A Simple Measure of Economic Complexity

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A Simple Measure of Economic Complexity

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A Simple Measure of Economic Complexity

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Abstract. The conventional view on economic development simplifies a country’s production to one aggregate variable, GDP. Yet product diversification matters for economic development, as recent, data-driven, “economic complexity” research suggests. A country’s product diversity reflects the country’s diversity of productive knowhow, or “capabilities”. Researchers derive from algorithms (inspired by network theory) metrics that measure the number of capabilities in an economy, notably the Economic Complexity Index (ECI), argued to predict economic growth better than traditional variables such as human capital, and the country Fitness index. This paper offers an alternative economic complexity measure (founded on information theory) that derives from a simple model of production as a combinatorial process whereby a set of capabilities combine with some probability to transform raw materials into a product. A country’s number of capabilities is given by the logarithm of its product diversity, as predicts the model, which also predicts a linear dependence between log-diversity, ECI, and log-fitness. The model’s predictions fit the empirical data well; its informational interpretation, we argue, is a natural theoretical framework for the complexity view on economic development.

Keywords: economic growth, economic development, product diversification, economic complexity metrics, entropy, information theory
1 Background

1.1 Product Diversification Matters for Economic Development

Contrary to a long tradition in economics according to which international prosperity is achieved when national economies specialize, product diversity is strongly correlated with economic development [1-6]. The “richest” countries make almost all types of products, from the most rudimentary to the most sophisticated ones; while the “poorest” countries make comparatively fewer and more rudimentary products (Table 1).

Table 1. The World’s Most and Least Diversified Economies (2018, 4-digit HS).1

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<tr>
<td>Country</td>
<td>Diversification</td>
</tr>
<tr>
<td>United States</td>
<td>1224</td>
</tr>
<tr>
<td>China</td>
<td>1221</td>
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<tr>
<td>India</td>
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<td>Japan</td>
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<td>United Kingdom</td>
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<td>France</td>
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At the lowest extreme of the economic complexity spectrum are small-population islands, which mostly export natural products (naturally occurring goods: fruits, vegetables, fish).2 Then come economies specialized in highly demanded raw materials (notably oil); these countries have higher incomes despite their low

1 See data description and source in Subsection 4.1. The number of products a country makes depends of course on the product nomenclature used (usually 2, 4, 5, or 6-digit product codes), notably the SITC (Standard International Trade Classification) and the HS (Harmonized System). The results from the two product nomenclatures and at different aggregation levels are very similar; hence we present no systematic comparison of the results based on product nomenclature.

2 The islands are, e.g., Bouvet (BVT), Netherlands Antilles (ANT), Kiribati (KIR), Northern Mariana (MNP), Micronesia (FSM), Pitcairn (PCN), South Geogia & Sandwich (SGS), Tuvalu (TUV), Wallis & Futuna (WLF).
product diversity (Figure 1). All in all, about 80% of countries’ GDP ranking can be explained by the mere product-diversity ranking (Figure 1) if we put aside natural resources, which, as documented throughout, are the main source of bias in this purely qualitative view on production. (Islands, which expectedly would appear as even greater outliers then oil-experts, are not included in Figure 1, due to missing GDP data.)

Figure 1. Countries’ GDP versus Diversification Rankings (4-digit HS). Red countries are countries with exporters with natural-resource rents (averaged across years) at least 10% of their GDP.  

These associations are not just correlations and can be explained from a basic combinatorial model of production. The complexity of a country’s production (the diversity and sophistication of its products) reveals a diversity of productive knowledge in that economy that combine to make various products. Qualitatively,

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3 Throughout the straight lines are least-square fits.
4 This paper’s combinatorial model of production differs by its simplicity from earlier ones [7-9].
products differ precisely by the amount of knowhow involved in their production: in theory, the spectrum of this knowhow content of products ranges from zero, for naturally occurring goods (a natural resource sold in the raw, for example), to a maximum value when all the available knowhows are involved in the making of the product (consider an aircraft, for example). A product’s (technological) sophistication or complexity can be defined by the amount of knowledge its production requires; and the (technological) complexity of an economy, by the total amount of knowledge involved in its output. But what precisely is an “amount of productive knowledge” and how can we measure it?

1.2 Productive Knowledge in Conventional Growth Models

In conventional economic theory, a country’s productive knowledge is summarized by an aggregate production function GDP = F(Capital, Labor), or the aggregate output (or income) the country can produce from any combination of aggregate labor and capital, where the function F is homogenous of degree 1, so that income per capita is a function of the stock of capital per worker: GDP/Labor = F(Capital/Labor, 1). But as Solow’s seminal contribution established [10, 11], capital and labor accumulation cannot account for much of economic growth, which Solow explains in terms of exogenous shifts of the production function F through a multiplicative factor (denoted “A” and later named “Total Factor Productivity”), whose growth rate is interpreted as measuring technological progress. Much of later development of standard growth theory consisted of attaching theoretical substance and identity to the “Solow residual” (in other words, to “endogenize” the part of economic development not explained by the level of capital per worker), the consensus being that the residual somehow captures productive knowledge or “technology”, often identified with human capital (years of schooling), innovation, or research and development [12-15].

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5 Capital is traditionally denoted in economic theory as K, which we reserve for “knowledge”. 
1.3 The Complexity View on Economic Development

In contrast to the aggregate production function approach to production is the above-mentioned, data-driven, “complexity” approach to economic development inspired by the empirical correlation between economic development and product diversification. The diversity of a country’s production, as noted above, reflects the diversity of its capabilities (elemental units of productive knowhow), which combine to make more and more sophisticated products.

![Network Model of Production](Image)

Figure 2. Network Model of Production. Countries make products using capabilities: (A) The country-capability-product network. (B) The country-product network. Capabilities are not directly observable: How to count them?

Thus, it should be possible to infer the amount of productive knowledge involved in an economy from its product diversification data. Researchers framed this problem in terms of network theory, modeling countries’ productions as a tripartite network connecting countries to the products they make, products to the capabilities their production requires, and countries to the capabilities they possess (Figure 2). Thus, the core problem of the network approach to production was conceived as one of reconstructing the partly unobservable country-product-capability network from its empirically observed bipartite country-product projection \[16, 17\]. Researchers conceived algorithms to that effect, notably the
Economic Complexity Index (ECI), which, as the authors argue, predict economic growth better than traditional variables such as human capital [16, 17]. The ECI is jointly computed with the Product Complexity Index (PCI) by an algorithm akin to that which the web search engine Google uses to rank webpages.\textsuperscript{6} Another algorithm produces alternative country and product complexity measures [20], named country Fitness $(F)$ and product Quality $(Q)$. Both algorithms will be presented shortly. (Section 4 offers a step-by-step derivation of the metrics and the basic logic underlying them, for the reader not familiar with this literature.)\textsuperscript{7}

The primary data of the network view on production is formally, the country-product binary matrix $M=[M_{cp}]$ connecting countries to the products they make:

$M_{cp}=1$ if country $c$ makes product $p$, and $M_{cp}=0$, otherwise; this simple product list data is not available, however; thus, one takes as proxy for countries’ product lists, the countries’ export lists. (More on the data description in Subsection 4.1.) Given the matrix $M$, the product diversity of country $c$ (the number of its products) and the ubiquity of product $p$ (the number of its producers) are respectively:\textsuperscript{8}

$D_c = \sum_p M_{cp},$ \hspace{1cm} (1)

$U_p = \sum_c M_{cp},$ \hspace{1cm} (2)

The complexity metrics are (up to norming) the solutions to the equations:

$D_c ECI_c = \sum_p M_{cp} PCI_p,$ \hspace{1cm} (3)

$U_p PCI_p = \sum_c M_{cp} ECI_c,$ \hspace{1cm} (4)

$F_c = \sum_c M_{cp} Q_p,$ \hspace{1cm} (5)

$Q_p = [\sum_c M_{cp} F_c^{-1}]^{-1}.$ \hspace{1cm} (6)

\textsuperscript{6} More precisely the ECI-PCI algorithm is more similar in spirit to an algorithm developed by J. Kleinberg [18, 19] and used by Ask.com. It is an eigenvector problem, as one can see from (3)-(4).

\textsuperscript{7} The complexity metrics are analyzed in various studies, some of which offer critiques, alternatives, or refinements of the metrics, including the one presented here, in an earlier draft [21-27].

\textsuperscript{8} The natural concept is not ubiquity per se, but its inverse, which can be called product rarity.
2 Results and Discussions

2.1 Measuring an Economy’s Knowhow by Counting Its Products

A product is some transformed natural resources, some raw materials to which is applied a set of knowhows to turn them into an economically valuable outcome; and knowledge comes in discrete elementary units, or capabilities, that combine to make more and more sophisticated knowledge. The results presented in this paper derives from these definitions and two simple assumptions about the constraints on knowledge sophistication and raw-material availability:

1. Any $S$ capabilities can be put together to transform raw materials into a valuable product only with probability $\tau^S$ (uniform across countries and products).
2. A country finds the raw materials needed for making a product involving $S$ capabilities only with probability $\nu^S$ (uniform across countries and products).

The two assumptions imply that a product tends to appear in a country’s product list with a probability that decays exponentially with the product’s sophistication. Thus, a product’s sophistication can be measured by its log-likelihood of appearing:

$$S = -\frac{\log \text{prob(product)}}{\log(\tau \nu)}.$$  \hspace{1cm} (7)

Moreover, one can show (Subsection 4.2) that the total number of capabilities in an economy that makes $D$ products is given by the country’s log-product-diversity: \hspace{1cm}

$$K = \frac{\log(D)}{\log(1 + \tau \nu)}. \hspace{1cm} (8)$$

The model predicts the following relationships between knowhow, fitness, and ECI that fit the data well up to the bias related to natural products (Figure 3-Figure 4):

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9 The derivation is straightforward if we assume away the model’s two constraints (1)-(2); then a country possessing $K$ capabilities makes $D=2^K$ products, whose sophistication range from 0 (for unprocessed natural resources sold) to $K$. Thus, $K$ is given by $\log D$ (up to a scaling constant).

10 Notation: mean(X) denotes a cross-country average of X (average X across all countries); std(X) means the cross-country standard deviation of X; later, we will use mean(X|c) to mean the average of X in a country c; and in equation (16)-(17), we write mean(X|K) for the average of X in a country with $K$ capabilities. The metrics are systematically compared in standardized form (namely their z-scores) in the figures below, unless otherwise indicated by the scale of the plot.
ECI = \frac{\log D - \text{mean}(\log D)}{\text{std}(\log D)}.

\frac{\log F}{\text{mean}(\log F)} = \frac{\log D}{\text{mean}(\log D)}.

(9)

(10)
Figure 3. The Three Country Complexity Measures Related: Data versus Model. (Top panel: 4-digit SITC; Bottom Panel: 2-digit HS.)
Figure 4. Log-Diversity vs ECI (top) vs Log-Fitness (bottom). (2-digit HS)
More specifically (Subsection 4.5), the model predicts the following relationships between knowhow, diversity, complexity, sophistication, fitness, and quality, assuming the two algorithms (3)-(6) measure accurately these variables:

\begin{align*}
D &= (1 + \nu \tau)^K, \\
F &= 2^K F(0), \\
Q &= (\tau \nu)^{-S} Q(0), \\
ECI &= \frac{K - \text{mean}(K)}{\text{std}(K)}, \\
PCI &= \frac{S - \text{mean}(S)}{\text{std}(S)}. \\
\text{mean}(S \mid K) &= \left(\frac{\tau \nu}{1 + \nu \tau}\right) K, \\
\text{mean}(Q \mid K) &= \left(\frac{2}{1 + \nu \tau}\right)^K, 
\end{align*}

where \( F(0) \) and \( Q(0) \) are merely normalizing constants, and the notation \( \text{mean}(X \mid K) \) stands for the (conditional) average of \( X \) in a country with \( K \) capabilities. The last two predictions (16)-(17) are particularly nontrivial and fit equally well the empirical data (Figure 5). They offer a simple criterion for assessing the accuracy of each complexity algorithms.\(^\text{11} \) The F-Q metrics are better fit by the model because this algorithm better deals with the bias related to natural products, whose rarity is due to natural reasons and not to knowhow, and which are mostly exported by island countries, the least complex economies: it is indeed an essential aspect (and motivation) of the F-Q algorithm to emphasize the lowest-complexity countries in the estimation of \( Q_p \), namely (6). (See Subsection 4.3.) In contrast, island countries are major outliers of the ECI versus log-diversity theoretical fit [Figure 3, bottom panel, (a)].

\(^{11}\) In an earlier draft (arxiv.org, 2016) we suggested that it takes more than a regression between \( \log D \) and \( \log F \) to assess the accuracy of the F-Q algorithm, because fitness being the average product quality multiplied by diversity, a strong correlation between \( \log F \) and \( \log D \) might be a fortuitous one (that holds even with random data, as confirms Figure 5: Bottom Panel). The nontrivial prediction (17) is the right criterion in this respect.
Figure 5. Countries’ Productive Knowhow (Log-Diversity) Can be Measured by their Average PCI or by their Log Average Product Quality. (4-digit HS.)
The model’s two parameters come down effectively to one, the joint probability
\[ \pi = \pi \nu. \]  
(18)

Once this probability is known, all variables are determined (including the scales or norming constants). This parameter is simply related to the slope between log-diversity and log-fitness, by virtue of the predictions (11) and (12), which imply
\[ \log D = \frac{\log(1 + \pi)}{\log 2} \log F - \frac{\log(1 + \pi)}{\log 2} \log F(0). \]  
(19)

Thus, we can estimate the model’s key probability parameter \( \pi \) and the norming constant \( F(0) \) through linear regression, which yields:\textsuperscript{12}
\[ \frac{\text{cov}(\log D, \log F)}{\text{var}(\log F)} = 2 \left( 1 - \frac{1}{2} \right), \]
(20)
\[ F(0) = \exp\left( \text{mean}(\log F) - \left[ \log(1 + \pi) \right]^{-1} \text{mean}(\log D) \log 2 \right). \]  
(21)

The number of capabilities in each country is then estimated through either one of the three economic complexity measures:
\[ K = \frac{\log D}{\log(1 + \pi)} = \frac{\log[F/F(0)]}{\log 2} = \text{mean}[\log D] + \text{std}[\log D] \text{ECI}. \]  
(22)

The probability \( \pi \) and the spread of the distribution of \( K \) (notably the maximum knowhow \( K_{\text{max}} \)) depends on the product nomenclature (Figure 6).

\textsuperscript{12} If one can manage to find \( F(0) \) explicitly in terms of the norming constants involved in the F-Q algorithm (Section 4.3), then one can sharpen the regression equation by regressing \( \log D \) on \( \log(F/F(0)) \), and split the error (or residual) term into a mean term, which would measure the bias due to raw products, and a pure noise term.
Figure 6. World Distribution of Country Knowhow. [HS: 2-digit (Top) versus 4-digit product (Bottom) categories.]
2.2 Measuring a Product’s Sophistication by Harmonically Counting Its Producers

The model’s two assumptions also amount effectively to one: the probability that a country $c$ makes a product $p$, which we write simply $\text{prob}(c \mid p)$, decays exponentially with the product’s sophistication $S_p$. That is:

$$\text{prob}(p \mid c) = \frac{M_{cp}}{D_c} = \pi^{S_p}. \quad (23)$$

The (unconditional) probability of finding a product $p$ with sophistication $S_p$ in the world economy (all countries combined) is (by the law of total probability):

$$\text{prob}(p) = \sum_{c=1}^{C} \text{prob}(p \mid c) \text{prob}(c) = \sum_{c=1}^{C} \text{prob}(p \mid c) \frac{1}{C}. \quad (23)$$

Given (23), we get

$$\text{prob}(p) = \pi^{S_p} = \frac{1}{C} \sum_{c=1}^{C} \frac{M_{cp}}{D_c}. \quad (24)$$

After rearranging the terms, we get:

$$\pi^{-S_p} = C \frac{1}{\sum_{c=1}^{C} \frac{M_{cp}}{D_c}}. \quad (25)$$

Thus, the model predicts that product sophistication is (up to norming) given by the formula:

$$S_p = \frac{1}{\log \frac{1}{\pi}} \log[\text{mean}_H \{M_{cp}D_c\} \frac{C}{U_p}], \quad (26)$$

where $\text{mean}_H$ stands for (cross-country) harmonic mean. By the same token, we have also established the following correspondence (up to norming):

$$(F, Q) \leftrightarrow (D, \pi^{-S_p}). \quad (27)$$

That is, if we replace fitness by diversity in the F-Q algorithm, then this latter yields $Q_p = Q(0)\pi^{-S_p}$. The three product complexity measures ($S$, PCI, and LogQ) are strongly associated (Figure 7).
Figure 7. The three product sophistication measures related. (Top panel: 2-digit HS; bottom panel: 4-digit HS: outlier products are mostly raw products.)
2.3 Diversity and Complexity are Orthogonal, but Not Independent

This subsection is an interlude suggested by a growing interpretation concerning the orthogonality between complexity and diversity, a misapprehension that seems to call into question the very dependence between these concepts (and hence everything we said so far). Diversity and complexity are related notions intuitively. The model predicts that product diversity is an exponential function of economic complexity. One can show mathematically that the dependence cannot be a linear one anyway, at least if complexity is measured by the ECI, or more precisely by its non-standardized version (an eigenvector associated with the country-product network: Subsection 4.2), which we denote $k_2$ below. This follows from a basic yet elegant mathematical result [23], establishing orthogonality of $k_2$ and the product diversity vector, which we denote $d = [D_c]$. By symmetry one can similarly establish orthogonality between product ubiquity $u = [U_p]$ and the non-standardized PCI, which we denote $s_2$.

Contrary to a spreading interpretation [23, 24, 27], however, orthogonality between the two vectors, be it reminded, merely implies that the two vectors are not linearly dependent (where linearity is to be taken in the strict mathematical sense, which excludes affine dependence, or inclusion of an intercept). Orthogonality of complexity and diversity is not incompatible with positive dependence between the two vectors: to the contrary, the orthogonality combined with the positive dependence merely put a constraint on average complexity. Thus $k_2 d = 0$ and $s_2 u = 0$ combined with $\text{cov}(k_2, d) > 0$ and $\text{cov}(s_2, u) < 0$ (which by now should be taken as well-established both empirically and theoretically) simply imply that

$$\text{mean}(k_2) < 0,$$

(28)

$$\text{mean}(s_2) > 0.$$

(29)
Since the orthogonality is true mathematically, the sign condition (28)-(29) simply reflects the dependence between complexity and diversity, which is strong as we already know, and which the sign conditions confirm (Figure 8).

Figure 8. Distribution of the (non-standardized) ECI and PCI. Top: Country Complexity. Bottom: Product Complexity. (2-digit HS).
3 Conclusion: Towards an Information Theory of Economic Growth

The results, in the final analysis, suggest that both the empirical data and the algorithms of the economic complexity literature can be simply rationalized by a combinatorial model of production in which bits of knowhow (or capabilities) combine, with some probability \( \pi \), to make more and more sophisticated knowhow. For simplicity we treated the model’s (effectively unique) parameter \( \pi \) as uniform across countries and across products; it is more accurate, however, to assume some cross-country and cross-product variability of \( \pi \) to account for the bias due to raw products (or natural resources).

Fundamentally, the model rests entirely on the assumption that knowledge comes in discrete units and that it expands combinatorially. This suggests a simple informational interpretation of the model that seems to be the natural language for the complexity view on economic development more generally. For our limited purpose here, information theory can be summarized by an informational interpretation of Boltzmann’s famous entropy formula:\(^{13}\)

\[
\text{Information Content of System} = \log \left( \text{Effective Number of Basic Information States} \right).
\]

That is, the amount of information that conveys an information source (a system to whose states can be associated meaning) is measured by a logarithm of the system’s effective number of basic states (those states that can carry information). A basic information state can be the realization of an event, for example: the information that conveys the realization of an event is then measured by the log-inverse-probability of the event, and the total information conveyed by the whole system (here a

\(^{13}\) Information theory becomes more intuitive (compared to its usual formulation based on probability as a primitive concept) if the combinatorial foundation of information is more explicitly emphasized, or even taken as the primitive concept, as seems to suggest Kolmogorov [28]. If indeed the basic nature of information is that it comes in discrete units and that it expands combinatorially (or exponentially, in the simplest case), then it is natural to measure the amount of information of an information system by the logarithm of its effective number of information states.
probability space) is obtained by averaging the information contents conveyed by the basic events (Shannon’s entropy formula); if the events are equally likely, then the overall information content is measured by the logarithm of the total number of possible events. Indeed Shannon’s entropy formula [29] can be viewed as a special case of the general information formula: in this sense, Shannon formula defines the effective number of information states to be the exponential of Shannon’s entropy.

Think of a country as an information source revealing information about the knowledge content of the products it makes; and think of the country’s products as events or messages that reveal information about the country’s (unobserved) productive knowledge endowment. Thus, the information (or knowledge) content of a country’s output is measured by the country’s log-product-diversity. Similarly, think of the set of all countries potentially producing a product as the information source; then each producer reveals partial information about the knowledge content (or sophistication) of the product: the product’s knowledge content is then obtained as an average of the partial information revealed by the producers, and is roughly measured by the product’s log-frequency (or log-ubiquity) among countries. However, this is only a rough measure that implicitly assumes uniform probability of basic states or events: hence the need for a generalized (or effective) diversity and ubiquity measures, theoretically given by the model’s predictions (8) and (26). The complexity algorithms, on the other hand, can be viewed as an empirical way of correcting diversity and ubiquity mutually, since a country’s knowhow is reflected in the products it makes, and vice versa.

A systematic analysis of the economic implications of the information theory of economic development is beyond this paper’s scope, since we choose to center the discussion entirely on the purely qualitative dimension of production (where the question is whether a country can make a product or not): more generally, a country can be considered to be rich either because of its productive knowhow (as reflected in its product diversity) or by the intensity of its production (or the average amount
of output the country is able to sell: the quantitative aspect of production determined by shorter-term factors such as demand).\textsuperscript{14}

\textsuperscript{14} A discussion of the economic implications of the model is postponed to a follow-up work, which contains a development accounting in terms of the two dimensions of production (diversity versus intensity of output), sketched in an earlier draft (arxiv.org, 2016), but that expanded in subtlety.
4 Method: Data and Model

4.1 The Data

In principle, the complexity view on growth requires very simple data (for each country, the list of products it makes), which are not yet available, however; hence one takes as proxy for countries’ product lists, their export lists. While there will inevitably be some error in centering the analysis on export data (for lack of detailed data on production), the bias has proved minor a posteriori, given the accuracy of the results (apparently, a country’s export mix is representative of its total output’s composition). The results presented throughout this paper are based on the proxy matrix:

\[
M_{cp} = \begin{cases}
1 & \text{if } X_{cp} > 0, \\
0 & \text{if } X_{cp} = 0,
\end{cases}
\]  

(30)

where \(X_{cp}\) is the amount country \(c\) exported in product \(p\), using the Comtrade data in HS (revision 2007), available for the years 1995-2018 [30].\(^{15}\) We also use for comparison the Comtrade data in SITC (revision 2) as compiled and corrected for mistakes by Feenstra et al. and available for the years 1962-2000 [31].

Unlike in this paper, the standard practice in the economic complexity literature is to define the \(M_{cp}\) matrix more restrictively as

\[
M_{cp} = \begin{cases}
1, & \text{if } RCA_{cp} > 1, \\
0, & \text{if } RCA_{cp} < 1,
\end{cases}
\]  

(31)

where \(RCA_{cp}\) is the revealed comparative advantage of a country \(c\) in product \(p\) and is defined as \(RCA_{cp} = \left( \frac{X_{cp}}{\sum_p X_{cp}} \right) / \left( \frac{\sum_c X_{cp}}{\sum_p \sum_c X_{cp}} \right) \).

\(^{15}\) The trade data are accessible through the Atlas of Economic Complexity Dataverse (Harvard University): [https://dataverse.harvard.edu/dataverse/atlas](https://dataverse.harvard.edu/dataverse/atlas). The income data are countries’ GDP in PPP (purchasing power parity) from the Penn World Table (PWT8); we use the RGDP variable (an output-oriented GDP estimate), though the other measures give very similar results. The PWT is accessible through the GGDC (Groningen Growth and Development Centre, University of Groningen): [https://www.rug.nl/ggdc/productivity/pwt/](https://www.rug.nl/ggdc/productivity/pwt/).
4.2 The ECI-PCI Algorithm

The ECI-PCI algorithm [16, 17] assumes that an economy’s knowhow is proportional to the average knowledge content of its products, and, vice versa, a product’s knowledge content is proportional to the average knowhow of its producers. Thus, if $k_c$ measures the amount of knowhow in country $c$, and $s_p$, the knowledge content of product $p$, then

$$K_c = \alpha \sum_p W_{cp} s_p,$$

$$s_p = \beta \sum_p W_{pc} k_c,$$

where $\alpha$ and $\beta$ are positive normalizing constants, and the weights

$$W_{cp} = \frac{M_{cp}}{\sum_p M_{cp}},$$

$$W_{pc} = \frac{M_{pc}}{\sum_p M_{pc}},$$

Collecting the variables and weights into the vectors and matrices $k = [K_c]$, $s = [s_p]$, $W = [W_{cp}]$, and $W' = [W_{pc}]$, (32) and (33) become $k = \alpha W s$ and $s = \beta W' k$. So we get

$$(W W') k = (\alpha \beta)^{-1} k.$$  \hfill (36)

$$(W' W) s = (\alpha \beta)^{-1} s.$$  \hfill (37)

That is, the complexities of countries and products are given by eigenvectors of the matrices $WW'$ and $W'W$, respectively, where the associated eigenvalue is $(\alpha \beta)^{-1}$. Because the averaging weights sum to 1, it is easy to see that any (positive) uniform vectors $k = [K, \ldots, K]'$ and $s = [S, \ldots, S]'$ are solutions to this eigenvector problem; these are the eigenvectors associated with the largest eigenvalue, which is 1 (by a known linear algebra result, the Perron-Frobenius theorem). Thus, the authors of this algorithm choose the eigenvectors associated with the second largest eigenvalue. Let $k_2$ and $s_2$ be these eigenvectors: then ECI and PCI are (up to the sign) the elements of the chosen eigenvectors given in standardized form:
ECI = +\text{sign}[\text{corr}(k_2, d)] \frac{k_2 - \text{mean}(k_2)}{\text{std}(k_2)}, \quad (38)

PCI = -\text{sign}[\text{corr}(s_2, u)] \frac{s_2 - \text{mean}(s_2)}{\text{std}(s_2)}. \quad (39)

We multiply by the signed correlation of the eigenvectors with country diversification vector $d$ and product ubiquity vector $u$, respectively, to ensure the signs are correct; this is simply because the sense of an eigenvector being arbitrary, the standardization specifies the metrics only up to the sign: for example, any chosen eigenvector $k$ is equivalent to any nonzero multiples $\gamma k$, so that

$$\frac{\gamma k - \text{mean}(\gamma k)}{\text{std}(\gamma k)} = \frac{\gamma (k - \text{mean}(k))}{|\gamma| \text{std}(k)}. \quad (40)$$

4.3 The Fitness-Quality Algorithm

In essence, this algorithm [20] measures the complexity of an economy by the total complexity of its products; and the complexity of a product, by the product’s inverse ubiquity, multiplied by the harmonic mean of the complexities of the producers. That is, the two metrics are jointly computed recursively as follows:

$$F^{(n+1)}_c = \frac{1}{\text{mean}(Q_p^{(n)})} \sum_p M_{cy} Q_p^{(n)}, \quad (41)$$

$$Q_p^{(n+1)} = \frac{1}{\text{mean}(F_c^{(n)})} \sum_c \frac{1}{M_{cy}} \frac{1}{F_c^{(n)}}. \quad (42)$$

The means are averages across all countries and all products, respectively, and the initial conditions are unit complexities for all countries and all products. The algorithm converges to a fix-point $(F^{(\infty)}, Q^{(\infty)})$, which, in normalized form, define the country Fitness and product Quality indices:

$$F = \frac{F^{(\infty)}}{\text{mean}(F^{(\infty)})}, \quad (43)$$

$$Q = \frac{Q^{(\infty)}}{\text{mean}(Q^{(\infty)})}. \quad (44)$$
The crucial novelty of this algorithm is the following ingenious observation: if a low-complexity country is among the producers of a product, this product is necessarily a low-sophistication product; but to know that a highly complex economy is among the producers of a product barely reveals any information about the product’s complexity (since such country makes almost all product types). Thus, highly complex economies should be discounted in the measure of product complexity, dominated by the more informative, lowest-complexity, producer: this is precisely what does the harmonic mean, whose following bounds are known:

\[ \min\{M_{cpc} \} \leq \text{mean}_p(M_{cpC}) \leq U_p \min\{M_{cpC} \}. \]

We know from the theoretical model why the harmonic mean is the natural choice.

4.4 The Model

For short, we refer to an $S$-sophisticated knowhow, an $S$-sophisticated product, and a $K$-sophisticated economy respectively as $S$-knowhow, $S$-product, and $K$-country.

We combine the model’s two assumptions into one:

_Assumption: Any $S$ random combination of knowhows among a country’s knowhow list corresponds to a product with probability $\pi^S$. _

That is, a $K$-country can explore up to \( \binom{K}{S} \) possible $S$-collections of skillsets, among which only a proportion given by $\tau^S$ are coherent productive skillsets. Thus, a $K$-country makes $\binom{K}{S}\pi^S$ $S$-products, and, in total, it makes a total number of products:

\[ D = \sum_{S=0}^{K} \binom{K}{S} \pi^S = (1 + \pi)^K. \] (45)

4.5 Model’s Predictions about the Complexity Algorithms

4.5.1 Model’s Prediction about ECI

As usual we index an empirical country and product by $c$ and $p$, and we index the theoretical counterparts by $K$ and $S$, respectively, and refer to them as $K$-country and

\[ \text{(See https://en.wikipedia.org/wiki/Harmonic_mean).} \]
S-product. A K-country makes \((1 + \pi)^K\) products among which \((\frac{\pi}{1 + \pi})^S\) are S-products. Thus, the distribution of product sophistication in a K-country is

\[
\text{prob}(S \mid K) = \frac{(\frac{\pi}{1 + \pi})^S}{(1 + \pi)^K}, \quad S = 0, \ldots, K.
\] (46)

The average product sophistication in a K-country is:

\[
\mathbb{E}(S \mid K) = \sum_{S=0}^{K} S \text{prob}(S \mid K) \quad \text{(by definition)}
\]

\[
= D^{-1} \sum_{S=1}^{K} S (\frac{\pi}{1 + \pi})^S
\]

\[
= D^{-1} \sum_{S=1}^{K} S \frac{K}{S} \left(\frac{\pi}{1 + \pi}\right)^S \quad \text{(by a known identity)}
\]

\[
= D^{-1} \pi K \sum_{S=1}^{K} \left(\frac{\pi}{1 + \pi}\right)^{S-1}
\]

\[
= D^{-1} \pi K \sum_{N=0}^{K-1} \left(\frac{\pi}{1 + \pi}\right)^N \quad \text{(by setting } N = S - 1)\]

\[
= D^{-1} \pi K (1 + \pi)^{K-1}.
\]

That is,

\[
\mathbb{E}(S \mid K) = \frac{\pi}{1 + \pi} K.
\] (47)

This theoretical result justifies the measurement of a country’s output complexity by its average product complexity (up to scaling): it explains why ECI works as a measure of knowhow. We can check the extent to which the ECI-PCI algorithm does effectively estimate a country’s knowhow as follows. Let the estimated country’s complexity as measured by the ECI-PCI algorithm be written (up to norming) as

\[
K_c^{(2)} = \text{mean}(S^{(2)} \mid c),
\] (48)

where \(K_c^{(2)}\) is the entry of the country complexity eigenvector \(k_2\) and \(S^{(2)} \mid c\) is the restriction of the product complexity eigenvector \(s_2\) to the products made by country \(c\). If product complexity \(S\) is accurately measured by \(s_2\) (which it can do only up to the scale of measurement of sophistication, and an error term that should average out), then

\[
S^{(2)} = \text{constant} \times S + \text{error}.
\] (49)
And if in addition the combinatorial model of production is accurate, then the ECI-PCI algorithm yields an $ECI_{\epsilon}$ that is as an estimate of the theoretical counterpart

$$ECI = \frac{K - \mathbb{E}(K)}{\text{std}(K)} = \frac{\log D - \mathbb{E}(\log D)}{\text{std}(\log D)}.$$  

(50)

4.5.2 Model’s Prediction about Fitness

Under the model’s predicted correspondence $(F, Q) \leftrightarrow (D, \pi^S)$, we have:

$$Q(S) = Q(0)\pi^S.$$  

(51)

Thus $Q_p$ is an empirical version of the theoretical counterpart:

$$Q(S) = Q(0)\pi^S,$$  

(52)

The model also predicts that the average product quality in a $K$-country is

$$\mathbb{E}(Q \mid K) = Q(0)\sum_{s=0}^{K} \pi^S \text{prob}(S \mid K)$$

$$= D^{-1}Q(0)\sum_{s=0}^{K} \pi^S(\binom{K}{s})\pi^s.$$  

That is,

$$\mathbb{E}(Q \mid K) = D^{-1}Q(0)2^K = Q(0)(\frac{2}{1+\pi})^K.$$  

(53)

Therefore, the F-Q algorithm produces (up to norming) a fitness index

$$F_{\epsilon} = \text{mean}(M_{q_p}Q_p)D_{\epsilon}$$

which is an estimate of the theoretical counterpart

$$F = Q(0)2^K.$$  

(54)
References