad j = 1, \ldots, n-1, \quad g(p) \neq 0. \tag{2}$$

We say that α_1 is a zero of Q of multiplicity 1 *if $\alpha_j \notin [\alpha_1]$ for $j = 2, \ldots, n$.*
We say that α_1 is a zero of Q of multiplicity $n \geq 2$ *if $\alpha_j \in [\alpha_1]$ for all $j = 2, \ldots, n$.*
If $\alpha_j \in \mathbb{R}$ we can repeat the notion of multiplicity of α_1 where (2) *holds by removing the assumption $\alpha_{j+1} \neq \bar{\alpha}_j$.*

The definition of multiplicity, in the case of a real zero, coincides with the standard notion of multiplicity since, in this case, the $\star$-product reduces to the pointwise product. Note that if a function has a sphere of zeros at $[\alpha]$ with multiplicity n, at most one point on $[\alpha]$ can have higher multiplicity.
As a consequence of Proposition 1.1 one has (see Corollary 3.3 in [4]):

Corollary 1.3 *If $\lim_{r \to 1} |f(re^{I\theta})| = 1$, for all I fixed in $\mathbb{S}$, then*

$$\lim_{r \to 1} |(f \star g)(re^{I\theta})| = |g(e^{I'\theta})|,$$

a.e. for $\theta \in [0, 2\pi)$ (and $I' \in \mathbb{S}$ depends on θ, I and f).

2. The Hardy space of the unit ball

The quaternionic Hardy space $H_2(\mathbb{B})$ of the unit ball $\mathbb{B}$ is defined as the space of square summable (left) slice regular power series, see [3]:

$$H_2(\mathbb{B}) = \left\{ f(p) = \sum_{k=0}^{\infty} p^k f_k : \|f\|^2_{H_2(\mathbb{B})} := \sum_{k=0}^{\infty} |f_k|^2 < \infty \right\}.$$

The space $H_2(\mathbb{B})$ is a right quaternionic Hilbert space if equipped with the inner product

$$\langle f, g\rangle = \sum_{k=0}^{\infty} \bar{g}_k f_k \quad \text{if} \quad f(p) = \sum_{k=0}^{\infty} p^k f_k, \;\; g(p) = \sum_{k=0}^{\infty} p^k g_k. \tag{3}$$

If f is as in (3) then for a fixed $I \in \mathbb{S}$ we have

$$\begin{aligned}\int_0^{2\pi} |f(re^{I\theta})|^2 d\theta &= \int_0^{2\pi} \left(\sum_{j,k=0}^{\infty} r^{k+j} \overline{f}_k e^{I(j-k)\theta} f_j \right) d\theta \\ &= \sum_{j,k=0}^{\infty} r^{k+j} \overline{f}_k \left(\int_0^{2\pi} e^{I(j-k)\theta} d\theta \right) f_j = 2\pi \cdot \sum_{n=0}^{\infty} r^{2n} |f_n|^2.\end{aligned}$$

This last formula implies that the norm in $H_2(\mathbb{B})$ can be equivalently defined as, see [4]:

$$\|f\|^2_{H_2(\mathbb{B})} = \sup_{0\leq r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta \tag{4}$$

where the value of the integral on the right is the same for each $I \in \mathbb{S}$. Observe that the supremum in (4) can be replaced by the limit as r tends to one. The integrand in (4) depends on $I \in \mathbb{S}$, so one could have put in the definition of the norm also the supremum with respect to $I \in \mathbb{S}$, however since the integral does not depend on the choice of I the supremum with respect to $I \in \mathbb{S}$ is not needed.

Remark 2.1 The space $H_2(\mathbb{B})$ can be alternatively characterized as the reproducing kernel Hilbert space with reproducing kernel

$$k_{H_2(\mathbb{B})}(p, q) = \sum_{n=0}^{\infty} p^n \overline{q}^n. \tag{5}$$

The function $k_{H_2(\mathbb{B})}(\cdot, q)$ belongs to $H_2(\mathbb{B})$ for every $q \in \mathbb{B}$ and the fact that $k_{H_2(\mathbb{B})}(p, q)$ is a reproducing kernel means for any function $f \in H_2(\mathbb{B})$ as in (3),

$$\left\langle f, k_{H_2(\mathbb{B})}(\cdot, q)\right\rangle_{H_2(\mathbb{B})} = \sum_{k=0}^{\infty} q^k f_k = f(q). \tag{6}$$

An important feature of slice hyperholomorphic functions which is important in the sequel is the following formula:

Theorem 2.2 *(Representation Formula) Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f : \Omega \to \mathbb{H}$ be a slice regular function. The following equality holds for all $q = x + Iy \in \Omega$, $J, K \in \mathbb{S}$:*

$$f(x + Iy) = (J - K)^{-1}[Jf(x + Jy) - Kf(x + Ky)] + I(J - K)^{-1}[f(x + Jy) - f(x + Ky)]. \tag{7}$$

This formula implies that if we have two points on a same sphere, the value of the function at any other point on the sphere is uniquely determined by the values in the first two points. Moreover we have:

Proposition 2.3 *The finite collection of functions* $\{k_{H_2(\mathbb{B})}(\cdot, q_i)\}$ *based on distinct points* $q_1, \ldots, q_k \in \mathbb{B}$ *is (right) linearly independent in* $H_2(\mathbb{B})$ *if and only if none three of these points belong to the same* 2*-sphere.*

The linear dependence of three functions $k_{H_2(\mathbb{B})}(\cdot, p_i)$ based on equivalent points explains in an alternative way that the restriction of any function $f \in H_2(\mathbb{B})$ to any 2-sphere is completely determined by the values of f at any two points of this sphere.
The following result, see [1, Theorem 3.3], [3, Theorem 6.2] shows that Schur functions are contractions on $H_2(\mathbb{B})$:

Theorem 2.4 *Let* $S : \mathbb{B} \to \mathbb{H}$*. The following are equivalent:*

(i) S *is slice regular on* $\mathbb{B}$ *and* $|S(p)| \leq 1$ *for all* $p \in \mathbb{B}$*.*

(ii) The operator M_S *of left* $\star$*–multiplication by* S

$$M_S : f \mapsto S \star f \tag{8}$$

is a contraction on $H_2(\mathbb{B})$*, that is,* $\|S \star f\|_{H_2(\mathbb{B})} \leq \|f\|_{H_2(\mathbb{B})}$ *for all* $f \in H_2(\mathbb{B})$*.*

The matrix-valued version of this result is in [4, Theorem 4.6]. In analogy to the complex case we introduce the Hardy space $H_\infty(\mathbb{B})$ of bounded slice regular functions on $\mathbb{B}$ with norm $\|S\|_\infty = \sup_{p\in\mathbb{B}} |S(p)| < \infty$ and the space $\mathcal{M}(H_2(\mathbb{B}))$ of bounded multipliers, that is, the functions $S : \mathbb{B} \to \mathbb{H}$ such that the operator M_S of left $\star$–multiplication (8) is bounded on $H_2(\mathbb{B})$. By its definition, the set of slice hyperholomorphic functions on $\mathbb{B}$ which are bounded by 1 and denoted by $\mathcal{R}(\mathbb{B}, \overline{\mathbb{B}})$ is the closed unit ball of $H_\infty(\mathbb{B})$, see [1, Corollary 3.5]. Moreover, we have the following consequence of Theorem 2.4.

Corollary 2.5 $H_\infty(\mathbb{B}) = \mathcal{M}(H_2(\mathbb{B}))$ *and* $\|S\|_\infty = \|M_S\|$ *for every* $S \in H_\infty(\mathbb{B})$*.*

2.1. Blaschke products

We now recall the notion of Blaschke factors and products, see [5, 4, 6]. According to the fact that the zeros of a slice hyperholomorphic functions can also be spheres, we will introduce two different types of Blaschke factors.

Definition 2.6 *Let* $a \in \mathbb{H}$*,* $|a| < 1$*. The function*

$$B_a(p) = (1 - p\bar{a})^{-\star} \star (a - p)\frac{\bar{a}}{|a|} \tag{9}$$

is called Blaschke factor at a*.*

Remark 2.7 Let $\lambda(p) = 1 - p\bar{a}$ and let us apply formula (1) to the products $\lambda^c(p) \star \lambda(p)$ and $\lambda^c(p) \star (a - p)$. We have

$$\begin{aligned} B_a(p) &= (\lambda^c(p) \star \lambda(p))^{-1}\lambda^c(p) \star (a - p)\frac{\bar{a}}{|a|} = (\lambda^c(p)\lambda(\tilde{p}))^{-1}\lambda^c(p)(a - \tilde{p})\frac{\bar{a}}{|a|} \\ &= \lambda(\tilde{p})^{-1}(a - \tilde{p})\frac{\bar{a}}{|a|} = (1 - \tilde{p}\bar{a})^{-1}(a - \tilde{p})\frac{\bar{a}}{|a|}, \end{aligned} \tag{10}$$

where $\tilde{p} = \lambda^c(p)^{-1}p\lambda^c(p)$. Formula (10) gives the Blaschke factor $B_a(p)$ in terms of the pointwise multiplication.

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The following property extends the analogous property in the complex case:

Theorem 2.8 *Let $a \in \mathbb{H}$, $|a| < 1$. The Blaschke factor $B_a(q)$ has the following properties:*

(i) it takes the unit ball $\mathbb{B}$ to itself;

(ii) it takes the boundary of the unit ball to itself;

(iii) it has a unique zero for $p = a$.

In [4, Theorem 5.6] we proved the following result:

Theorem 2.9 *Let $\{a_j\} \subset \mathbb{B}$, $j = 1, 2, \dots$ be a sequence of nonzero quaternions such that $[a_i] \neq [a_j]$ if $i \neq j$ and assume that $\sum_{j\geq 1}(1 - |a_j|) < \infty$. Then the function*

$$B(p) := \Pi^\star_{j\geq 1}(1 - p\bar{a}_j)^{-\star} \star (a_j - p)\frac{\bar{a}_j}{|a_j|}, \tag{11}$$

where $\Pi^\star$ denotes the $\star$-product, converges uniformly on the compact subsets of $\mathbb{B}$.

The Blaschke factor vanishing at the sphere $[a]$ is given in the following definition:

Definition 2.10 *Let $a \in \mathbb{H}$, $|a| < 1$. The function*

$$B_{[a]}(p) = (1 - 2\mathrm{Re}(a)p + p^2|a|^2)^{-1}(|a|^2 - 2\mathrm{Re}(a)p + p^2) \tag{12}$$

is called Blaschke factor at the sphere $[a]$.

The definition of $B_{[a]}(p)$ does not depend on the choice of the point a that identifies the 2-sphere. Indeed all the elements in the sphere $[a]$ have the same real part and module. We also have, see [6]:

Proposition 2.11 *The $\star$-inverse of B_a and $B_{[a]}$ are $B_{\bar{a}^{-1}}$, $B_{[a^{-1}]}$ respectively.*

The following result is the analogue of Theorem 2.9:

Proposition 2.12 *A Blaschke product having zeros at the set of spheres*

$$Z = \{([c_1], \nu_1), ([c_2], \nu_2), \dots\}$$

where $c_j \in \mathbb{B}$, the sphere $[c_j]$ is a zero of multiplicity ν_j, $j = 1, 2, \dots$ and $\sum_{j\geq 1} \nu_j(1 - |c_j|) < \infty$ is given by

$$\prod_{j\geq 1}(B_{[c_j]}(p))^{\nu_j}.$$

By Theorem 2.9 and 2.12 we can prove the following general result, see [4, 6]:

Theorem 2.13 *A Blaschke product having zeroes at the set*

$$Z = \{(a_1, n_1), \dots, ([c_1], m_1), \dots\}$$

where $a_j \in \mathbb{B}$, a_j have respective multiplicities $n_j \geq 1$, $a_j \neq 0$ for $j = 1, 2, \dots$, $[a_i] \neq [a_j]$ if $i \neq j$, $c_i \in \mathbb{B}$, the spheres $[c_j]$ have respective multiplicities $m_j \geq 1$, $j = 1, 2, \dots$, $[c_i] \neq [c_j]$ if $i \neq j$ and

$$\sum_{i,j\geq 1} \Big(n_i(1 - |a_i|) + 2m_j(1 - |c_j|)\Big) < \infty \tag{13}$$

is of the form

$$\prod_{i\geq 1}(B_{[c_i]}(p))^{m_i} \prod^{\star}_{i\geq 1} \prod_{j=1}^{\star n_i} (B_{\alpha_{ij}}(p)),$$

where $n_j \geq 1$, $\alpha_{11} = a_1$ and α_{ij} are suitable elements in $[a_i]$, and if $\alpha_{ij} \in \mathbb{H} \setminus \mathbb{R}$ $\alpha_{i\,j+1} \neq \overline{\alpha_{ij}}$, for $j = 2, 3, \dots$.

In the case in which one has to construct a Blaschke product having a zero at a_i with multiplicity n_i by prescribing the factors $(p - a_{i1}) \star \cdots \star (p - a_{in_i})$, $a_{ij} \in [a_i]$ for all $j = 1, \ldots, n_i$, the factors in the Blaschke product must be chosen accordingly, see [4, 6].

Definition 2.14 *A Blaschke product of the form*

$$B(p) = \prod_{i=1}^{r} (B_{[c_i]}(p))^{m_i} \prod_{i=1}^{\star s} \prod_{j=1}^{\star n_i} (B_{\alpha_{ij}}(p)), \tag{14}$$

is said to have degree $d = \sum_{i=1}^{r} 2m_i + \sum_{j=1}^{s} n_j$.

Let us denote by $\mathcal{H}(B)$ the quaternionic Hilbert space with reproducing kernel K_B. We have, see [6]:

Theorem 2.15 *Let $B(p)$ be a Blaschke product as in (14). Then* $\dim(\mathcal{H}(B)) = \deg B$.

Theorem 2.16 *Let B_a be a Blaschke factor. The operator*

$$M_a \ : \ f \mapsto B_a \star f$$

is an isometry from $\mathbf{H}_2(\mathbb{B})$ *into itself.*

The following problem is the simplest Beurling-Lax type problem in the present setting and we show below how to solve it:

Problem 2.17 *Given N points $a_1, \ldots, a_N \in \mathbb{B}$, and M spheres $[c_1], \ldots, [c_M]$ in $\mathbb{B}$ such that the spheres $[a_1], \ldots [a_N], [c_1], \ldots, [c_M]$ are pairwise non-intersecting, find all $f \in \mathbf{H}_2(\mathbb{B})$ such that*

$$f(a_i) = 0, \quad i = 1, \ldots, N, \tag{15}$$

and

$$f([c_j]) = 0, \quad j = 1, \ldots, M. \tag{16}$$

Theorem 2.18 *There is a Blaschke product B such that the solutions of Problem 2.17 are the functions $f = B \star g$, when g runs through* $\mathbf{H}_2(\mathbb{B})$.

3. The Hardy space of the half-space

Let us consider the half-space $\mathbb{H}_+$ of the quaternions q such that $\mathrm{Re}(q) > 0$ and set $\Pi_{+,I} = \mathbb{H}_+ \cap \mathbb{C}_I$. We will denote by f_I the restriction of a function f defined on $\mathbb{H}_+$ to $\Pi_{+,I}$. We define

$$\mathbf{H}_2(\Pi_{+,I}) = \{f \text{ slice hyperholomorphic in } \mathbb{H}_+ \ : \ \int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy < \infty\},$$

where $f(Iy)$ denotes the nontangential value of f at Iy. Note that these value exist almost everywhere, in fact any $f \in \mathbf{H}_2(\Pi_{+,I})$ when restricted to a complex plane $\mathbb{C}_I$ can be written as $f_I(x + Iy) = F(x + Iy) + G(x + Iy)J$ where J is any element in $\mathbb{S}$ orthogonal to I, and F, G are $\mathbb{C}_I$-valued holomorphic functions. Since the nontangential values of F and G exist almost everywhere at Iy, also the nontangential value of f exists at Iy a.e. on $\Pi_{+,I}$, see [2].

Remark 3.1 In alternative, we could have defined $\mathbf{H}_2(\Pi_{+,I})$ as the set of slice hyperholomorphic functions f such that $\sup_{x>0}\int_{-\infty}^{+\infty}|f_I(x+Iy)|^2dy < \infty$. However note that $f_I(x+Iy) = F(x+Iy)+G(x+Iy)J$, see the above discussion, and so $|f_I(x+Iy)|^2 = |F(x+Iy)|^2+|G(x+Iy)|^2$. Thus, from the result in the complex case, we have

$$\begin{aligned}\sup_{x>0}\int_{-\infty}^{+\infty}|f_I(x+Iy)|^2dy &= \sup_{x>0}\int_{-\infty}^{+\infty}|F(x+Iy)|^2dy+\sup_{x>0}\int_{-\infty}^{+\infty}|G(x+Iy)|^2dy\\ &= \int_{-\infty}^{+\infty}|F(Iy)|^2dy+\int_{-\infty}^{+\infty}|G(Iy)|^2dy \qquad (1)\\ &= \int_{-\infty}^{+\infty}|f_I(Iy)|^2dy.\end{aligned}$$

In $\mathbf{H}_2(\Pi_{+,I})$ we define the scalar product

$$\langle f,g\rangle_{\mathbf{H}_2(\Pi_{+,I})} = \int_{-\infty}^{+\infty}\overline{g_I(Iy)}f_I(Iy)dy,$$

where $f_I(Iy)$, $g_I(Iy)$ denote the nontangential values of f,g at Iy on $\Pi_{+,I}$. This scalar product gives the norm

$$\|f\|_{\mathbf{H}_2(\Pi_{+,I})} = \left(\int_{-\infty}^{+\infty}|f_I(Iy)|^2dy\right)^{\frac{1}{2}},$$

(which is finite by our assumptions).

Proposition 3.2 *Let f be slice hyperholomorphic in $\mathbb{H}_+$ and assume that $f\in\mathbf{H}_2(\Pi_{+,I})$ for some $I\in\mathbb{S}$. Then for all $J\in\mathbb{S}$ the following inequalities hold*

$$\frac{1}{2}\|f\|_{\mathbf{H}_2(\Pi_{+,I})}\leq\|f\|_{\mathbf{H}_2(\Pi_{+,J})}\leq 2\|f\|_{\mathbf{H}_2(\Pi_{+,I})}.$$

An immediate consequence of this result is:

Corollary 3.3 *A function $f\in\mathbf{H}_2(\Pi_{+,I})$ for some $I\in\mathbb{S}$ if and only if $f\in\mathbf{H}_2(\Pi_{+,J})$ for all $J\in\mathbb{S}$.*

We now introduce the Hardy space of the half space $\mathbb{H}_+$, see [2, Definition 4.4]:

Definition 3.4 *We define $\mathbf{H}_2(\mathbb{H}_+)$ as the space of slice hyperholomorphic functions on $\mathbb{H}_+$ such that*

$$\sup_{I\in\mathbb{S}}\int_{-\infty}^{+\infty}|f(Iy)|^2dy<\infty. \qquad (2)$$

We have:

Proposition 3.5 *The function*

$$k(p,q) = (\bar p+\bar q)(|p|^2+2\mathrm{Re}(p)\bar q+\bar q^2)^{-1} \qquad (3)$$

is slice hyperholomorphic in p and $\bar q$ on the left and on the right, respectively in its domain of definition, i.e. for $p\notin[\bar q]$. The restriction of $\frac{1}{2\pi}k(p,q)$ to $\mathbb{C}_I\times\mathbb{C}_I$ coincides with $k_{\Pi_+}(z,w)$. Moreover we have the equality:

$$k(p,q) = (|q|^2+2\mathrm{Re}(q)p+p^2)^{-1}(p+q). \qquad (4)$$

The function $k(p,q)$ can also be constructed by taking the left $\star$-inverse with respect to the variable p or the right $\star$-inverse with respect to the variable q, that is

$$k(p,q) = (|q|^2 + 2\mathrm{Re}(q)p + p^2)^{-1}(p+q) = (p+\bar{q})^{-\star}$$

or

$$k(p,q) = (\bar{p}+\bar{q})(|p|^2 + 2\mathrm{Re}(p)\bar{q} + \bar{q}^2)^{-1} = (p-\bar{q})^{-\star_r}.$$

Proposition 3.6 *The kernel* $\frac{1}{2\pi}k(p,q)$ *is reproducing, i.e. for any* $f \in \mathbf{H}_2(\mathbb{H}_+)$

$$f(p) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(p, Iy) f(Iy) dy.$$

For some computations, it is useful to know that the kernel $k(p,q)$ satisfies (see [2, Proposition 4.7])

$$pk(p,q) + k(p,q)\bar{q} = 1.$$

We know that if $\{\phi_n(z)\}$ is an orthonormal basis for $\mathbf{H}_2(\Pi_{+,I})$, for some $I \in \mathbb{S}$, then

$$k(z,w) = \sum_{n=1}^{\infty} \phi_n(z)\overline{\phi_n(w)}, \tag{5}$$

and so the kernel $k(z,w)$ is positive definite. In [2, Proposition 4.8] we have proved the following:

Proposition 3.7 *Let* $\{\phi_n(z)\}$ *be an orthonormal basis for* $\mathbf{H}_2(\Pi_{+,I})$*, for some* $I \in \mathbb{S}$*, and let* $\{\Phi_n(q)\} = \{\mathrm{ext}(\phi_n(z))\}$ *be the sequence of the slice hyperholomorphic extensions of its elements. Then* $\{\Phi_n(q)\}$ *is an orthonormal basis for* $\mathbf{H}_2(\mathbb{H}_+)$*, and*

$$k(p,q) = \sum_{n=0}^{\infty} \Phi_n(p)\overline{\Phi_n(q)}.$$

Example 3.8 As an example of decomposition using an orthonormal basis, we consider

$$\Phi_n(p) = \sqrt{2}(p+1)^{-n-1}(p-1)^n.$$

We have

$$k(p,q) = \sum_{n=0}^{\infty} \Phi_n(p)\overline{\Phi_n(q)} = 2\sum_{n=0}^{\infty}(p+1)^{-n-1}(p-1)^n(\bar{q}-1)^n(\bar{q}+1)^{-n-1}.$$

We now introduce the Blaschke factors in the half space $\mathbb{H}_+$, see [2].

Definition 3.9 *For* $a \in \mathbb{H}_+$ *set*

$$b_a(p) = (p+\bar{a})^{-\star} \star (p-a).$$

The function $b_a(p)$ *is called Blaschke factor at* a *in the half space* $\mathbb{H}_+$.

Remark 3.10 The function $b_a(p)$ is defined outside the sphere $[-a]$ as it can be easily seen by rewriting it as

$$b_a(p) = (p^2 + 2\mathrm{Re}(a)p + |a|^2)^{-1}(p+a) \star (p-a) = (p^2 + 2\mathrm{Re}(a)p + |a|^2)^{-1}(p^2 - a^2)$$

and it has a zero for $p = a$.

We have the following result which characterizes the convergence of a Blaschke product:

Theorem 3.11 *Let* $\{a_j\} \subset \mathbb{H}_+$, $j = 1, 2, \dots$ *be a sequence of quaternions such that* $\sum_{j\geq 1} \mathrm{Re}(a_j) < \infty$. *Then the function*

$$B(p) := \Pi^{\star}_{j\geq 1}(p + \bar{a}_j)^{-\star} \star (p - a_j), \tag{6}$$

converges uniformly on the compact subsets of $\mathbb{H}_+$.

As in the unit ball case, we have two kinds of Blaschke factors. In fact, products of the form

$$b_a(p) \star b_{\bar{a}}(p) = ((p + \bar{a})^{-\star} \star (p - a)) \star ((p + a)^{-\star} \star (p - \bar{a}))$$

can be written as

$$b_a(p) \star b_{\bar{a}}(p) = (p^2 + 2\mathrm{Re}(a)p + |a|^2)^{-1}(p^2 - 2\mathrm{Re}(a)p + |a|^2),$$

and they admit the sphere $[a]$ as set of zeros. Thus if we want to construct a Blaschke product vanishing at some assigned spheres, it is convenient to introduce the following:

Definition 3.12 *For* $a \in \mathbb{H}_+$ *set*

$$b_{[a]}(p) = (p^2 + 2\mathrm{Re}(a)p + |a|^2)^{-1}(p^2 - 2\mathrm{Re}(a)p + |a|^2).$$

The function $b_a(p)$ *is called Blaschke factor at the sphere* $[a]$ *in the half space* $\mathbb{H}_+$.

Note that the definition is well posed since it does not depend on the choice of the point a. As a consequence of Theorem 3.11 we have:

Corollary 3.13 *Let* $\{c_j\} \subset \mathbb{H}_+$, $j = 1, 2, \dots$ *be a sequence of quaternions such that* $\sum_{j\geq 1} \mathrm{Re}(c_j) < \infty$. *Then the function*

$$B(p) := \Pi_{j\geq 1}(p^2 + 2\mathrm{Re}(c_j)p + |c_j|^2)^{-1}(p^2 - 2\mathrm{Re}(c_j)p + |c_j|^2), \tag{7}$$

converges uniformly on the compact subsets of $\mathbb{H}_+$.

Thus we can prove the following [2, Theorem 4.14]:

Theorem 3.14 *A Blaschke product having zeros at the set*

$$Z = \{(a_1, \mu_1), (a_2, \mu_2), \dots, ([c_1], \nu_1), ([c_2], \nu_2), \dots\}$$

where $a_j \in \mathbb{H}_+$, a_j *have respective multiplicities* $\mu_j \geq 1$, $[a_i] \neq [a_j]$ *if* $i \neq j$, $c_i \in \mathbb{H}_+$, *the spheres* $[c_j]$ *have respective multiplicities* $\nu_j \geq 1$, $j = 1, 2, \dots$, $[c_i] \neq [c_j]$ *if* $i \neq j$ *and*

$$\sum_{i,j\geq 1} \Big(\mu_j(1 - |a_j|) + 2\nu_i(1 - |c_i|)\Big) < \infty$$

is given by

$$\prod_{i\geq 1}(b_{[c_i]}(p))^{\nu_i} \prod_{j\geq 1}^{\star} \prod_{k=1}^{\star\mu_j}(b_{a_{jk}}(p))^{\star\mu_j},$$

where $a_{11} = a_1$ *and* $a_{jk} \in [a_j]$ *are such that* $\alpha_{j+1} \neq \bar{\alpha}_j$, $j = 1, \dots, n-1$, *if* $\alpha_j \in \mathbb{H} \setminus \mathbb{R}$, $k = 1, 2, 3, \dots, \mu_j$.

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We conclude this section by proving that the operator of multiplication by a Blaschke factor is an isometry. In the proof we are in need of the notion of conjugate of a function f. Given a slice hyperholomorphic function f consider its restriction to a complex plane $\mathbb{C}_I$ and write it, as customary, in the form $f_I(z) = F(z) + G(z)J$ where J is an element in $\mathbb{S}$ orthogonal to I and F, G are $\mathbb{C}_I$-valued holomorphic functions. Define $f^c(p) = \mathrm{ext}(\overline{F(\bar{z})} - G(z)J)$ as the unique extension of the function $\overline{F(\bar{z})} - G(z)J$, see e.g. [10]. Note that if $f(p) = \sum_{n\geq 0} p^n a_n$ then $f^c(p) \sum_{n\geq 0} p^n \bar{a}_n$. We have the following, see [2]:

Lemma 3.15 *Let* $f \in \mathbf{H}_2(\mathbb{H}_+)$. *Then* $\|f\|_{\mathbf{H}_2(\mathbb{H}_+)} = \|f^c\|_{\mathbf{H}_2(\mathbb{H}_+)}$.

Theorem 3.16 *Let* b_a *be a Blaschke factor. The operator*

$$M_{b_a} \;:\; f \mapsto b_a \star f$$

is an isometry from $\mathbf{H}_2(\mathbb{H}_+)$ *into itself.*

More generally, a function S slice hyperholomorphic in the right-half-plane will be such that M_S is a contraction from the Hardy space of the right half-plane into itself if and only it is bounded by one in modulus there. The operator range $\sqrt{I - M_S M_S^*}$ with the lifted norm is then the associated de Branges Rovnyak space. For more information see [2], and [8].

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