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On a class of quaternionic positive definite functions and their derivatives

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In this paper, we start the study of stochastic processes over the skew field of quaternions. We discuss the relation between positive definite functions and the covariance of centered Gaussian processes and the construction of stochastic processes and their derivatives. The use of perfect spaces and strong algebras and the notion of Fock space are crucial in this framework. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4977082]

I. INTRODUCTION

In the present work, we present some constructions pertaining to stochastic processes in the quaternionic setting. We begin with a brief overview in the real and complex cases. Let $K(t,s)$ be a complex valued continuous function positive definite on $\mathbb{R}\times\mathbb{R}$. It is well known (see, for instance, Refs. 34 and 37, pp. 38-39) that K is the covariance function of a Gaussian process $(X_t)_{t\in\mathbb{R}}$ defined in a probability space, say $(\Omega,\mathcal{B},P)$. The assumed continuity insures that the associated reproducing kernel Hilbert space is made of continuous functions (this follows easily from the Cauchy-Schwarz inequality) and is separable (see Ref. 23, Lemma 4.10, p. 347 for the latter). There are a number of ways to construct the probability space $(\Omega,\mathcal{B},P)$, and some connections between the various approaches have been studied in Ref. 9. A first approach, see Refs. 37 (pp. 38-39), 29 and Section II, is to consider the space $\times_{j\in J}\mathbb{R}$, where J is an index set with the same power as the power of the reproducing kernel Hilbert space $\mathcal{H}(K)$ associated with K, with the sigma-algebra generated by cylinders and as measure the corresponding product of $N(0,1)$ laws. Another way is to take for Ω the space of continuous functions on the real line. Another convenient way to build the process is to take Ω to be the space of real tempered distributions and use Hida's white noise space; see Refs. 22, 24, and 33 for the latter. Viewing Hida's white noise space as the center of a Gelfand triple is the approach allowed in Refs. 3 and 4 to define the derivative of X_t in a space of stochastic distributions introduced by Yuri Kondratiev space for positive definite functions of the form

$$r(t)+\overline{r(s)}-r(t-s). \tag{1.1}$$

Such functions, and the associated Gaussian stochastic processes, play an important role in various fields of mathematics. The case $r(t)=|t|^{2H}$ with $H\in(0,1)$ corresponds to the fractional Brownian motion and, in particular, to the Brownian motion for $H=1/2$. Covariance functions of the form (1.1) were studied by Schoenberg, von Neumann, and Krein, see Refs. 32 and 36, and are exactly functions of the form

$$\int_{\mathbb{R}}\frac{e^{iut}-1}{u}\frac{e^{-ius}-1}{u}d\sigma(u),$$

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where σ is a positive Borel measure on the real line subject to the condition

$$\int_{\mathbb{R}} \frac{d\sigma(u)}{u^2+1} < \infty.$$

The measure σ and the function r are related by

$$r(t) = -\int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2+1} \right\} \frac{d\sigma(u)}{u^2},$$

which reduces to

$$r(t) = 2\int_0^{\infty} \frac{1-\cos ut}{u^2} d\sigma(u)$$

for even σ.

When going to the non commutative version of the previous analysis and to the case of the case of free processes, one replaces the white noise space by the full Fock space associated with $\mathbf{L}_2(\mathbb{R}, dx)$. Now the values of the free stochastic processes are continuous operators in the free white noise space. In particular, the non commutative version of the Brownian motion can be constructed, see Refs. 40 and 42. In this paper, we consider another non commutative setting, namely, the quaternions. In Refs. 10 and 13, free processes with covariance function (1.1) and their derivatives were constructed. To construct the derivatives, one associates with the free white noise space a Gelfand triple, and the derivatives are continuous operators from a space of non commutative test functions to a space of non commutative distributions; the latter is an example of algebras of a special form, introduced and called in Refs. 10 and 13–15 strong algebras. In the present paper, we develop the quaternionic version of this analysis to define quaternionic processes with covariance functions (1.1) and, under appropriate hypothesis on the function $r(t)$, their derivative. We build in particular the fractional Brownian motion in this setting. The fact that we are in the quaternionic setting would allow, in principle, to consider positive definite functions of the form

$$\int_{\mathbb{S}} d\mu(\mathbf{i}) \left(\int_{\mathbb{R}} \frac{e^{\mathbf{i}ut}-1}{u} \frac{e^{-\mathbf{i}us}-1}{u} d\sigma(u) \right), \tag{1.2}$$

where $d\mu$ is a positive measure on the sphere $\mathbb{S}$ of quaternionic square roots of unity (see Ref. 6 for a similar phenomenon).

To the best of our knowledge, this paper and Ref. 6 are among the first to consider infinite dimensional analysis in the quaternionic setting. Among related papers, though more related to mathematical physics aspects, we mention Refs. 30 and 35. Quaternions are important in physics, for example, in classical mechanics, see Ref. 21 (where also more general hypercomplex algebras are treated), but also in quantum mechanics, see Refs. 2, 16, and 5 for a recent development.

This paper is not merely a generalization of the analogous results in the complex setting: the quaternionic framework requires different tools and methods. For example, one needs a different notion of spectrum (the so-called S-spectrum, see Ref. 18), one has to define a suitable way for a product to obtain the counterpart of the tensor product and even the notion of algebra, being constructed on the skew field of quaternions differs from the classical one. Thus it is interesting that we could generalize some results, despite these crucial differences.

The paper consists of six sections besides the present Introduction, and its outline is as follows: In Section II we extend to the quaternionic case the well known one-to-one relationship between positive definite functions and covariances of centered Gaussian processes. The main features of a perfect topological vector in the quaternionic setting (in which being compact is equivalent to being bounded and closed) are studied in Section III. In Section IV we consider quaternionic Fock spaces. Strong algebras are considered in Section V, while Secs. VI and VII consider constructions of stochastic processes and their derivatives.

II. POSITIVE DEFINITE FUNCTIONS AND GAUSSIAN PROCESSES

Let J be an arbitrary set of indices (note that the set J need not be countable), and consider the space $\Omega = \times_J \mathbb{R}$ endowed with the cylinder algebra $\mathcal{C}$, and with the product measure $P = \times_J \gamma_1$, where

$$d\gamma_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

The coordinate system of the probability space $(\Omega, \mathcal{C}, P)$ defines a family $(x_j)_{j\in J}$ of independent $N(0,1)$ real-valued random variables indexed by J, see, e.g., Ref. 37, pp. 38-39 and also Ref. 9, Definition 3.29 for a related discussion. The real Gaussian Hilbert space associated with the random process $(x_j)_{j\in J}$ is the closed linear span of these variables in $\mathbf{L}_2(\Omega, \mathcal{C}, P)$ (see Ref. 37, p. 39, Ref. 28, Example 1.9, p. 6), and by definition of $\mathcal{C}$ the symmetric Fock space associated with $\mathcal{H}$ is $\mathbf{L}_2(\Omega, \mathcal{C}, P)$; see Ref. 28, Chapter 2.

Definition 2.1. A quaternionic random variable X is a measurable map from $(\Omega, \mathcal{C}, P)$ into $\mathbb{H}$ (equivalently the four real components of X are measurable). We denote by $\mathbf{L}_2(\Omega, \mathcal{C}, P) \otimes \mathbb{H}$ the space of quaternionic random variables such that $|X| \in \mathbf{L}_2(\Omega, \mathcal{C}, P)$.

We now define a quaternionic Gaussian process on $(\Omega, \mathcal{C}, P)$ (the definition will be the same for any other probability space).

Definition 2.2. The $\mathbb{H}$-valued stochastic process on $(\Omega, \mathcal{C}, P)$ is called Gaussian if the closed linear space of its real components is a real Gaussian subspace of $\mathbf{L}_2(\Omega, \mathcal{C}, P)$.

Theorem 2.3. *Let T be a set, and let $K(t,s)$, $t, s \in T$, be a $\mathbb{H}$-valued positive definite function. Then there exists a quaternionic Gaussian process $(X_t)_{t\in T}$ such that*

$$\mathbb{E}(X_t \overline{X_s}) = K(t,s), \tag{2.1}$$

where $\mathbb{E}$ denotes the expectation.

Proof. Let $\mathcal{H}(K)$ be the reproducing kernel Hilbert space associated with K of $\mathbb{H}$-valued functions defined on T. Let $(e_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}(K)$.

For every $t, s \in T$, we have

$$K(t,s) = \sum_{j\in J} e_j(t)\overline{e_j(s)}, \quad t, s \in T, \tag{2.2}$$

and in particular $(e_j(t))_{j\in J} \in \ell_2(J, \mathbb{H})$. It follows that the series

$$X_t(\omega) = \sum_{j\in J} e_j(t) x_j(\omega) \tag{2.3}$$

converges in $\mathbf{L}_2(\Omega, \mathcal{C}, P)$, where $(\Omega, \mathcal{C}, P)$ is as in the introduction to this section, and that (2.1) holds. □

Remark 2.4. Under appropriate hypothesis we will embed $\mathbf{L}_2(\Omega, \mathcal{A}, P)$ into a strong algebra (see Section V for the definition) in which both the formal derivative

$$X_t'(\omega) = \sum_{j\in j} \frac{d}{dt} e_j(t) x_j(\omega) \tag{2.4}$$

and the integral

$$\int_a^b f(t) X_t'(\omega) dt \tag{2.5}$$

make sense.

Definition 2.5. The function $F(t) = (\overline{e_j(t)})_{j\in J} \in \ell_2(J, \mathbb{H})$ is called a representer of K.

Remark 2.6. We have

$$K(t,s) = \langle F(s), F(t)\rangle_{\ell_2(J,\mathbb{H})}, \quad t,s \in T, \tag{2.6}$$

and the representer is unique up to multiplication by a unitary operator on the right.

To give the next definition, we need some more notations. We begin with a given function $F \in \ell_2(J,\mathbb{H})$. Then, we set iteratively $F_1(t_1) = F(t_1)$ and

$$F_n(t_1,\ldots,t_n) = F(t_n) \otimes F_{n-1}(t_1,\ldots,t_{n-1}), \quad n = 2,3,\ldots$$

and

$$K_n(t_1,\ldots,t_n,s_1,\ldots,s_n) = \langle F_n(s_1,\ldots,s_n), F_n(t_1,\ldots,t_n)\rangle_{\ell_2(J,\mathbb{H})}, \quad n = 1,2,\ldots.$$

Definition 2.7. The space

$$\mathbb{H} \oplus \oplus_{n=1}^{\infty} \mathcal{H}(K_n),$$

where $\mathcal{H}(K_n)$ is the reproducing kernel Hilbert space associated with K_n, is the Fock space associated with F.

As is well known, the product of two complex-valued positive definite functions is still positive definite. In the case of matrix-valued (or operator-valued) functions, the pointwise product has to be replaced by the tensor product. This result does not hold in the quaternionic case. The product of two $\mathbb{H}$-valued positive definite functions need not be Hermitian, let alone positive definite. We here propose the following definition for such a product.

Proposition 2.8. Let K_1 and K_2 be two positive definite functions on the sets E_1 and E_2 and let $F_1 = (e_j)_{j\in J}$ be a representer of K_1. The function

$$\sum_{j\in J} e_j(t_1)K_2(t_2,s_2)\overline{e_j(s_1)} \tag{2.7}$$

is positive definite in $E_1 \times E_2$.

Proof. It suffices to remark that (2.7) is a sum of positive definite kernels. □

Remark 2.9.

(a) Definition (2.7) depends on the choice of F_1, as is illustrated by the example

$$K_1(t_2,s_2) \equiv 1 \quad \textit{and} \quad F_1(t) = c, \quad c \in \mathbb{H} \textit{ such that } |c| = 1.$$

Then (2.7) becomes $cK_2(t_2,s_2)\overline{c} \neq K_2(t_2,s_2)$ in general.

(b) If F_2 is a representer of K_2, then $F_1 \otimes F_2$ is a representer of (2.7).

(c) If $K_1 = K_2$ and $E_1 = E_2$, (2.7) becomes

$$\sum_{j\in J}\sum_{j\in J} e_j(t_1)K_1(t_2,s_2)\overline{e_j(s_1)}$$

and in particular is positive definite on $E_1 \times E_1$.

III. PERFECT SPACES

The arguments in the paper make use, in the quaternionic setting, of a number of facts from the theory of topological vector spaces and in particular of the notion of perfect spaces, nuclear spaces, and Gelfand triple. We consider these notions in the present section in the quaternionic setting and use as sources (which are set for complex and real vector spaces) the books.[19,20,25] Most of the arguments can be naturally adapted, and then we do not provide proofs. Two important differences should be pointed out: First the dual of a left (right) quaternionic vector space is a right (left) quaternionic vector space. Next, the notion of the spectrum of a linear operator was only recently defined in 2007; see Refs. 7 and 18 and the references therein. This notion is needed to adapt arguments where the spectrum intervenes.

In the quaternionic version of the approach of Gelfand and Shilov, the starting point is a right linear space, say $\mathcal{V}$, endowed with a countable family of increasing norms, say $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots$, which are pairwise compatible. This means that if one is given a sequence of elements of $\mathcal{V}$ which is a Cauchy sequence with respect to two of these norms and if it converges to 0 with respect to one of the norms, it also converges to 0 in the second norm.

Definition 3.1. We denote by $\mathcal{V}_p$ the completion of $\mathcal{V}$ with respect to $\|\cdot\|_p$.

The fact that the norms are compatible insures that the natural map from $\mathcal{V}_q$ into $\mathcal{V}_p$ when $q \geq p$ is one-to-one.

We endow the intersection $\cap_{p=0}^{\infty}\mathcal{V}_p$ with the smallest topology with respect to which all the norms $\|\cdot\|_p$ are continuous.

Proposition 3.2. (see Ref. 20, Theorem p. 17 for the complex setting case) With $\mathcal{V}$ and $\mathcal{V}_p$ as above, then $\mathcal{V}$ is complete if and only if $\mathcal{V} = \cap_{p=0}^{\infty}\mathcal{V}_p$ is a Fréchet space, and its topology can be defined by the metric

$$d(u,v) = \sum_{p=0}^{\infty} \frac{1}{2^p} \frac{\| u - v\|_p}{1 + \| u - v\|_p}.$$

Definition 3.3. (see Ref. 20, p. 53 for the complex setting case) The space $\mathcal{V}$ as above is called perfect if it is complete and if a set is compact if and only if it is closed and bounded in $\mathcal{V}$.

For the following result in the complex case, see Ref. 20, *Theorem p. 55.*

Proposition 3.4. If there is a sequence $p_1 < p_2 < \cdots$ such that the inclusions $\mathcal{V}_{p_{j+1}} \to \mathcal{V}_{p_j}$ are compact, the space is perfect.

The dual of a countably normed space is characterized as follows (see [20, p. 34]).

Proposition 3.5. An element φ belongs to $\mathcal{V}'$ if and only if there exists a $p \in \mathbb{N}$ and $C > 0$ such that

$$|\varphi(v)| \leq C \| v\|_p, \quad v \in \mathcal{V}. \tag{3.1}$$

Dual of perfect spaces have specific properties (see Ref. 20, Section 6.4), and we mention in particular:

Proposition 3.6. (see Ref. 20, p. 45). A set is bounded in $\mathcal{V}'$ if and only if it is bounded in one of the $\mathcal{V}_p'$ in the corresponding norm.

Theorem 3.7. *Let $\mathcal{V}$ be perfect. Then weak and strong convergences of sequences are equivalent in the dual.*

Theorem 3.8. *In the dual $\mathcal{V}'$ of a perfect space $\mathcal{V}$, a sequence converges if and only if it converges in one of the spaces $\mathcal{V}_p'$ in the corresponding norm.*

IV. TENSOR PRODUCTS, THE FOCK SPACE, AND SECOND QUANTIZATION

We refer to Ref. 27 for information on tensor products of modules. Tensor products of quaternionic spaces have been studied in a number of places; see, for instance, Refs. 11, 26, and 38. Here we use a concrete construction of tensor products of quaternionic spaces of sequences. Specifically, we let

$$\ell_2(\mathbb{N},\mathbb{H}) = \left\{ (a_1, a_2, \ldots) \in \mathbb{H}^{\mathbb{N}} \, : \, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\},$$

and for $a, b \in \ell_2(\mathbb{N},\mathbb{H})$ we define $a \otimes b \in (\ell_2(\mathbb{N},\mathbb{H}))^{\mathbb{N}}$ by

$$\begin{aligned} a \otimes b &= (a_1 b, a_2 b, \ldots) \\ &= (a_1 b_1, a_1 b_2, \ldots, a_2 b_1, a_2 b_2, \ldots, a_3 b_1, \ldots). \end{aligned} \tag{4.1}$$

Remark 4.1. The space $(\ell_2(\mathbb{N},\mathbb{H}))^{\mathbb{N}}$ is canonically identified with $\ell_2(\mathbb{N},\mathbb{H})$, and we define the inner product and norm accordingly. In particular, for $a, b, c, d \in \ell_2(\mathbb{N},\mathbb{H})$, we have

$$\langle a \otimes b, c \otimes d \rangle_{(\ell_2(\mathbb{N},\mathbb{H}))^{\mathbb{N}}} = \langle (a_1 b, a_2 b, \ldots)\,,\, (c_1 d, c_2 d, \ldots) \rangle_{(\ell_2(\mathbb{N},\mathbb{H}))^{\mathbb{N}}} = \sum_{u=1}^{\infty} \langle a_u b, c_u d \rangle_{\ell_2(\mathbb{N},\mathbb{H})}. \tag{4.2}$$

Proposition 4.2. One has the inner product formula

$$\langle a \otimes b, c \otimes d \rangle_{(\ell_2(\mathbb{N},\mathbb{H}))^{\mathbb{N}}} = \langle \langle a, c \rangle b, d \rangle,$$

where the brackets $\langle \cdot, \cdot \rangle$ *denote the inner product in* $\ell_2(\mathbb{N},\mathbb{H})$*, and in particular we have*

$$\| a \otimes b \|_{(\ell_2(\mathbb{N},\mathbb{H}))^{\mathbb{N}}} = \| a \|_{\ell_2(\mathbb{N},\mathbb{H})} \cdot \| b \|_{\ell_2(\mathbb{N},\mathbb{H})}. \tag{4.3}$$

Proof. By (4.2) we have

$$\begin{aligned} \langle a \otimes b, c \otimes d \rangle_{(\ell_2(\mathbb{N},\mathbb{H}))^{\mathbb{N}}} &= \langle (a_1 b, a_2 b, \ldots)\,,\, (c_1 d, c_2 d, \ldots) \rangle_{(\ell_2(\mathbb{N},\mathbb{H}))^{\mathbb{N}}} \\ &= \sum_{u=1}^{\infty} \langle a_u b, c_u d \rangle_{\ell_2(\mathbb{N},\mathbb{H})} \\ &= \sum_{u=1}^{\infty} \langle \overline{c_u} a_u b, d \rangle_{\ell_2(\mathbb{N},\mathbb{H})} \\ &= \left\langle \left(\sum_{u=1}^{\infty} \overline{c_u} a_u \right) b, d \right\rangle_{\ell_2(\mathbb{N},\mathbb{H})} \\ &= \left\langle \left(\langle a, c \rangle_{\ell_2(\mathbb{N},\mathbb{H})} \right) b, d \right\rangle_{\ell_2(\mathbb{N},\mathbb{H})}. \end{aligned}$$

Formula (4.3) follows directly. □

Definition 4.3. We denote by $\ell_2(\mathbb{N},\mathbb{H}) \otimes \ell_2(\mathbb{N},\mathbb{H})$ *the right Hilbert space generated by the span of elements of the form (4.1) in* $(\ell_2(\mathbb{N},\mathbb{H}))^{\mathbb{N}}$*. We define iteratively*

$$\ell_2(\mathbb{N},\mathbb{H})^{\otimes n} = \ell_2(\mathbb{N},\mathbb{H}) \otimes \ell_2(\mathbb{N},\mathbb{H})^{\otimes n-1}$$

and

$$\Gamma(\ell_2(\mathbb{N},\mathbb{H})) = \mathbb{H} \oplus \left(\oplus_{n=1}^{\infty} \ell_2(\mathbb{N},\mathbb{H})^{\otimes n} \right).$$

Each of the spaces $\ell_2(\mathbb{N},\mathbb{H})^{\otimes n}$ is a right quaternionic Hilbert space, and $\Gamma(\ell_2(\mathbb{N},\mathbb{H}))$ is their Hilbert space direct sum.

Definition 4.4. We call $\Gamma(\ell_2(\mathbb{N},\mathbb{H}))$ *the full Fock space associated with* $\ell_2(\mathbb{N},\mathbb{H})$*. It will also be denoted by* $\mathcal{F}$.

The space $\mathcal{F}$ was introduced and studied in the quaternionic setting in Ref. 8.

We denote by $\widetilde{\ell}$ the free non commutative monoid generated by $\mathbb{N}$ (see Ref. 1 for the quaternionic case). We write an element of $\widetilde{\ell}$ as a finite sequence of pairs

$$\alpha = (i_1, n_1), (i_2, n_2), \ldots, (i_N, n_N),$$

where $i_1, \ldots, i_N$ are integers such that $i_j \neq i_{j+1}, j = 1, \ldots, N-1$, and $n_1, \ldots, n_N \in \mathbb{N}$. We set

$$|\alpha| = \sum_{j=1}^{N} n_j. \tag{4.4}$$

We will also use the notation

$$\widetilde{z}^{\alpha} = z_{i_1}^{n_1} \cdots z_{i_N}^{n_N}, \tag{4.5}$$

where $z_{i_1}, \ldots, z_{i_N}$ are non commuting variables.

Proposition 4.5. The space $\ell_2(\mathbb{N}, \mathbb{H})^{\otimes n}$ can be seen as the right linear span of elements of the form $\widetilde{z}^{\alpha}$ with $|\alpha| = n$.

The space $\mathcal{F}$ consists of the functions of the form

$$f = f_0 + \sum_{\alpha \in \widetilde{\ell}} \widetilde{z}^{\alpha} f_{\alpha}, \tag{4.6}$$

where the coefficients belong to $\mathbb{H}$ and are such that

$$|f_0|^2 + \sum_{\alpha \in \widetilde{\ell}} |f_{\alpha}|^2 < \infty.$$

Proof. Let $a^{(1)}, a^{(2)}, \ldots, a^{(n)} \in \ell_2(\mathbb{N}, \mathbb{H})$. The elementary tensor $a^{(1)} \otimes (a^{(2)} \otimes (\cdots))$ of $\ell_2(\mathbb{N}, \mathbb{H})^{\otimes n}$ is a sequence of quaternions of the form

$$a_{m_1}^{(1)} a_{m_2}^{(2)} \cdots a_{m_n}^{(n)}.$$

We associate in a unique way the sequence $(m_1, \ldots, m_n)$ with an element in the monoid $\widetilde{\ell}$ as follows: if $m_1 = \cdots = m_{n_1}$ and m_{n1} is different from the following index, the first component in the associated monoid element is (i_1, n_1), with $m_1 = i_1$ and $n_1 = n$. We then reiterate with the index following m_{n1}. □

Definition 4.6. Let $h \in \ell_2(\mathbb{N}, \mathbb{H})$. The creation operator is defined by

$$\ell_h(f) = h \otimes f, \quad f \in \mathcal{F}.$$

Proposition 4.7. It holds that

$$\ell_k^* \ell_h = \langle h, k \rangle I, \quad f, g \in \mathcal{F}, \tag{4.7}$$

and in particular

$$\langle \ell_k^* \ell_h f, g \rangle_{\mathcal{F}} = \langle \langle h, k \rangle f, g \rangle_{\ell_2(\mathbb{N}, \mathbb{H})}. \tag{4.8}$$

Proof. The map ℓ_k sends $\ell_2(\mathbb{N}, \mathbb{H})^{\otimes n}$ into $\ell_2(\mathbb{N}, \mathbb{H})^{\otimes (n+1)}$. Thus for $m \neq n$, $f \in \ell_2(\mathbb{N}, \mathbb{H})^{\otimes n}$, and $g \in \ell_2(\mathbb{N}, \mathbb{H})^{\otimes m}$, we have

$$\langle \ell_h f, \ell_k g \rangle_{\mathcal{F}} = 0.$$

Now, for $f, g \in \ell_2(\mathbb{N}, \mathbb{H})^{\otimes n}$ we have

$$\begin{aligned}
\langle \ell_k^* \ell_h f, g \rangle_{\ell_2(\mathbb{N}, \mathbb{H})^{\otimes n}} &= \langle \ell_h f, \ell_k g \rangle_{\mathcal{F}} \\
&= \langle \ell_h f, \ell_k g \rangle_{\ell_2(\mathbb{N}, \mathbb{H})^{\otimes (n+1)}} = \langle (h_1 f, h_2 f, \ldots)\ (k_1 g, k_2 g, \ldots) \rangle_{\ell_2(\mathbb{N}, \mathbb{H})^{\otimes (n+1)}} \\
&= \sum_{u=1}^{\infty} \langle h_u f,\ k_u g \rangle_{\ell_2(\mathbb{N}, \mathbb{H})^{\otimes (n+1)}} \\
&= \sum_{u=1}^{\infty} \langle \overline{k_u} h_u f,\ g \rangle_{\ell_2(\mathbb{N}, \mathbb{H})^{\otimes (n+1)}} \\
&= \left\langle \left(\sum_{u=1}^{\infty} \overline{k_u} h_u \right) f,\ g \right\rangle_{\ell_2(\mathbb{N}, \mathbb{H})^{\otimes n}} \\
&= \langle \langle h, k \rangle f,\ g \rangle_{\ell_2(\mathbb{N}, \mathbb{H})^{\otimes n}},
\end{aligned}$$

and hence the result. □

It follows from formula (4.7) that

$$\langle \ell_h 1, \ell_k 1\rangle_{\mathcal{F}} = \langle h, k\rangle, \quad h, k \in \mathcal{F}. \tag{4.9}$$

Definition 4.8. The linear span of the operators ℓ_h endowed with the inner product (4.9) is a pre-Hilbert space. We will denote by $\boldsymbol{L}_2$ the associated Hilbert space.

More generally, for a sequence $m = (m_n)_{n\in\mathbb{N}}$ of strictly positive numbers (a weight function), we define $\ell_2(\mathbb{N}, \mathbb{H}, m)$ to be the Hilbert space of sequences $(f_n)_{n\in\mathbb{N}}$ of quaternions such that

$$\sum_{n=1}^{\infty} m_n |f_n|^2 < \infty$$

and define for $\alpha = ((i_1, \alpha_1), (i_2, \alpha_2), \ldots) \in \widetilde{\ell}$,

$$m_\alpha = \prod m_{i_k}^{\alpha_k} = \prod m_j^{\sum_{k\,;\,i_k=j}\alpha_k}.$$

This construction leads to a full Fock space $\Gamma(\ell_2(\mathbb{N}, \mathbb{H}, m))$, which we will also denote by $\mathcal{F}(m)$.

Proposition 4.9. The quaternionic *right vector space $\mathcal{F}(m)$ consists of the series of the form (4.6) such that*

$$|f_0|^2 + \sum_{\alpha\in\widetilde{\ell}} m_\alpha |f_\alpha|^2 < \infty.$$

Notation 4.10. For $p \in \mathbb{Z}$ we denote by m^p the sequence (m_n^p) and by $\mathcal{F}(m^p)$ the corresponding Fock space.

The proofs of the following two results follow the complex case and will be omitted.

Theorem 4.11. *Let T be a bounded linear operator from $\ell_2(\mathbb{N}, \mathbb{H}, m)$ into itself. Then $T^{\otimes n}$: $(\ell_2(\mathbb{N}, \mathbb{H}, m))^{\otimes n} \to (\ell_2(\mathbb{N}, \mathbb{H}, m))^{\otimes n}$ is defined iteratively by*

$$T^{\otimes n}(u_1 \otimes \cdots \otimes u_n) = Tu_1 \otimes (Tu_2 \cdots \otimes Tu_n)\,.$$

The operator $T^{\otimes n}$ is bounded. When T is a contraction, it induces a bounded linear operator $\Gamma(\ell_2(\mathbb{N}, \mathbb{H})) \to \Gamma(\ell_2(\mathbb{N}, \mathbb{H}))$, denoted by $\Gamma(T)$ and called the second quantization of T.

Definition 4.12. Let T *be a bounded linear operator from $\ell_2(\mathbb{N}, \mathbb{H}, m)$ into itself which is a contraction. The bounded linear operator*

$$\Gamma(T):\ \Gamma(\ell_2(\mathbb{N}, \mathbb{H})) \to \Gamma(\ell_2(\mathbb{N}, \mathbb{H}))$$

is called the second quantization of T.

Let (λ_n) be a sequence of non-negative numbers. For $\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n} \in \widetilde{\ell}$ (where $i_1 \neq i_2 \neq \cdots \neq i_n$), we denote

$$\lambda_{\mathbb{N}}^{\alpha} = \prod_{k=1}^{n} \lambda_{i_k}^{\alpha_k} = \prod_{j\in\{i_1,\ldots,i_n\}} \lambda_j^{\left(\sum_{k:i_k=j}\alpha_k\right)}.$$

We recall that if $T: \mathcal{H}_1 \to \mathcal{H}_2$ is a compact operator between two separable right quaternionic Hilbert spaces, then

$$Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n\rangle h_n,$$

where $(e_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and where (λ_n) is a non-negative sequence converging to zero. Conversely, any such decomposition defines a compact operator $\mathcal{H}_1\to\mathcal{H}_2$ (see, for instance, Ref. 39).

Theorem 4.13. *Let $T:\mathcal{H}_1\to\mathcal{H}_2$ be a compact contraction operator between two separable quaternionic right Hilbert spaces with*

$$Tf=\sum_{n=1}^{\infty}\lambda_n\langle f,e_n\rangle h_n,$$

where $(e_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and where (λ_n) is a non-negative sequence converging to zero. Let $\Gamma(T)$ be its second quantization as in Definition 4.12. Then,

(a) *it holds that*

$$\Gamma(T)f=\sum_{\alpha\in\widetilde{\ell}}\lambda_{\mathbb{N}}^{\alpha}\langle f,e_\alpha\rangle h_\alpha,$$

where $(e_\alpha)_{\alpha\in\widetilde{\ell}}$ and $(h_\alpha)_{\alpha\in\widetilde{\ell}}$ are orthonormal bases of $\Gamma(\mathcal{H}_1)$ and $\Gamma(\mathcal{H}_2)$, respectively.

(b) *if furthermore T is an Hilbert-Schmidt operator, i.e., $(\lambda_n)\in\ell^2(\mathbb{N})$, then*

$$\|\Gamma(T)\|_{HS}^2=\sum_{n=0}^{\infty}\|T\|_{HS}^{2n}.$$

In particular, $\Gamma(T)$ is a Hilbert-Schmidt operator if and only if T is a Hilbert-Schmidt operator with $\|T\|_{HS}<1$ and in this case we obtain

$$\|\Gamma(T)\|_{HS}=\frac{1}{\sqrt{1-\|T\|_{HS}^2}}.$$

V. STRONG ALGEBRAS

Motivated by an algebra of stochastic distributions defined in Ref. 31, but see also Ref. 24, p. 81, strong algebras were introduced and studied in the series of papers.[10,12–15] We here follow the special case[14] and not the most general setting described in Ref. 15. The notion of algebra can be defined in the quaternionic setting, but it differs from the classical one.

Definition 5.1. Let $\mathcal{V}$ be a quaternionic right vector space endowed with a product $\cdot:\ \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ such that $(v_1,v_2)\mapsto v_1\cdot v_2$ and satisfying the following:

(1) *associative property: $(v_1\cdot v_2)\cdot v_3=v_1\cdot(v_2\cdot v_3)$ for all $v_1,v_2,v_3\in\mathcal{V}$;*
(2) *distributive properties: $u\cdot(v_1+v_2)=u\cdot v_1+u\cdot v_2$ and $(v_1+v_2)\cdot u=v_1\cdot u+v_2\cdot u$, for all $u,v_1,v_2\in\mathcal{V}$;*
(3) *right linearity in the second factor: $(v_1\cdot v_2)q=v_1\cdot(v_2q)$, for all $v_1,v_2\in\mathcal{V}$ and $q\in\mathbb{H}$.*

Then $\mathcal{V}$ is said to be a quaternionic right associative algebra. The algebra is said to be unital or with unity if there exists an element $1\in\mathcal{V}$ such that $v\cdot 1=1\cdot v=v$ for all $v\in\mathcal{V}$.

Remark 5.2. If $\mathcal{V}$ is a quaternionic left vector space, property (3) has to be substituted with the left linearity in the first factor, i.e., $q(v_1\cdot v_2)=(qv_1)\cdot v_2$ for all $v_1,v_2\in\mathcal{V}$ and $q\in\mathbb{H}$. If $\mathcal{V}$ is a quaternionic two-sided vector space, property (3) becomes the left linearity in the first factor and the right linearity in the second factor, i.e., $q(v_1\cdot v_2)=(qv_1)\cdot v_2$ and $(v_1\cdot v_2)q=v_1\cdot(v_2q)$, for all $v_1,v_2\in\mathcal{V}$ and $q\in\mathbb{H}$. It should be noted that the standard definition of associative algebra is given for vector spaces over a field or, more in general, over a commutative division ring and property (3) expresses the bilinearity of the product in both the factors. Since $\mathbb{H}$ is a skew field, the standard definition cannot be applied and the bilinearity is meant as the linearity on the left in the first factor and/or the linearity on the right in the second factor according to the fact that $\mathcal{V}$ is a left or right or two-sided vector space over $\mathbb{H}$.

Definition 5.3. Let $(\mathcal{H}_p, \|\cdot\|_p)_{p\in\mathbb{N}_0}$ *be an increasing sequence of quaternionic right Hilbert spaces with decreasing norms, and assume that the union* $\mathcal{V}=\cup_{p\in\mathbb{N}_0}\mathcal{H}_n$ *is the dual of a nuclear Fréchet space. Then* $\mathcal{V}$ *is called a strong algebra if it is endowed with a product, which makes it into an algebra with the following property: there is an integer* $d\in\mathbb{N}_0$ *such that for all* $p,q\in\mathbb{N}_0$ *such that* $q\geq p+d$ *and all* $f\in\mathcal{H}_p$ *and* $g\in\mathcal{H}_q$*, both fg and gf belong to* $\mathcal{H}_q$ *and*

$$\|fg\|_q \leq A(p,q)\,\|f\|_p\|g\|_q, \tag{5.1}$$

$$\|gf\|_q \leq B(p,q)\,\|f\|_p\|g\|_q, \tag{5.2}$$

where A(p,q) and B(p,q) depend only on p and q.

Example 5.4. Let $\mathcal{G}_p=\ell_2(\mathbb{N},(2^{-np})_{n\in\mathbb{N}},\mathbb{H})$ and let $\mathcal{G}=\cup_{p\in\mathbb{N}}\mathcal{G}_p$. Then $\mathcal{G}$ endowed with the convolution of sequences is a strong algebra.

Strong algebras can be built using Fock spaces. For $p\in\mathbb{Z}$ and for a given weight function $m=(m_n)_{n\in\mathbb{N}}$, recall that

$$\ell_2(\mathbb{N},\mathbb{H},m^p)=\left\{(f_n)\in\mathbb{H}^{\mathbb{N}}\ :\ \sum_{n=1}^{\infty}|f_n|^2m_n^p<\infty\right\},\quad p\in\mathbb{Z}$$

and

$$\mathcal{V}=\cap_{p\in\mathbb{N}_0}\ell_2(\mathbb{N},\mathbb{H},m^p)\quad\text{and}\quad\mathcal{V}'=\cup_{p\in\mathbb{N}_0}(\ell_2(\mathbb{N},\mathbb{H},m^p))'$$

Proposition 5.5.

$$(\ell_2(\mathbb{N},\mathbb{H},m^p))'=\ell_2(\mathbb{N},\mathbb{H},m^{-p}),$$

and in particular

$$\mathcal{V}'=\cup_{p\in\mathbb{N}_0}\ell_2(\mathbb{N},\mathbb{H},m^{-p}).$$

Proof. Let $\varphi\in\mathcal{V}'$. By definition of the topology of $\mathcal{V}$ there exists $p\in\mathbb{N}$ such that

$$|\varphi(f)|\leq K\,\|f\|_p,$$

see Ref. 20, Section 4.3 for the complex case; the proof in the quaternionic case is the same.

By Riesz representation theorem (which still holds for quaternionic Hilbert spaces, see Ref. 17), there is $h^{(p)}\in\ell_2(\mathbb{N},\mathbb{H},m^p)$ such that

$$\varphi(f)=\langle f,h^{(p)}\rangle_{\ell_2(\mathbb{N},\mathbb{H},m^p)}.$$

Let

$$g_n^{(p)}=h_n^{(p)}m_n^p.$$

Then $g^{(p)}\in\mathcal{G}_{-p}$ since

$$\sum_{n=1}^{\infty}\frac{|h_n^{(p)}m_n^p|^2}{m_n^p}=\sum_{n=1}^{\infty}|h_n^{(p)}|^2m_n^p$$

and

$$\varphi(f)=\langle f,g^{(p)}\rangle_{\ell_2(\mathbb{N},\mathbb{H},m^{-p})}.$$

□

Proposition 5.6. The quaternionic right vector space $\mathcal{V} = \cap \ell_2(\mathbb{N}, \mathbb{H}, m^p)$ is a Fréchet space. It is nuclear if and only if there exists $d \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} m_n^{-d} < \infty.$$

Proof. Let $p, q \in \mathbb{Z}$ be such that $q \geq p$, and consider the embedding $I_{q,p}: \mathcal{V}_q \hookrightarrow \mathcal{V}_p$. We have

$$\|I_{q,p} m_n^{-q/2} e_n\|_p = m_n^{-(q-p)/2} \|m_n^{-p/2} e_n\|_q,$$

and hence

$$\|I_{q,p}\|_{HS} = \sqrt{\sum_{n\in\mathbb{N}} m_n^{-(q-p)}}.$$

□

Remark 5.7. The dual of a Fréchet space is nuclear if and only if the initial space is nuclear. Thus, $\bigcup_{p\in\mathbb{N}} \ell_2(\mathbb{N}, \mathbb{H}, m^{-p})$ is nuclear if and only if $\bigcap_{p\in\mathbb{N}} \ell_2(\mathbb{N}, \mathbb{H}, m^p)$ is nuclear. This in turn will hold if and only if for any p there is some $q > p$ such that $\|T_{q,p}\|_{HS} < \infty$, where $T_{p,q}$ denotes the injection from $\ell_2(\mathbb{N}, \mathbb{H}, m^q)$ into $\ell_2(\mathbb{N}, \mathbb{H}, m^p)$, that is, if and only if there exists some $d > 0$ such that $\sum_{n\in\mathbb{N}} m_n^{-d}$ converges. We note that in this case, d can be chosen so that

$$\sum_{n\in\mathbb{N}} m_n^{-d} < 1. \tag{5.3}$$

Definition 5.8. We call the smallest integer d which satisfies the inequality (5.3) the index of $\bigcup_{p\in\mathbb{N}} \ell_2(\mathbb{N}, \mathbb{H}, m^{-p})$.

The following theorem appears in Ref. 13. We repeat the proof for completeness; the argument (in the commutative case) can be found in Ref. 24, p. 129 and is first referred to in Ref. 41.

Theorem 5.9. *Assume (5.3) in force. Then $\bigcup_{p\in\mathbb{N}} \Gamma(\ell_2(\mathbb{N}, \mathbb{H}, m^{-p}))$ is a strong algebra: it holds*

$$\|f \otimes g\|_q \leq c_{p-q} \|f\|_p \|g\|_q \text{ and } \|g \otimes f\|_q \leq c_{p-q} \|f\|_p \|g\|_q$$

for all $q \geq p + d$, where $\|\cdot\|_p$ is the norm associated with $\Gamma(\ell_2(\mathbb{N}, \mathbb{H}, m^{-p}))$ and where

$$c_{p-q} = \sum_{\alpha\in\widetilde{\ell}} m_{\mathbb{N}}^{-\alpha(q-p)} = \frac{1}{1 - \sum_{n\in\mathbb{N}} a_m^{-(q-p)}} < \infty.$$

Proof. Denoting $b_\alpha = a_{\mathbb{N}}^{\alpha}$, we have that

$$\Gamma(\ell_2(\mathbb{N}, \mathbb{H}, m^{-p})) = \left\{ (f_\alpha)_{\alpha\in\widetilde{\ell}} : \sum_{\alpha\in\widetilde{\ell}} |f_\alpha|^2 b_\alpha^{-p} < \infty \right\}.$$

Since for any $\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n} \in \widetilde{\ell}$ and $\beta = z_{j_1}^{\beta_1} z_{j2}^{\beta_2} \cdots z_{i_m}^{\beta_m} \in \widetilde{\ell}$ it holds that

$$b_\alpha b_\beta = a_{\mathbb{N}}^{\alpha} a_{\mathbb{N}}^{\beta} = \prod_{k=1}^{n} a_{i_k}^{\alpha_k} \cdot \prod_{l=1}^{m} a_{i_l}^{\beta_l} = a_{\mathbb{N}}^{\alpha\beta} = b_{\alpha\beta},$$

for any $f \in \Gamma(\ell_2(\mathbb{N}, \mathbb{H}, m^{-p}))$ and $g \in \Gamma(\ell_2(\mathbb{N}, \mathbb{H}, m^{-q}))$ we obtain

$$\begin{aligned}
\|f \otimes g\|_q^2 &= \sum_{\gamma \in \widetilde{\ell}} \left| \sum_{\alpha \leq \gamma} f_\alpha g_{\alpha^{-1}\gamma} b_\gamma^{-q/2} \right|^2 \\
&\leq \sum_{\gamma \in \widetilde{\ell}} \left(\sum_{\alpha \leq \gamma} |f_\alpha| b_\alpha^{-q/2} |g_{\alpha^{-1}\gamma}| b_{\alpha^{-1}\gamma}^{-q/2} \right)^2 \\
&= \sum_{\gamma \in \widetilde{\ell}} \left(\sum_{\alpha, \alpha' \leq \gamma} |f_\alpha| b_\alpha^{-q/2} |f_{\alpha'}| b_{\alpha'}^{-q/2} |g_{\alpha^{-1}\gamma}| b_{\alpha^{-1}\gamma}^{-q/2} |g_{(\alpha')^{-1}\gamma}| b_{(\alpha')^{-1}\gamma}^{-q/2} \right) \\
&\leq \sum_{\alpha, \alpha' \in \widetilde{\ell}} \left(|f_\alpha| b_\alpha^{-q/2} |f_{\alpha'}| b_{\alpha'}^{-q/2} \sum_{\gamma \geq \alpha, \alpha'} |g_{\alpha^{-1}\gamma}| b_{\alpha^{-1}\gamma}^{-q/2} |g_{(\alpha')^{-1}\gamma}| b_{(\alpha')^{-1}\gamma}^{-q/2} \right) \\
&\leq \left(\sum_{\beta \in \widetilde{\ell}} |f_\beta| b_\beta^{-p/2} \right)^2 \left(\sum_{\beta \in \widetilde{\ell}} |g_\beta|^2 b_\beta^{-q} \right)^{\frac{1}{2}} \left(\sum_{\beta \in \widetilde{\ell}} |g_\beta|^2 b_\beta^{-q} \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\beta \in \widetilde{\ell}} b_\beta^{-(q-p)} \right) \left(\sum_{\beta \in \widetilde{\ell}} |f_\beta|^2 b_\beta^{-p} \right) \left(\sum_{\beta \in \widetilde{\ell}} |g_\beta|^2 b_\beta^{-q} \right) \\
&= \|\Gamma(T_{q,p})\|_{HS}^2 \, \|f\|_p^2 \|g\|_q^2.
\end{aligned}$$

The second inequality is obtained in the same manner. □

The definition of a topological algebra requires only separate continuity of the product. In a strong algebra, the product is in fact jointly continuous, see Ref. 15; the proof stays the same in the quaternionic setting. This allows to simplify somewhat some of the arguments of Ref. 4 in the proof of the following result.

Theorem 5.10. *Let $\mathcal{V} = \cup_{p \in \mathbb{N}_0} \mathcal{H}_p$ be a strong algebra. Let f and g be two continuous functions from $[0,1]$ into $\mathcal{V}$. Then the integral $\int_0^1 f(t)g(t)dt$ exists as a Riemann integral in the strong topology of $\mathcal{V}$.*

Proof. The product in a strong algebra is jointly continuous, see Ref. 15, Theorem 3.3, p. 215, and so the map $t \mapsto f(t)g(t)$ is continuous on the compact set $[0,1]$; its range is thus a compact subset of $\mathcal{V}$ and hence included in one of the spaces $\mathcal{H}_p$, and f is uniformly continuous from $[0,1]$ into $\mathcal{H}_p$ (see Ref. 4, Theorem 3.1, p. 405). □

VI. GENERALIZED STOCHASTIC PROCESSES

Let $K(t,s)$ be a quaternionic-valued positive definite function on $\mathbb{R}$, and let $\mathcal{H}(K)$ be the associated reproducing kernel right quaternionic Hilbert space with reproducing kernel $K(t,s)$. We assume that K is continuous, and so $\mathcal{H}(K)$ is separable. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}(K)$. We have

$$K(t,s) = \sum_{n=1}^{\infty} e_n(t)\overline{e_n(s)}. \tag{6.1}$$

For every $t \in \mathbb{R}$ the sequence $h_t = (e_n(t))_{n\in\mathbb{N}}$ belongs to $\ell_2(\mathbb{N}, \mathbb{H})$.

To state the next result, we recall that the strong algebra $\mathcal{G}$ has been defined in Example 5.4.

Proposition 6.1. Let $(e_j(t))_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}(K)$, assume that $e_j(t)$ are continuously differentiable on [0,1] and that for $t \in [0, 1]$

$$|e_j'(t)| \leq K j^M,$$

for some $K > 0$ and $M \in \mathbb{N}$. Then the function $t \mapsto (e_j(t))$ is differentiable in $\mathcal{G}$ with derivative equal to $(e_j'(t))$.

Proof. Let $t_0 \in (0, 1)$ (if $t_0 = 0$ or $t_0 = 1$ the arguments are readily adapted). We have

$$\frac{e_j(t) - e_j(t_0)}{t - t_0} - e_j'(t_0) = \frac{\int_{t_0}^{t} (e_j'(u) - e_j'(t_0))du}{t - t_0} = e_j'(\xi) - e_j'(t_0),$$

where $\xi \in [t_0, t]$. Hence,

$$|\frac{e_j(t) - e_j(t_0)}{t - t_0} - e_j'(t_0)|^2 \leq 2K^2 j^{2M},$$

and so the limit

$$\lim_{t \to t_0} \frac{(e_j(t))_{j \in \mathbb{N}_0} - (e_j(t_0))_{j \in \mathbb{N}_0}}{t - t_0} = (e_j'(t_0))_{j \in \mathbb{N}_0}$$

holds in $\mathcal{G}_1$. This means that it holds in $\mathcal{G}$ as is seen by using Theorem 3.8. □

Theorem 5.10 allows to define stochastic integrals of the type (2.5).

VII. CONSTRUCTION OF PROCESSES USING FOCK SPACES

Fock spaces are relevant for the construction of strong algebras but not only. In this section, we show how to construct stochastic processes using Fock spaces. Next result generalizes to the quaternionic setting Theorem 7.4 in Ref. 10. For the notation in use, we refer the reader to Section IV and

$$\mathcal{F}[m] = \cup_{p \in \mathbb{N}} \mathcal{F}(m^p), \qquad \widetilde{\mathcal{F}}[m] = \cap_{p \in \mathbb{N}} \mathcal{F}(m^{-p}).$$

Theorem 7.1. *Consider a sequence of differentiable functions on* [0,1] *such that $a(t) = (a_n(t))_{n \in \mathbb{N}_0} \in \ell_2(\mathbb{N}, \mathbb{H})$ and such that*

$$|a_n(t) - a_n(s)| \leq |t - s| \cdot \varphi(n), \quad t, s \in [0, 1], \tag{7.1}$$

where

$$\varphi(n) = \begin{cases} C \cdot (n + 1)^M & \text{(case1)}, \\ C \cdot M^n & \text{(case 2)}, \end{cases} \tag{7.2}$$

C and M being strictly positive constants. Let $(X_a)(t) = 2\mathrm{Re}\, \ell_{a(t)}$. Then there exists a $\mathcal{L}(\mathcal{F}[m], \widetilde{\mathcal{F}}[m])$-valued function W_a such that

$$\frac{d}{dt} X_a(t) f = W_a(t) f, \quad f \in \mathcal{F}[m] \tag{7.3}$$

in the topology of $\widetilde{\mathcal{F}}[m]$.

Proof. We divide the proof into steps.

STEP 1: *There exists a weight function such that $(a_n(t))_{n \in \mathbb{N}_0} \in \ell_2(\mathbb{N}_0, \mathbb{H}, m^{-p})$ for some $p \in \mathbb{N}$.*

In case 1, take $m_n = (n + 1)$. Then $p > 2M + 2$ will do. In case 2, take $m_n = 2^n$. Then we need to chose p such that $M^2 < 2^p$.

STEP 2: *There exists a weight function such that $(a_n'(t))_{n \in \mathbb{N}_0} \in \ell_2(\mathbb{N}_0, \mathbb{H}, m^{-p})$ for some $p \in \mathbb{N}$.*

This follows from step 1, since $|a_n'(t)| \leq \varphi(n)$ in view of (7.1).

STEP 3: *Let $f \in \ell_2(\mathbb{N}_0, \mathbb{H}, m^{-p})$ for some $p \in \mathbb{N}$. Then the associated creation operator ℓ_f is bounded from $\mathcal{F}(m^{-p})$ into itself.*

Indeed, let $u=(u_\alpha)_{\alpha\in\widetilde{\ell}}\in\mathcal{F}(m^{-p})$. Then, $\ell_f u=(f_n u_\alpha)_{\substack{n\in\mathbb{N}_0\\ \alpha\in\widetilde{\ell}}}$ and

$$\|\ell_f u\|^2_{\mathcal{F}(m^p)}=\sum_{n,\alpha}|f_n|^2|u_\alpha|^2 m_n^2 m_\alpha^2=\|f\|_{\ell_2(\mathbb{N},\mathbb{H},m^{-p})}\cdot\|u\|^2_{\mathcal{F}(m^{-p})}.$$

It follows from the previous step that

STEP 4: *$X_f=\ell_f+\ell_f^*$ is bounded from $\mathcal{F}(m^p)$ into $\mathcal{F}(m^{-p})$, with norm less or equal to* $2\,\|f\|_{\ell_2(\mathbb{N},\mathbb{H},m^{-p})}$.

The remainder of the proof consists of applying the first steps to $f=a(t)$ and is as in Ref. 10. □

The two cases considered in the previous theorem are just sample cases; depending on the growth condition of the functions $a_n'(t)$, other strong algebras can be considered.

As in Sec. VI, Theorem 7.1 allows to define stochastic integrals of the type (2.5).

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