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A Class of N-Player Colonel Blotto Games with Multidimensional Private Information

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Abstract. In this paper, we study N-player Colonel Blotto games with incomplete information about battlefield valuations. Such games arise in job markets, research and development, electoral competition, security analysis, and conflict resolution. For $M \geq N + 1$ battlefields, we identify a Bayes-Nash equilibrium in which the resource allocation to a given battlefield is strictly monotone in the valuation of that battlefield. We also explore extensions such as heterogeneous budgets, the case $M \leq N$, full-support type distributions, and network games.

Keywords. Colonel Blotto games · Private information · Bayes-Nash equilibrium · Generalized Dirichlet distributions · Networks

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1 Introduction

In a Colonel Blotto game, players simultaneously and independently allocate their endowments of a resource across a set of battlefields. The player that deploys the largest amount of the resource to a given battlefield scores a win and enjoys a gain in utility equivalent to her valuation of that battlefield. Thus, a player's utility corresponds to the sum of the valuations of all battlefields won by the player. Colonel Blotto games naturally arise in a large number of applied settings, such as in job markets, research and development, electoral competition, security analysis, and conflict resolution. Colonel Blotto games also have been among the first games seriously studied in the theoretical literature [7, 8, 9]. While the case of complete information is fairly well understood [21, 20, 25, 26, 18, 29], progress has been more limited in the case of incomplete information, with very few exceptions [1, 17, 13, 3, 15].

This paper studies N-player Colonel Blotto games with M battlefields and multi-dimensional incomplete information regarding battlefield valuations. We assume that valuation vectors are private information and independently distributed across players. Only the ex-ante distribution of valuation vectors is common knowledge. Each player maximizes the expected sum of valuations of battlefields won, where resource budgets are fixed and homogeneous across players, and where unused resources do not have any positive value. In the case where the number of battlefields strictly exceeds the number of players, i.e., for $M \geq N+1$, we identify a Bayes-Nash equilibrium in which any player's resource allocation to a battlefield is strictly monotone increasing in her valuation of that battlefield. The construction of equilibria for more than two players relies on a new distributional assumption. Specifically, we exploit the particular properties of generalized Dirichlet and Liouville distributions in finite-dimensional vector spaces equipped with a

p-norm.

We also explore several extensions. First, we touch upon the case of heterogeneous budgets. While a complete solution is beyond the scope of the present paper, we find new classes of Bayes-Nash equilibria. In one example, a player with a substantially larger budget outbids her opponent on her preferred $(M-1)$ battlefields, while the player with the smaller budget bids only on a single preferred battlefield. Next, we seek equilibria in the case excluded by our assumptions so far, i.e., for the case $M \leq N$. We find equilibria in the "crowded" case where the number of battlefields is sufficiently small compared to the number of players. These equilibria, in which all players bid on their preferred battlefield only, are shown to exist under a fairly flexible assumption on ex-ante type distributions. Third, we study distributions with full support, which allows us to extend existing results. Fourth and finally, we discuss network games in which players may be active only in a subset of all battlefields.

While the Colonel Blotto game has a certain similarity with a single-unit all-pay auction [30, 12, 5, 6, 16], our analysis draws especially on three prior contributions. Kovenock and Roberson [17] presented an example with two players and three battlefields. Private valuations of battlefields are drawn independently from a uniform distribution over a two-dimensional surface in Euclidean space. Since, in that case, marginal type distributions are uniform, the budget constraint may be kept by bidding the squared valuation on each battlefield. It turns out that this strategy constitutes a symmetric Bayes-Nash equilibrium. Hortala-Vallve [13] solved the case $N = M = 2$, where bidding exclusively on one of the highest-valuation battlefields is a weakly dominant strategy. Akyol [3] noted that rescaling a valuation vector by a positive factor does not affect a player's best response set. He offered an extension to any number of battlefields by assuming that individual battlefield valuations follow a generalized gamma distribution. However, he still focused on the case of two players, which may be restrictive, e.g., in a job market environment. The analysis of the present paper subsumes all results obtained in prior work. Moreover, we construct equilibria with more than two players, where we use novel distributional assumptions to deal with the case $M \geq N + 1$. Thus, the present paper goes beyond existing work by considering a wider class of examples of multi-player Colonel Blotto games with incomplete information about valuations.

There are also a number of less closely related papers. In a model with N players and private information about budgets, Adamo and Matros [1] identified a symmetric monotone Bayes-Nash equilibrium. A higher budget allows a player to scale up her resource allocation, while the share of the resource allocated to individual battlefields remains constant. This leads to a tractable one-dimensional problem. Powell [23] studied a signaling game with private information about vulnerability. Next, in a model of price setting with menu costs for multiproduct firms, Alvarez and Lippi [4] made use of the marginals of a uniform distribution on a higher-dimensional Euclidean sphere that represents a vector of price changes. They, however, studied the problem of a monopolist, i.e., there is no Colonel Blotto game. Tang and Zhang [28] considered mixed extensions of normal-form games where mixed strategies correspond to points on a Euclidean sphere. Paarporn et al. [22] assumed one-sided incomplete information in a Colonel Blotto game with a finite state space. In our discussion of generalized Dirichlet and Liouville distributions, we follow Hashorva et al. [11] and Song and Gupta [27]. See also Richter [24] and Ahmadi-Javid and Moeini [2]. Gupta and Richards [10] offer an insightful historical account of Dirichlet and Liouville distributions.

The rest of this paper is structured as follows. Section 2 introduces the model. Section 3 presents the main result. Extensions are discussed in Section 4. Section 5 concludes. An Appendix offers formal detail omitted from the body of the paper.

2 The model

2.1 Set-up and notation

There are $N \geq 2$ risk-neutral players, denoted by $i \in \{1, \ldots, N\}$, and $M \geq 2$ battlefields, denoted by $j \in \{1, \ldots, M\}$. Each player is endowed with an identical budget of a perfectly divisible resource. For convenience, we normalize budgets to one. A player's *resource allocation* is a vector

$$\mathbf{b} = (b_1, \ldots, b_M),$$

where $b_j \geq 0$ denotes the amount of the resource allocated to battlefield j. We call a resource allocation $\mathbf{b} = (b_1, \ldots, b_M)$ *feasible* if

$$\textstyle\sum_{j=1}^{M} b_j \leq 1.$$

Denote by $\mathcal{B} = \mathcal{B}^M$ the set of feasible resource allocations over M battlefields.

Before deciding about the resource allocation, each player privately learns her respective vector of battlefield valuations,

$$\mathbf{v} = (v_1, v_2, \ldots, v_M).$$

The vector $\mathbf{v}$ is commonly known to be drawn, independently across players, from a given probability measure μ on (the Borel subsets of) $\mathbb{R}_+^M$, where $\mathbb{R}_+ = [0, \infty)$. Let $\mathcal{V}$ denote the support of μ. Specific assumptions on μ and $\mathcal{V}$ will be imposed in the statements of the subsequent results.

A *strategy* is a (measurable) mapping $\beta : \mathcal{V} \to \mathcal{B}$. When adhering to strategy β, type $\mathbf{v}$'s resource allocation is

$$\beta(\mathbf{v}) = (\beta_1(\mathbf{v}), \ldots, \beta_M(\mathbf{v})) \in \mathcal{B}.$$

Any strategy of an opponent induces a probability measure over feasible resource allocations. Therefore, given strategies for the $(N-1)$ opponents, type $\mathbf{v}$'s resource allocation translates into a vector of winning probabilities, and hence, into an expected payoff for type $\mathbf{v}$.

The N players simultaneously and independently choose feasible resource allocations. In each battlefield, the player that allocates the largest amount of the resource wins. In the case of a tie in battlefield j, each of the players that allocated the largest amount of the resource to battlefield j wins in that battlefield with equal probability. Each player's payoff equals the sum of her valuations of the battlefields won.

A strategy β^* will be referred to as a *symmetric Bayes-Nash equilibrium strategy* if, for any type realization $\mathbf{v} \in \mathcal{V}$, the resource allocation $\beta^*(\mathbf{v})$ maximizes the expected payoff of type $\mathbf{v}$ under the assumption that the other $(N-1)$ players individually adhere to strategy β^*.

2.2 Heuristic discussion of the player's problem

Suppose that all opponents of Player 1 adhere to strategy $\beta = \beta(\mathbf{v})$. Then, the marginal distribution of bids on each battlefield is identical across players $i \in \{2, \ldots, N\}$. We denote the distribution function of this common probability distribution by $G(b_j) = \Pr(\beta_j(\mathbf{v}) \leq b_j)$. Provided there are no mass points in G, type $\mathbf{v}$'s problem reads

$$\max_{(b_1,\ldots,b_M)\in\mathcal{B}} \sum_{j\in\{1,\ldots,M\}} F(b_j)v_j, \tag{1}$$

where, by independence of types across players, the cumulative distribution function of the highest bid is given as

$$F(b_j) = G(b_j)^{N-1}.$$

To grasp the nature of the problem, suppose that the solution to problem (1) is interior and characterized by first-order conditions, and that F is continuously differentiable in an open neighborhood of the optimal bid b_j, for $j \in \{1, \ldots, M\}$, with a strictly declining derivative f. Then, the optimal allocation $\beta(\mathbf{v})$ satisfies

$$f(\beta_j(\mathbf{v}))v_j - \lambda(\mathbf{v}) = 0,$$

for $j \in \{1, \ldots, M\}$, where $\lambda(\mathbf{v})$ is the Lagrange parameter of the budget constraint. Thus, provided that $v_j > 0$, Player 1's best response is given by

$$\beta_j^{\mathrm{br}}(\mathbf{v}) = f^{-1}\left(\tfrac{\lambda(\mathbf{v})}{v_j}\right),$$

where f^{-1} denotes the inverse of f, and $\lambda(\mathbf{v})$ is implicitly characterized by

$$f^{-1}\left(\tfrac{\lambda(\mathbf{v})}{v_1}\right) + \ldots + f^{-1}\left(\tfrac{\lambda(\mathbf{v})}{v_M}\right) = 1.$$

To solve for a symmetric Bayesian equilibrium strategy means identifying a bid function β such that $\beta^{\mathrm{br}} = \beta$. Even under the simplifying assumptions imposed above, the general solution to this problem is not known. E.g., Hortala-Vallve and Llorente-Saguer [14] assumed $N = 2$ and $M \in \{2, 3, 6\}$, with valuation vectors drawn from a uniform distribution on a discrete simplex. While they present an analytic solution for the case $M = 2$ (see the next section), they resorted to numerical methods in the cases $M = 3$ and $M = 6$.

2.3 Examples

We illustrate the set-up with the help of some examples.

Example 1 (Kovenock and Roberson [17]). *Suppose that μ is the uniform distribution on the sphere segment*

$$\mathcal{V} = \{\mathbf{v} \in \mathbb{R}_+^3 : (v_1)^2 + (v_2)^2 + (v_3)^2 = 1\}.$$

Then,

$$\beta^*(\mathbf{v}) = ((v_1)^2, (v_2)^2, (v_3)^2)$$

is a symmetric Bayes-Nash equilibrium strategy.

Example 2 (Hortala-Vallve [13]; Hortala-Vallve and Llorente-Saguer [14]). *Suppose that $N = M = 2$. Then, for any $\varkappa \in [0,1]$,*

$$\beta^*(\mathbf{v}) = \begin{cases} (1,0) & \text{if } v_1 > v_2 \\ (\varkappa, 1-\varkappa) & \text{if } v_1 = v_2 \\ (0,1) & \text{if } v_1 < v_2 \end{cases}$$

is weakly dominant, and hence, forms a symmetric Bayes-Nash equilibrium strategy for any type distribution μ on $\mathbb{R}^2_+$.

Example 3 (Akyol [3]). *Suppose that $N = 2$, $M \geq 3$, and that μ is a generalized Gamma distribution on $\mathbb{R}^M_+$, with componentwise independent density*

$$\phi(v) = \tfrac{M-1}{(M-2)\Gamma(1/(M-1))} \cdot v^{-\frac{M-3}{M-2}} \exp\left(-v^{\frac{M-1}{M-2}}\right).$$

Then,

$$\beta^*(\mathbf{v}) = \left(\frac{v_1^{\frac{M-1}{M-2}}}{\sum_{j=1}^M v_j^{\frac{M-1}{M-2}}}, \ldots, \frac{v_M^{\frac{M-1}{M-2}}}{\sum_{j=1}^M v_j^{\frac{M-1}{M-2}}} \right)$$

is a symmetric Bayes-Nash equilibrium strategy.

3 The case of N players

3.1 Distributional assumptions

For $M \geq 1$ and $p \geq 1$, we equip $\mathbb{R}^M$ with the *p-norm*

$$\|\mathbf{y}\|_p = (|y_1|^p + \ldots + |y_M|^p)^{1/p}.$$

Within the resulting normed space, we consider the *sphere segment*

$$\mathcal{V}^{M,p} = \{\mathbf{v} \in \mathbb{R}^M_+ : \|\mathbf{v}\|_p = 1\}$$

of vectors of p-norm one in $\mathbb{R}^M_+$. The set $\mathcal{V}^{M,p}$ is a bordered $(M-1)$-dimensional manifold embedded in $\mathbb{R}^M$. Figure 1 illustrates this fact for $M = p = 3$.

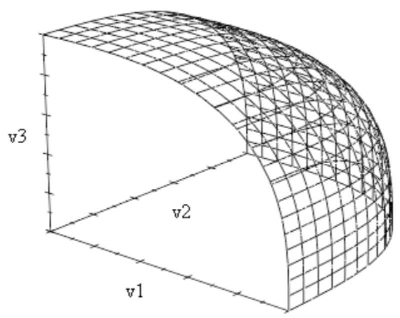

Figure 1. The sphere segment $\mathcal{V}^{M,p}$

To specify a probability measure with support $\mathcal{V}^{M,p}$, we parameterize the manifold using the first $(M-1)$ variables.

Definition 1 (Hashorva et al. [11]). *The p-**norm Dirichlet distribution** with parameter $\alpha > 0$ is defined by the density*

$$\psi(v_1, \dots, v_{M-1}) = \frac{p^{M-1}\Gamma(M\alpha)}{\Gamma(\alpha)^M} \left(1 - \sum_{j=1}^{M-1} v_j^p\right)^{\alpha-1} \cdot \prod_{j=1}^{M-1} v_j^{p\alpha-1},$$

where $v_1, \dots, v_{M-1} \in (0,1)$ such that $\sum_{j=1}^{M-1} v_j^p < 1$.

In the special case $p = 1$, Definition 1 characterizes the classic Dirichlet distribution on the simplex of dimension $(M-1)$. For general p, the distribution is derived from the Dirichlet distribution on the simplex by taking each component of the random vector to the power of $1/p$. The distribution characterized by Definition 1 is, therefore, invariant under arbitrary permutations of the players. Moreover, random variables following this distribution are easy to construct numerically [11]. For $\alpha p = 1$, the p-norm Dirichlet distribution with parameter $\alpha > 0$ corresponds to the uniform distribution on the sphere segment [27].

3.2 Statement of the main result

The main result of the present paper is the following.

Proposition 1. *Suppose that $M \geq N + 1$, and that each player's vector of battlefield valuations is drawn independently from a p-norm Dirichlet distribution with parameter α, where $p = \frac{M-1}{M-N}$ and $\alpha = \frac{1}{M-1}$. Then, the bid strategy defined through*

$$\beta^*(\mathbf{v}) = ({v_1}^p, \ldots, {v_M}^p),$$

is a symmetric Bayes-Nash equilibrium strategy.

Proof. See the next section. □

Proposition 1 extends existing equilibrium characterizations for Colonel Blotto games with incomplete information about valuations. In particular, Example 1 is contained as a special case where $N = 2$ and $M = 3$. Extensions covering Examples 2 and 3 will be presented later in the paper.

3.3 Proof of Proposition 1

Suppose that each player $i \in \{2, \ldots, N\}$ adheres to strategy β^*. Then, each type $\mathbf{v}$ bids $b_j = v_j^p$. By assumption, $(M-1)\alpha = 1$. Hence, by Lemma A.1 in the Appendix, the marginal distribution of valuations on any battlefield is a power function distribution with density $h\,(v) = p\alpha v^{p\alpha-1}$ for $v \in (0, 1)$, and cumulative distribution function $H(v) = v^{p\alpha}$. Clearly, $G(b) = H(b^{1/p}) = b^\alpha$. Therefore, $F(b) = b^{\alpha(N-1)}$, with density

$$f(b) = \alpha(N-1)b^{\alpha(N-1)-1}.$$

The inverse of f is given by

$$f^{-1}(x) = \left(\frac{x}{\alpha(N-1)}\right)^{\frac{1}{\alpha(N-1)-1}}.$$

By assumption, $p = \frac{1}{1-\alpha(N-1)}$. As before, let $\lambda \equiv \lambda(\mathbf{v})$ denote the shadow cost of the budget constraint in Player 1's problem. Then, as discussed above,

$$b_j(\mathbf{v}) = f^{-1}\left(\tfrac{\lambda(\mathbf{v})}{v_j}\right) = \left(\tfrac{\alpha(N-1)}{\lambda(\mathbf{v})}\right)^p v_j^p.$$

Clearly, in an optimal allocation, no resources remain unused, i.e., $b_1 + \ldots + b_M = 1$. Hence,

$$\lambda(\mathbf{v}) = \alpha(N-1)\left(\textstyle\sum_{j=1}^M v_j^p\right)^{1/p} = \alpha(N-1).$$

Thus, it is indeed optimal for type $\mathbf{v}$ of Player 1 to allocate the resource as prescribed by the symmetric equilibrium strategy $\mathbf{b}^*$. Obviously, the same is true for players $i \in \{2, \ldots, N\}$. This concludes the proof of Proposition 1.

4 Extensions

This section discusses several ways to generalize the framework considered so far.

4.1 Heterogeneous budgets

In this section, we explore the case of heterogeneous budgets. This is strategically equivalent to assuming the same biased contest technology in all battlefields. While symmetry is lost, the set-up and equilibrium notion of Bayes-Nash equilibrium generalize in a straightforward way. A simple observation is the following.

Proposition 2. *Suppose that $N \geq 2$ and $M \geq 1$, and that Player 1's budget is more than M times as large as any other player's budget. Then, it is a weakly dominant strategy for Player 1 to distribute the resource evenly across all battlefields.*

Proof. Suppose that Player 1 splits her budget evenly across all battlefields $j \in \{1, \ldots, M\}$. Then, Player 1 wins battlefield j even if all of her opponents concentrate their entire budget on battlefield j. $\square$

The situation becomes more interesting if players' relative positions are less definite. As the complete analysis of this case goes beyond the scope of the present paper, we confine ourselves to the presentation of an example.

Example 4. *Suppose that $N = 2$ and $M \geq 2$. Suppose also that type distributions are symmetric across battlefields and give probability zero to valuation ties across battlefields. Suppose, finally, that the budget of Player 1 is $X \in (M - \frac{1}{2}, M]$, while the budget of Player 2 is one. Then, the following is a Bayes-Nash equilibrium. For $\varepsilon > 0$ small, Player 1 places bids of $(1 + \varepsilon)$ on $(M - 1)$ battlefields for which she holds the highest valuations, and the remainder $X - (1 + \varepsilon)(M - 1) > \frac{1}{2}$ on the battlefield which she values least. Player 2 bids one on the battlefield that she values most.*

The equilibrium property is easy to check. Sticking to her strategy, Player 1 certainly wins the $(M - 1)$ battlefields for which she holds the highest valuations. Moreover, with probably $\frac{M-1}{M}$, she also wins the remaining battlefield. This is optimal even in the borderline case $X = M$. Indeed, in that case, splitting the budget evenly would win $(M - 1)$ randomly selected battlefields with probability one, and the remaining battlefield with probability $\frac{1}{2} \leq \frac{M-1}{M}$. Thus, Player 1's strategy is a best response. Player 2 is unable to win two battlefields, but may win one battlefield with probability $\frac{1}{M}$. Therefore, also Player 2's strategy is optimal, and the strategy profile described above is indeed a Bayes-Nash equilibrium.

4.2 The case $M \leq N$

Next, we discuss the case where the number of players is weakly larger than the number of battlefields.

Proposition 3. *Suppose that $N \geq 2$ and that*

$$M \leq M^*(N) \equiv \frac{1}{1 - (1/N)^{1/(N-1)}}.$$

Suppose also that each player's vector of valuations is distributed symmetrically across battlefields (but not necessarily across players), and gives probability zero to valuation ties across battlefields. Then, bidding one on any of the highest-valuation battlefields is a symmetric Bayes-Nash equilibrium strategy. Conversely, if $M > M^*(N)$, then there exists a distribution of types such that bidding exclusively on a highest-valuation battlefield does not constitute a symmetric Bayes-Nash equilibrium strategy.

Proof. See the Appendix. □

Proposition 3 gives a sharp threshold such that bidding exclusively on a highest-valuation battlefield constitutes a symmetric Bayes-Nash equilibrium strategy. Lemma A.2(i) in the Appendix shows that, for $N \geq 2$, the upper bound $M^*(N)$ satisfies $M^*(N) \leq N$, which justifies the heading of this section. Compared to Proposition 1, the distributional assumptions in Proposition 3 are more flexible. Indeed, the only requirement is that each player's type distribution is symmetric across battlefields and gives probability zero to valuation ties across battlefields.

Adamo and Matros ([1], Cor. 1) found that, in their model with incomplete information about budgets, all players compete for all prizes. Proposition 3 shows that this conclusion does not hold, in general, under incomplete information about valuations.

The case $N = M = 2$ deserves some attention. In Example 2, which does not even impose any restrictions on type distributions, we have seen that bidding one on the highest-valuation battlefield, or dividing the budget in the case of valuations being equal across battlefields, is a weakly dominant strategy [14, 13]. The reason is that each player wins precisely one battlefield in expectation, regardless of the strategy chosen. Hence, a player never "regrets" having placed a positive bid on either of two identically-valued battlefields. However, as the following example illustrates, the property of weak dominance does not generalize to the case of more than two players.

Example 5. *Suppose that $N \geq 3$ and $M \geq 2$. Suppose that Player 1's type $\mathbf{v} = (v_1, \ldots, v_M)$ satisfies $v_1 > 0$ and $v_2 \in (\frac{1}{N-1}v_1, v_1)$. Suppose also that players $j = 2, \ldots, N$ all bid one on Battlefield 1. Then, Player 1's expected payoff from bidding exclusively on Battlefield 1 is $\frac{1}{N} \cdot (v_1 + \ldots + v_M)$, whereas the expected payoff from bidding exclusively on Battlefield 2 is strictly higher, viz. $v_2 + \frac{1}{N} \cdot (v_3 + \ldots + v_M)$.*

Thus, while bidding exclusively on one of the highest-valuation battlefields remains a Bayes-Nash equilibrium under the assumption of Proposition 3, the property of weak dominance breaks down once there are more than two players.

Next, we study what happens in "crowded" Colonel Blotto games, i.e., if the number of players N is much larger than the number of battlefields M. We show in the Appendix that $M^*(N) \to \infty$ as N grows above all bounds. Therefore, the assumptions of Proposition 3 may be satisfied for any given number of battlefields $M \geq 2$. We arrive at the following observation.

Corollary 1. *Let the number of battlefields $M \geq 2$ be fixed. Suppose also that each player's vector of battlefield valuations is symmetric across battlefields, giving probability* *zero to valuation ties. Then, for any sufficiently large N, bidding exclusively on any of the highest-valuation battlefields is a symmetric Bayes-Nash equilibrium strategy.*

Proof. See the text above. $\square$

4.3 Alternative distributional assumptions

In this section, we extend our results to specific type distributions with support $\mathbb{R}_+^M$. For the construction, we exploit the fact that the player's problem is invariant if all battlefield valuations are multiplied by the same positive constant. This allows the extension to generalized Liouville distributions.

Proposition 4. *Suppose that $N \geq 2$ and $M \geq N+1$. For $i \in \{1, \ldots, N\}$, let player i's distribution of types on $\mathbb{R}_+^M$ be given by a density*

$$\eta_i(\mathbf{v}) = c_i \cdot \rho_i \left(\sum_{j=1}^M v_j^{\frac{M-1}{M-N}} \right) \cdot \left(\prod_{j=1}^M v_j \right)^{\frac{1}{M-N}-1},$$

where $c_i > 0$ is a constant, and ρ_i is an arbitrary positive (measurable) function such that

$$\int_0^\infty \rho_i(r) r^{\frac{1}{M-1}} dr < \infty.$$

Then,

$$\beta^*(\mathbf{v}) = \left(\frac{v_1^{\frac{M-1}{M-N}}}{\sum_{j=1}^M v_j^{\frac{M-1}{M-N}}}, \ldots, \frac{v_M^{\frac{M-1}{M-N}}}{\sum_{j=1}^M v_j^{\frac{M-1}{M-N}}} \right)$$

is a symmetric Bayes-Nash equilibrium strategy.

Proof. For any given valuation vector $\mathbf{v} \neq 0$, the player's problem (1) remains unchanged if the objective function is rescaled. Therefore, for $p = \frac{M-1}{M-N}$, we may consider instead the problem

$$\max_{(b_1,\ldots,b_M)\in\mathcal{B}} \sum_{j\in\{1,\ldots,M\}} F(b_j)\widehat{v}_j^p,$$

where $\widehat{v}_j = v_j / \|\mathbf{v}\|_p$. By Hashorva et al. ([11], Thm. 1), the vector $\widehat{\mathbf{v}} = (\widehat{v}_1, \ldots, \widehat{v}_M) \in \mathcal{V}^{M,p}$ follows a p-norm Dirichlet distribution with parameter $\alpha = \frac{1}{M-1}$. The claim is now immediate from Proposition 1. □

Proposition 4 extends Akyol's ([3], Prop. 8) main equilibrium characterization to the case of more than two players.

As explained by Hashorva et al. [11], there are numerous examples of distributions that are consistent with the assumptions of Proposition 4, including generalized Dirichlet, Kotz Type I through III, Pearson Type VII, and Kummer-Beta. The generalized beta distribution assumed in Example 3, for example, is a special case of the Kotz Type I distribution. Using Proposition 4, we may generalize the example to the multi-player case as follows.

Example 6. *Suppose that $N \geq 2$ and $M \geq N+1$. Suppose that each player's battlefield valuation v is drawn independently, across both players and battlefields, from a generalized gamma distribution with density $\frac{p}{\Gamma(\alpha/p)} v^{\alpha-1} \exp(-v^p)$, where α and p are specified as in the proof above. Then, the conclusion of Proposition 4 holds true.*

Given that the player's problem is homogeneous of degree zero, Proposition 4 looks like a natural extension of Proposition 1. However, there are noteworthy implications for expected payoffs and efficiency. Under the assumptions of Proposition 1, the expected payoff of a player does not depend on the type realization. Indeed, the equilibrium payoff of type $\mathbf{v} \in \mathcal{V}$ is given by

$$\Pi^* = \textstyle\sum_{j=1}^{M} F\left(\beta_j^*(\mathbf{v})\right) \cdot v_j = \sum_{j=1}^{M} v_j^p = 1.$$

Thus, differences in information rents [19] are seen to net out across battlefields. Under the assumptions of Proposition 4, however, equilibrium payoffs are homogeneous of degree one in $\|\mathbf{v}\|_p$. Therefore, types with a larger (smaller) p-norm of the valuation vector enjoy a higher (smaller) equilibrium payoff. This difference in payoffs is also reflected in the efficiency analysis. Clearly, the symmetric equilibrium strategy identified in Proposition 1 has the property that the amount of the resource deployed in any given battlefield increases strictly in the player's valuation of that battlefield. Thus, the identified Bayes-Nash equilibrium leads to an efficient selection of battlefield winners, just as in the symmetric single-unit all-pay auction with independent types. However, under the assumption of Proposition 4, the Colonel Blotto game is not efficient because the equilibrium allocation maximizes $\widehat{v}_j$ rather than v_j in each battlefield j.

4.4 Networks

As a final extension, we consider networks of Colonel Blotto games with $N \cdot K$ players and $M \cdot K$ battlefields, where $K \geq 1$ is an integer. Each player is restricted to be active in M given battlefields, and draws a type, e.g., from the p-norm Dirichlet distribution with parameter α, where p and α are set as in Proposition 1. For example, any triangulation of a globe, say, may be understood as a network of Blotto games, where each triangle represents a player, and each edge shared with a neighboring triangle represents a battlefield. In this case, $N = 2$ and $K \geq 2$. Figure 2(a) illustrates this for $K = 5$. Another example is a cube where each side represents a player, and each adjacent node represents a battlefield. In this case, $N = 3$, $M = 4$, and $K = 2$. See Figure 2(b) for illustration. It is immediate to see that examples exist for any combination of N and M for which Propositions 1 or 4 characterize a symmetric Bayesian equilibrium strategy, and for any $K \geq 2$. The equilibrium analysis extends in a straightforward way. Intuitively, a player does not care whether she is facing, in any two distinct battlefields, the same opponent or two different opponents. Indeed, the player's best response depends only on the marginal distribution of bids in each battlefield.

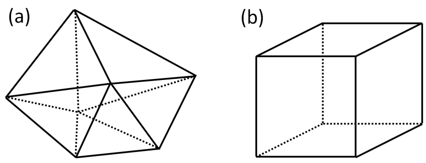

Figure 2. Networks of Colonel Blotto games

5 Concluding remark

The methods of this paper may also be used to construct new classes of mixed-strategy equilibria in two-player Colonel Blotto games with complete information, extending the construction of the disc solution [8, 9, 21, 29]. Specifically, one considers a p-norm hemisphere $\mathcal{H}^{M,p}$ in $\mathbb{R}_+ \times \mathbb{R}^{M-1}$, where $M \geq 3$ and $p = M-1$. On $\mathcal{H}^{M,p}$, one defines a uniform distribution [27, 11]. Then, a random vector from $\mathcal{H}^{M,p}$ is projected on the hyperplane $\{0\} \times \mathbb{R}^{M-1}$. The image of $\mathcal{H}^{M,p}$ under the projection is an $(M-1)$-dimensional sphere in the p-norm. It should be noted that the image is not rotation-invariant unless $M = 3$. Still, connecting all corners of the $(M-1)$ dimensional cube $\{0\} \times [-1, 1]^{M-1}$ with the projection point divides the cube into $2(M-1)$ hyperpyramids. As the $(M-1)$-dimensional volume of any such hyperpyramid is proportional to its height, the volumes are uniformly distributed by Lemma A.1 in the Appendix, and may be used to determine a player's share of the budget allocated to a battlefield. Thus, we indeed obtain a Nash equilibrium in a two-player Colonel Blotto game with $2(M-1)$ battlefields and complete information. The resulting equilibrium bids perfectly negatively correlate within pairs of battlefields, as discussed in Laslier and Picard [21]. Notwithstanding, for $M \geq 4$, they differ from existing constructions.

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Appendix

This appendix contains auxiliary results and the proof of Proposition 3.

The proof of Proposition 1 relies on the following characterization of the marginal distribution of the p-norm Dirichlet distribution, for which we could not find a suitable reference.

Lemma A.1 (Marginal density). *The univariate marginal density of the p-norm Dirichlet distribution with parameter α with respect to any of the components $v_1, \ldots, v_M$ is given as*

$$h(v) = \tfrac{p\Gamma(M\alpha)}{\Gamma(\alpha)\Gamma((M-1)\alpha)} \left(1 - v^p\right)^{(M-1)\alpha - 1} v^{p\alpha - 1},$$

where $v \in (0,1)$. In particular, if $(M-1)\alpha = 1$, then the univariate marginal is a power function distribution on $[0,1]$.

Proof. We follow the steps of the proof of Song and Gupta ([27], Thm. 2.1). Fix $v_1, \ldots, v_{M-2}$, and let

$$A_{M-1} = \left(1 - \textstyle\sum_{j=1}^{M-2} v_j^p\right)^{1/p} > 0.$$

Then,

$$\begin{aligned}
&\int_0^{A_{M-1}} \psi(v_1, \ldots, v_{M-1}) dv_{M-1} \\
&= \tfrac{p^{M-1}\Gamma(M\alpha)}{\Gamma(\alpha)^M} \left(\textstyle\prod_{j=1}^{M-2} v_j^{p\alpha-1}\right) \\
&\cdot \int_0^{A_{M-1}} \left(A_{M-1}^p - v_{M-1}^p\right)^{\alpha-1} v_{M-1}^{p\alpha-1} dv_{M-1}.
\end{aligned}$$

Using the substitution $\widetilde{v}_{M-1} = v_{M-1}/A_{M-1}$, it follows that

$$\begin{aligned}
&\int_0^{A_{M-1}} \psi(v_1, \ldots, v_{M-1}) dv_{M-1} \\
&= \tfrac{p^{M-1}\Gamma(M\alpha)}{\Gamma(\alpha)^M} \cdot \left(\prod_{j=1}^{M-2} v_j^{p\alpha-1}\right) \cdot A_{M-1}^{p(\alpha-1)+p\alpha} \\
&\underbrace{\cdot \int_0^1 \left(1 - \widetilde{v}_{M-1}^p\right)^{\alpha-1} \widetilde{v}_{M-1}^{p\alpha-1} d\widetilde{v}_{M-1}}_{=\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)p}} \\
&= \tfrac{p^{M-2}\Gamma(M\alpha)}{\Gamma(\alpha)^{M-2}\Gamma(2\alpha)} \cdot \left(\prod_{j=1}^{M-2} v_j^{p\alpha-1}\right) \cdot \left(1 - \sum_{j=1}^{M-2} v_j^p\right)^{2\alpha-1}.
\end{aligned}$$

In a second step, we find that

$$\begin{aligned}
&\int_0^{A_{M-2}} \int_0^{A_{M-1}} \psi(v_1, \ldots, v_{M-1}) dv_{M-1} dv_{M-2} \\
&= \tfrac{p^{M-2}\Gamma(M\alpha)}{\Gamma(\alpha)^{M-2}\Gamma(2\alpha)} \cdot \left(\prod_{j=1}^{M-3} v_j^{p\alpha-1}\right) \cdot A_{M-2}^{p(2\alpha-1)+p\alpha} \\
&\underbrace{\cdot \int_0^1 \left(1 - \widetilde{v}_{M-2}^p\right)^{2\alpha-1} \cdot \widetilde{v}_{M-2}^{p\alpha-1} d\widetilde{v}_{M-2}}_{=\frac{\Gamma(\alpha)\Gamma(2\alpha)}{\Gamma(3\alpha)p}} \\
&= \tfrac{p^{M-3}\Gamma(M\alpha)}{\Gamma(\alpha)^{M-3}\Gamma(3\alpha)} \cdot \left(\prod_{j=1}^{M-3} v_j^{p\alpha-1}\right) \cdot \left(1 - \sum_{j=1}^{M-3} v_j^p\right)^{3\alpha-1}.
\end{aligned}$$

After a total of $(M-2)$ iterations, we arrive at

$$\begin{aligned}
&\int_0^{A_2} \cdots \int_0^{A_{M-1}} \psi(v_1, \ldots, v_{M-1}) dv_{M-1} \ldots dv_2 \\
&= \tfrac{p\Gamma(M\alpha)}{\Gamma(\alpha)\Gamma((M-1)\alpha)} v_1^{p\alpha-1} \left(1 - v_1^p\right)^{(M-1)\alpha-1},
\end{aligned}$$

which proves the lemma. □

Proof of Proposition 3. Suppose that all opponents of Player 1 adhere to the candidate equilibrium strategy, i.e., bid the entire budget on one of the highest-valuation battlefields. Suppose first that Player 1 likewise follows the candidate equilibrium strategy. Then, Player 1 wins her selected battlefield with probability $\frac{1}{n+1}$ if precisely n other players bid on it, where $n \in \{0, \dots, N-1\}$. Moreover, Player 1 wins any other battlefield with probability $\frac{1}{N}$ if no other player bids on it. Denote by $v_{(j)}$ Player 1's j-th highest valuation, where $j \in \{1, \dots, M\}$. Then, Player 1's expected payoff from following the candidate strategy is

$$\begin{aligned}
\Pi^* &= \textstyle\sum_{n=0}^{N-1} \frac{v_{(1)}}{n+1} \binom{N-1}{n} \left(\frac{1}{M}\right)^n \left(1 - \frac{1}{M}\right)^{N-1-n} \\
&+ \textstyle\sum_{j=2}^{M} \frac{v_{(j)}}{N} \left(1 - \frac{1}{M}\right)^{N-1} \\
&= \tfrac{v_{(1)}}{N} \textstyle\sum_{n=0}^{N-1} \binom{N}{n+1} \left(\frac{1}{M}\right)^n \left(1 - \frac{1}{M}\right)^{N-1-n} \\
&+ \tfrac{1}{N} \left(1 - \frac{1}{M}\right)^{N-1} \textstyle\sum_{j=2}^{M} v_{(j)} \\
&= \tfrac{v_{(1)} M}{N} \textstyle\sum_{n=1}^{N} \binom{N}{n} \left(\frac{1}{M}\right)^n \left(1 - \frac{1}{M}\right)^{N-n} \\
&+ \tfrac{1}{N} \left(1 - \frac{1}{M}\right)^{N-1} \textstyle\sum_{j=2}^{M} v_{(j)} \\
&= \tfrac{v_{(1)} M}{N} \left\{1 - \left(1 - \frac{1}{M}\right)^N\right\} + \tfrac{1}{N} \left(1 - \frac{1}{M}\right)^{N-1} \textstyle\sum_{j=2}^{M} v_{(j)}.
\end{aligned}$$

Suppose next that Player 1 deviates. For the winning probability of a bid, it matters only if the bid is zero, one, or strictly between zero and one. Moreover, winning probabilities are weakly increasing in the bid. Therefore, it suffices to check the deviation that distributes the budget evenly over all M battlefields. In that case, Player 1 wins a battlefield j if and only if no other player bids on that battlefield. Hence, the resulting payoff from this deviation is

$$\Pi^d = \textstyle\sum_{j=1}^{M} \left(1 - \frac{1}{M}\right)^{N-1} v_j = \left(1 - \frac{1}{M}\right)^{N-1} \textstyle\sum_{j=1}^{M} v_{(j)}.$$

We have $\Pi^* \geq \Pi^d$ if and only if

$$\begin{aligned}
&\tfrac{v_{(1)} M}{N} \left\{1 - \left(1 - \frac{1}{M}\right)^N\right\} \\
&\geq \left(1 - \frac{1}{M}\right)^{N-1} v_{(1)} + \left(1 - \frac{1}{N}\right) \left(1 - \frac{1}{M}\right)^{N-1} \textstyle\sum_{j=2}^{M} v_{(j)}.
\end{aligned}$$

For this to hold for any distribution of types, it is necessary and sufficient that

$$\begin{aligned}
&\tfrac{M}{N}\left\{1-\left(1-\tfrac{1}{M}\right)^{N}\right\} \\
&\geq \left(1-\tfrac{1}{M}\right)^{N-1}+\left(1-\tfrac{1}{N}\right)\left(1-\tfrac{1}{M}\right)^{N-1}(M-1).
\end{aligned}$$

This is equivalent to $\frac{1}{N} \geq \left(1-\frac{1}{M}\right)^{N-1}$, which in turn is equivalent to $M \leq M^*(N)$. This proves the proposition. $\square$

The following lemma collects properties of the threshold $M^*(N)$ defined in the statement of Proposition 3.

Lemma A.2 (Properties of $M^*(N)$).

(i) $N \geq 2$ *implies* $M^*(N) \leq N$.

(ii) $\lim_{N\to\infty} M^*(N) = \infty$.

Proof. (i) A straightforward calculation shows that $M^*(N) \leq N$ is equivalent to $\left(1-\frac{1}{N}\right)^{N-1} \geq \frac{1}{N}$, which in turn follows from Bernoulli's inequality. (ii) It suffices to recall that $\lim_{N\to\infty} \sqrt[N]{N} = 1$. $\square$

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