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An Empirical Study of the Sentiment Capital Asset Pricing Model

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Comments

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Abstract

What is market sentiment? This paper takes a new approach to this question and derives a formula for market sentiment as a function of the risk-free rate, the price/dividend ratio, and the conditional stock market volatility. The formula is derived from a representative agent with a prospect theory probability weighting function. We estimate the model and find that our sentiment measure correlates positively with the leading sentiment indexes. The model matches the equity premium while generating a low and stable risk-free rate with low risk aversion. We also apply the model to explain other anomalies for the aggregate stock market.

JEL Classification: G40, G41

Keywords: Sentiment; Prospect Theory; Equity Premium Puzzle; Pricing Kernel Puzzle; Sentiment Indexes.

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1 Introduction

Sentiment in the form of optimism or pessimism has driven speculation in financial markets for centuries. Yet, active academic discussion on the role of investor sentiment in financial markets is recent. Empirical research on sentiment has rapidly developed since the introduction of the empirical index of market sentiment constructed by Baker and Wurgler (2006, 2007). However, from a theoretical perspective, two fundamental questions about investor sentiment remain: First, what is investor sentiment? In a review paper on measuring sentiment, Zhou (2018) writes, "true investor sentiment is almost always unobservable, and all computed measures are proxies." This elusiveness of defining and observing sentiment poses a significant theoretical challenge, and places the notion of sentiment at risk of being a 'dark matter' assumption (Cochrane, 2017) that is not well understood. Second, what role, if any, does sentiment have in explaining basic characteristics of the stock market such as the large historical equity premium, the predictability of the price/dividend ratio, or the shape of the pricing kernel?

In this paper, we consider a simple generalization of the classical consumption capital asset pricing model (CCAPM) that has a representative agent with one of the most promising prospect theory probability weighting functions (Wakker, 2010), which decomposes probability weights into the true probability and a sentiment component. We show that by specifying the risk-free rate and the market return, the sentiment component can be identified as the residual that satisfies the Euler equation for the equity premium. We show that this residual provides a structural formula for sentiment as a function of observable macroeconomic quantities, and we find that it is positively and significantly related to the empirical sentiment indexes in the literature (including the Baker and Wurgler (2006) indexes, the Michigan Consumer Sentiment Index, and the Consumer Confidence Index). Motivated by the structural formula for sentiment, we introduce a simple linear three-factor model of systematic sentiment that depends on the three primary factors of the derived sentiment index (the risk-free rate, the price-dividend ratio, and the conditional market volatility). We find that for our sample period spanning more than thirty years of monthly data, the three-factor model of sentiment explains approximately 60% of the variation (R-squared) in the Consumer Confidence Index, 50% of the variation in the Michigan Index, and roughly 30% of the variation in the Baker-Wurgler indexes. We further show that after removing the systematic component of sentiment predicted by the theory from the empirical sentiment indexes, the positive correlations between different empirical sentiment indexes become negative or insignificant.

Since we match the Euler equation for the equity premium exactly, our approach also matches the mean and volatility of the equity premium, the market Sharpe ratio, and the observed time variation in risk premia. Our approach also predicts a low and stable risk-free rate and generates a large equity premium and low risk free rate with low risk aversion (e.g., log utility) and with a small deviation from expected utility theory (a small weight on the sentiment component of the weighting function). We show that our approach also explains other effects of sentiment on the aggregate stock market such as the effect of sentiment on the mean-variance relationship documented by Yu and Yuan (2011), and the effect of sentiment on the risk-neutral distribution documented by Han (2007). We further show that the model provides explanations for two fundamental phenomena that are of broad interest in asset pricing: the time series predictability of the price/dividend ratio (Campbell and Shiller, 1988; Fama and French, 1988), and the non-monotonicity of the pricing kernel (the pricing kernel puzzle), as revealed by its empirically observed U-shape (Bakshi et al., 2010; Sichert, 2018). As the model we study provides a simple way to incorporate sentiment into asset pricing theory, we refer to it as the *Sentiment CAPM.*

Despite its simplicity, the Sentiment CAPM ties together four strands of the modern asset pricing literature by incorporating a role for market sentiment, model uncertainty, positive skewness and disaster risk.

Other sentiment-based models have also been developed for the aggregate stock market. However, Barberis et al. (2015), note in their Table 1 that many of the leading models of sentiment including De Long et al. (1990a,b), Campbell and Kyle (1993), Barberis et al. (1998), Cutler et al. (1990), Hong and Stein (1999), Barberis and Shleifer (2003), and Barberis et al. (2015) do not account for the equity premium puzzle, the large historical excess returns on stocks over bonds documented by Mehra and Prescott (1985). It is also interesting that none of these foundational papers on sentiment is directly linked to another pillar of behavioral finance: prospect theory (Kahneman and Tversky, 1979; Barberis, 2018).

Apart from the sentiment-based models cited above, Shefrin (2008) provides a general framework for studying the effects of sentiment on asset prices. Barone-Adesi et al. (2017) use that framework to jointly estimate sentiment, risk aversion, and time preference from option prices and historical returns. Drawing from robust control theory, Hansen and Sargent (2001) consider the role of model uncertainty and uncertainty aversion in explaining asset returns. Kraus and Litzenberger (1976), Harvey and Siddique (2000), Brunnermeier et al. (2007), Mitton and Vorkink (2007), Barberis and Huang (2008), and Bordalo et al. (2013) study the effects of a preference for positive skewness on asset returns. Rietz (1988), Barro (2006), Gabaix (2012), and Wachter (2013) consider the impact of rare economic disasters on asset prices. The Sentiment CAPM provides a simple analytical framework that links these four departures from the CCAPM.

The representative agent we study was previously considered analytically by Chateauneuf et al. (2007) and Zimper (2012) who both noted that the model can generate a larger equity premium than the CCAPM. However, the real challenge posed by the equity premium puzzle is whether a model can generate the full magnitude of the historical equity premium with plausible levels of risk aversion. This is ultimately an empirical question. Yet surprisingly there has been no empirical study of the Sentiment CAPM. This paper fills that gap.

We close by providing the Sentiment CAPM with microfoundations through (i) demonstrating that the same parameter values that satisfy the Euler equation also explain laboratory evidence on choices under risk, and (ii) by establishing an aggregation result. In particular, we show that a market with some standard expected utility traders and some noise traders can generate the same prices as a different economy with a representative agent that has a textbook prospect theory probability weighting function.

The paper is organized as follows: Section 2 introduces the representative agent. Section 3 presents the Sentiment CAPM and derives a general formula that indicates how sentiment is related to observable economic quantities. Section 4 applies the Sentiment CAPM to explain the equity premium puzzle. Section 5 constructs a GARCH model for the Sentiment CAPM and derives a more precise formula for sentiment as a function of fundamental variables. There we also estimate the derived sentiment index from historical data and correlate the derived index with the leading sentiment indexes in the literature. We then use the factors of the derived sentiment index to construct a three-factor model of sentiment that we apply to explain variation in the leading sentiment indexes. Section 6 demonstrates that the Sentiment CAPM predicts other empirical relationships between sentiment and the aggregate stock market that have been documented in the literature. Section 7 considers microfoundations for the Sentiment CAPM. Section 8 concludes.

2 Robust Optimization with Investor Sentiment

We consider an economy with one risky asset (a stock), that represents the aggregate stock market, and one risk-free asset (a bond). There is a representative agent as in Definition 1 who has non-negative holdings of the risky asset. In the behavioral economics literature, deviations from the predictions of expected utility theory (EU) are often explained by prospect theory (Kahneman and Tversky, 1979). We do not employ the full machinery of prospect theory, but we consider an agent who deviates from EU via a prospect theory probability weighting function that overweights the tails of the distribution. Wakker (2010) notes that the probability weighting function that we use (embedded in the agent's preferences in Definition 1) is among the most promising families of weighting functions in the literature and that "the interpretation of its parameters is clearer and more convincing than with other families" (p. 210). We refer to such an agent as an EU-Hurwicz agent. Versions of EU-Hurwicz preferences are advocated by Ellsberg (2001) and Chateauneuf et al. (2007), and they are formally a special case of cumulative prospect theory (Tversky and Kahneman, 1992) when restricted to choices over non-negative outcomes. Let there be a set S of possible states of nature, a set $\mathcal{C}$ of consumption levels, and a set $\mathcal{F}$ of acts in the sense of Savage (1954) where an act, $f : S \to \mathcal{C}$ assigns a consumption level to each state.

Definition 1. An *EU-Hurwicz agent* has the following value function for an act, f:

$$V(f) = \gamma E[u(C)] + (1-\gamma)[\alpha u(\overline{C}) + (1-\alpha)u(\underline{C})] \tag{1}$$

In (1), $E[u(C)]$ is the agent's expected utility from consumption, $u(\overline{C})$ and $u(\underline{C})$ are, respectively, the utility from the best-case and worst-case consumption levels across the possible states of nature, γ can be interpreted as the agent's degree of confidence in the accuracy of her prior distribution across states, and α represents the agent's degree of optimism toward uncertainty. These preferences maximize the convex combination of the agent's expected utility and a measure of robustness (represented by the Hurwicz optimism-pessimism criterion). The preferences achieve a separation of the agent's beliefs (represented by a unique subjective probability distribution) and the agent's ambiguity attitudes (represented by α) ranging from extreme pessimism ($\alpha = 0$) to extreme optimism ($\alpha = 1$). As the agent becomes less confident in her beliefs, γ decreases and greater weight is placed on her preference for robustness to model uncertainty (the Hurwicz criterion (Hurwicz, 1951), which does not depend on the agent's probability distribution). Formula (1) for the representative agent thus incorporates a role for sentiment (represented by α), model uncertainty (represented by γ), positive skewness (represented by $\overline{C}$), and disaster risk (represented by $\underline{C}$), thereby linking four primary strands of the literature.

The EU-Hurwicz agent exhibits two general principles of behavior that have posed a challenge for EU since its inception: (i) ambiguity aversion and (ii) positive skewness preference. One form of ambiguity aversion is an aversion to prospects that are less robust to incorrect probability models (model uncertainty). A classical approach to capture such a preference for robustness to mis-specified probabilities is Wald's maximin rule (Wald, 1950) that remains widely used in the field of robust optimization and which selects alternatives that have better worst-case scenarios.

Models of ambiguity aversion such as those due to Schmeidler (1989), Gilboa and Schmeidler (1989), Hansen and Sargent (2001), and Klibanoff et al. (2005) have been used to explain buying-selling price gaps in markets (Dow and da Costa Werlang, 1992) and the equity premium puzzle (Collard et al., 2018; Gollier, 2011; Ju and Miao, 2012; Maenhout, 2004). Like models of habit-formation (Campbell and Cochrane, 1999), loss aversion (Barberis et al., 2001), long-run risk (Bansal and Yaron, 2004), and disaster risk (Barro, 2006), models of ambiguity aversion systematically overweight bad outcomes, relative to expected utility theory. However, such approaches do not account for another determinant of asset prices - positive skewness preference.

The inability of EU to provide a plausible explanation for the observation that many people purchase both insurance policies and lottery tickets has been known since Friedman and Savage (1948). More broadly, most risk-seeking behavior that is observed in real market settings is often of the form of low-probability, high-payoff prospects, revealing a positive skewness preference. Examples include the over-pricing of long-shots in betting markets (Weitzman, 1965) and the overvaluation of positively skewed stocks in financial markets (Barberis and Huang, 2008).

The literatures on ambiguity aversion and skewness preference have largely developed separately. However, given the important role that both of these biases have in market contexts, it seems that a more complete model of asset valuation could incorporate both of these deviations from the classical model.

2.1 Properties of Expected Utility-Hurwicz Preferences

The EU-Hurwicz preference model has several appealing features:

1. **Separating Beliefs and Ambiguity Preferences:** EU-Hurwicz preferences achieve a separation of the decision maker's subjective beliefs (represented by a unique subjective prior distribution over states) and the decision maker's ambiguity attitudes (represented by α). An EU-Hurwicz agent has two objectives: maximizing expected utility with respect to her subjective prior, and making investment decisions that are robust to a mis-specified prior. The parameter γ determines how the agent trades off these two objectives.

2. **Separating the Elasticity of Intertemporal Substitution and Risk Aversion:** EU-Hurwicz preferences provide a partial separation between risk aversion and the elasticity of intertemporal substitution which does not hold under the standard EU model.

3. **Axiomatic Foundations:** EU-Hurwicz preferences satisfy basic normative properties including transitivity and first order stochastic dominance with respect to the agent's probability distribution over states. They have a theoretical foundation as they satisfy

the axioms of two basic models of choice under uncertainty, the multiple priors model (Gilboa and Schmeidler, 1989; Ghirardato et al., 2004), and the Choquet expected utility model (Schmeidler, 1989; Chateauneuf et al., 2007). EU-Hurwicz preferences are also given an explicit axiomatic characterization by Chateauneuf et al. (2007).

4. **Prospect Theory Probability Weighting:** An EU-Hurwicz agent has a textbook prospect theory probability weighting function (Tversky and Kahneman, 1992; Wakker, 2010) that overweights the tails of the distribution.

5. **Ambiguity Aversion and Skewness Preference:** EU-Hurwicz preferences incorporate a bias toward prospects that are robust to model uncertainty (by overweighting the worst outcome), and a bias toward prospects that have high potential (by overweighting the best outcome). That is, EU-Hurwicz preferences capture forms of both ambiguity aversion and positive skewness preference.

3 The Sentiment CAPM

A representative *EU-Hurwicz* agent faces the following two-date optimization problem:

$$\begin{aligned} &\max_{\{C_t,B_t,S_t\}} u(C_t) + \gamma E_t \beta u(C_{t+1}) + (1-\gamma)\beta[\alpha_t u(\overline{C}_{t+1}) + (1-\alpha_t)u(\underline{C}_{t+1})] \\ &\text{s.t.} \quad C_t + S_t P_t + B_t = S_{t-1}(P_t + D_t) + R^f_{t-1} B_{t-1} + \Omega_t, \end{aligned} \tag{2}$$

where S_t and B_t are holdings of the stock and bond, respectively, R^f_t is the real risk-free rate, and P_t and D_t are the stock price and dividend. Ω_t represents other sources of income. Assuming that the space of outcomes next period is compact, one can write $\overline{C}_{t+1} = \max_{s\in S} C_{s,t+1}$, $\underline{C}_{t+1} = \min_{s\in S} C_{s,t+1}$. In case of stochastic outcomes with an unbounded support like a normal distribution, we assume that the agent has a rule to truncate the tails of the distribution. Finally, $\alpha_t \in [0,1]$ is our measure of sentiment that represents optimism. An economy with a representative EU-Hurwicz agent was first considered in Chateauneuf et al. (2007) and Zimper (2012). It is easy to show that the Euler equation for the gross market return $R_{t+1} = \frac{P_{t+1}+D_{t+1}}{P_t}$ is

$$\gamma E_t M_{t+1} R_{t+1} + (1-\gamma)\left(\alpha_t \overline{M}_{t+1} \overline{R}_{t+1} + (1-\alpha_t)\underline{M}_{t+1}\underline{R}_{t+1}\right) = 1, \tag{3}$$

where $M_{t+1} \equiv \beta \frac{u'(C_{t+1})}{u'(C_t)}$. We use $\overline{M}_{t+1}$ and $\overline{R}_{t+1}$ to denote the next period's values of M_{t+1} and the market return in the optimistic scenario, and $\underline{M}_{t+1}$ and $\underline{R}_{t+1}$ represent these variables in the pessimistic scenario. Similarly, the Euler equation for the gross risk-free rate R_t^f is

$$\gamma E_t M_{t+1} R_t^f + (1-\gamma)\left(\alpha_t \overline{M}_{t+1} R_t^f + (1-\alpha_t)\underline{M}_{t+1} R_t^f\right) = 1. \tag{4}$$

Subtracting (4) from (3) gives us the equity premium for the EU-Hurwicz agent. After rearranging the terms we have

$$\begin{aligned} E_t R_{t+1} - R_t^f = &- \frac{Cov_t(M_{t+1}, R_{t+1})}{E_t M_{t+1}} + \\ &\left(\frac{1-\gamma}{\gamma}\right)\left[\alpha_t \frac{\overline{M}_{t+1}(R_t^f - \overline{R}_{t+1})}{E_t M_{t+1}} + (1-\alpha_t)\frac{\underline{M}_{t+1}(R_t^f - \underline{R}_{t+1})}{E_t M_{t+1}}\right]. \end{aligned} \tag{5}$$

The risk premium can be expressed as a linear factor model. Defining

$$\beta_1 \equiv \frac{Cov_t(M_{t+1}, R_{t+1})}{Var_t(M_{t+1})}, \qquad \beta_2 \equiv R_t^f - \overline{R}_{t+1}, \qquad \beta_3 \equiv \underline{R}_{t+1} - R_t^f,$$

the equity premium in (5) can be written as a three-factor asset pricing model in (6):

Corollary 2. *In equilibrium, for a representative EU-Hurwicz agent, the equity premium is*

$$E_t R_{t+1} - R_t^f = \beta_1 \left[-\frac{Var_t(M_{t+1})}{E_t\left[M_{t+1}\right]}\right] + \beta_2 \left[\frac{\alpha_t(\frac{1-\gamma}{\gamma})\overline{M}_{t+1}}{E_t\left[M_{t+1}\right]}\right] + \beta_3 \left[-\frac{(1-\alpha_t)(\frac{1-\gamma}{\gamma})\underline{M}_{t+1}}{E_t\left[M_{t+1}\right]}\right] \tag{6}$$

In (6), the equity premium depends on the index of dispersion for M_{t+1} and the covariance of asset returns with M_{t+1}, as in the standard CCAPM, as well as on a bull sentiment factor and a bear sentiment factor. In addition, β_2 quantifies the asset's exposure to bullish sentiment, and β_3 quantifies the asset's exposure to bearish sentiment. In particular, the asset has greater exposure to bullish sentiment if it has high potential (high $\overline{R}_{t+1}$) in the optimistic scenario, resulting in lower equilibrium expected returns. The asset has greater exposure to bearish sentiment if it has extreme negative returns (low $\underline{R}_{t+1}$) in the pessimistic scenario, resulting in higher equilibrium expected returns. The bullish sentiment factor becomes more negative as optimism (α_t) increases. The bearish sentiment factor becomes larger as optimism (α_t) decreases.

Lu and Murray (2019) find that a particular form of bear market risk, the risk-neutral probability of future bear market states is a priced risk factor. Consistent with this finding, note that in (5), $\frac{(1-\alpha_t)\underline{M}_{t+1}}{E_t[M_{t+1}]}$ is a priced risk factor and $(1-\alpha_t)$ is a component of the risk-neutral probability of a future bear market state for a representative EU-Hurwicz agent.

Since the three-factor model in (5) generalizes the classical Consumption CAPM to include a role for investor sentiment, we refer to (5) as the *Sentiment CAPM*.

4 The Historical Equity Premium

We next find the range of EU-Hurwicz parameters that satisfy the unconditional long-run moments for the equity-premium and the risk-free rate. The Euler equations for the risk-free rate and the equity premium are given by formulas (4) and (5), respectively. We further assume that the EU-Hurwicz agent has a time separable constant relative risk aversion (CRRA) period utility and we denote the risk aversion parameter by θ. We use the following simple specification for the future consumption growth and market return.

Assumption 3. *The next period consumption growth and market return have conditional log-normal distributions:*

$$\Delta c_{t+1} \equiv \log \frac{C_{t+1}}{C_t} \sim \mathcal{N}(\mu, \sigma^2)$$
$$r_{t+1} \equiv \log(R_{t+1}) \sim \mathcal{N}(x_t, q_t^2).$$

Next, we specify how the EU-Hurwicz agent estimates the best and worst-case scenarios.

Assumption 4. *The EU-Hurwicz representative agent truncates the tails of a normal distribution based on the (conditional) standard deviation from the (conditional) mean:*

$$\overline{\Delta c_{t+1}} = \mu + \overline{\xi}\sigma, \qquad \overline{r_{t+1}} = x_t + \overline{\xi} q_t$$
$$\underline{\Delta c_{t+1}} = \mu - \underline{\xi}\sigma, \qquad \underline{r_{t+1}} = x_t - \underline{\xi} q_t,$$

where $\underline{\xi}$ and $\overline{\xi}$ are fixed numbers.

The variable x_t represents the best prediction of the market return. In Section 6 we show that under the Sentiment CAPM, the price-dividend ratio must have return predictability in response to changes in sentiment, making the price-dividend ratio a natural predictor for the market return in our setting. Given the CRRA utility function, $M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\theta} = e^{\ln\beta - \theta\Delta c_{t+1}} = e^{-\rho-\theta\Delta c_{t+1}}$, where $\rho \equiv -\ln\beta$. The log-normality assumptions imply

$$E_t M_{t+1} = e^{-\rho-\theta\mu+\frac{1}{2}\theta^2\sigma^2}, \quad \sigma_t(M_{t+1}) = E_t M_{t+1}\sqrt{e^{\theta^2\sigma^2}-1}$$
$$E_t R_{t+1} = e^{x_t+\frac{1}{2}q_t^2}, \qquad \sigma_t(R_{t+1}) = E_t R_{t+1}\sqrt{e^{q_t^2}-1},$$

and hence, the covariance term in the equity premium equation equals

$$-\frac{Cov_t(M_{t+1}, R_{t+1})}{E_t M_{t+1}} = \eta\theta\sigma q_t(1 + x_t + \frac{1}{2}q_t^2),$$

where η is the time invariant correlation between the log market return and log consumption growth, and the equality follows from the approximation $e^x = 1 + x$, for small values of x. Since we use this approximation frequently, we write it as an extra assumption.

Assumption 5. *For small values of x we have $e^x = 1 + x$.*

Using the above assumptions 3-5 on (i) conditional log-linearity of future returns and consumption growth; (ii) assuming that the EU-Hurwicz agent truncates normal distributions;

and (iii) approximating e^x for small x, we can rewrite the equity premium (5) as (7):

$$xr_t + \frac{1}{2}q_t^2 = \eta\theta\sigma q_t(1 + x_t + \frac{1}{2}q_t^2) - \\ \frac{1-\gamma}{\gamma}\left[\alpha_t\big(1 - \frac{1}{2}\theta^2\sigma^2 - \theta\overline{\xi}\sigma\big)(xr_t + \overline{\xi}q_t) + (1-\alpha_t)\big(1 - \frac{1}{2}\theta^2\sigma^2 + \theta\underline{\xi}\sigma\big)(xr_t - \underline{\xi}q_t)\right], \quad (7)$$

where $R_t^f = 1 + r_t^f$, $r_{f,t} \equiv \ln(1 + r_t^f)$ and $xr_t \equiv x_t - r_{f,t}$.

This result allows us to specify the sentiment as the residual α_t that satisfies equity premium equation (5). Corollary 6 presents the residual sentiment.

Corollary 6. *In an economy with an EU-Hurwicz representative agent and under assumptions 3-5, the sentiment α_t that satisfies the equation for the equity premium is*

$$\alpha_t = \frac{(\underline{\xi}q_t - xr_t)(1 - \frac{1}{2}\theta^2\sigma^2 + \theta\underline{\xi}\sigma) - \left(\frac{\gamma}{1-\gamma}\right)\left[xr_t + \frac{1}{2}q_t^2 - \eta\theta\sigma q_t(1 + x_t + \frac{1}{2}q_t^2)\right]}{(\underline{\xi} + \overline{\xi})q_t(1 - \frac{1}{2}\theta^2\sigma^2) + \theta\sigma\left[q_t(\underline{\xi}^2 - \overline{\xi}^2) - xr_t(\underline{\xi} + \overline{\xi})\right]}. \quad (8)$$

Except for the model parameters $\{\gamma, \underline{\xi}, \overline{\xi}, \theta\}$ the right hand side of (8) is all data. Corollary 6 shows the required deviation from the expected utility theory in order to match the equity premium. We can use data to calibrate the parameters $\underline{\xi}$ and $\overline{\xi}$ based on the maximum and minimum values of consumption growth and the market return in comparison to their mean. Since we are using the exponential approximation, it is instructive to see the effect of shrinking the time period (*i.e.*, dropping the second-oder terms). In that case, (8) reduces to the following simple equation

$$\alpha \approx \frac{\underline{\xi}}{\underline{\xi} + \overline{\xi}} - \frac{1}{(\underline{\xi} + \overline{\xi})(1-\gamma)}\frac{xr}{q}, \quad (9)$$

where we have also dropped the t subscript because we use this equation only for the long-run (with unconditional moments). Notice that dropping the second order terms reveals that sentiment is approximately a function of the Sharpe ratio, and that the value of the risk aversion parameter θ is not very important to match the equity premium. However, as we see shortly, we can use this parameter (in conjunction with ρ) to match the risk-free rate.

Next, we focus on the risk-free rate. Using the Euler equation (4), we can derive the risk-free rate as a function of sentiment.

Corollary 7. *The risk-free rate with the EU-Hurwicz representative agent and the CRRA utility is*

$$r_{f,t} = \rho + \theta\mu - \gamma\frac{1}{2}\theta^2\sigma^2 + (1-\gamma)\theta\sigma\left[\alpha_t(\underline{\xi} + \overline{\xi}) - \underline{\xi}\right]. \tag{10}$$

Proof. See appendix A. □

As we did with the equity premium, we can derive the sentiment that satisfies the risk-free rate. Dropping the second-order terms, and solving (4) for α, we have

$$\alpha \approx \frac{\underline{\xi}}{\underline{\xi} + \overline{\xi}} - \frac{1}{(\underline{\xi} + \overline{\xi})(1-\gamma)} \frac{\rho + \theta\mu - r_f}{\theta\sigma}. \tag{11}$$

Comparing (9) and (11) shows that it is possible to find an α that satisfies both the equity premium and the risk-free rate for every value of γ. The condition that we need to satisfy is the following

$$\frac{xr}{q} = \frac{\rho + \theta\mu - r_f}{\theta\sigma},$$

which can be written as the θ parameter in terms of ρ.

Corollary 8. *In order to satisfy both the equity premium and the risk-free rate, we need to have the following relationship between parameters that determine the inter-temporal elasticity of substitution θ and the temporal discount-rate ρ:*

$$\theta = \frac{rf - \rho}{\mu - \sigma\frac{xr}{q}}. \tag{12}$$

It is instructive to compare (12) with the risk-aversion parameter θ_{EU} that one gets from the standard Euler equation of the risk-free rate with expected-utility[1]:

$$\theta_{EU} \equiv \frac{rf - \rho}{\mu} < \frac{rf - \rho}{\mu - \sigma\frac{xr}{q}}.$$

Clearly, the EU-Hurwicz agent needs a higher risk-aversion parameter θ to match *both* the risk-free rate and the equity-premium, and the deviation from the standard EU risk-aversion θ_{EU} depends on the key data moment $\sigma\frac{xr}{q}$. Still, one can use the historical moments to see that the EU-Hurwicz risk-aversion parameter cannot be much higher than one.

In order to show the loci of parameters (α, γ) and (θ, ρ) that satisfy both the equity premium and the risk-free rate, we use the historical moments of the U.S. data to depict equations (9) and (12). For illustration, we take the widely used set of statistics for the post-war period from Cochrane (2009) in which the real market return is about 9% with a standard deviation of about 16%. The real return on treasury bills has been about 1%, and so are the mean and standard deviation of the real per capita consumption growth (measured as non-durables plus services). It remains to calibrate $\{\underline{\xi}, \overline{\xi}\}$, for which we use the monthly real per capita consumption growth data. We use the statistics $\frac{\mu(\Delta c)-\min(\Delta c)}{\sigma(\Delta c)}$ and $\frac{\max(\Delta c)-\mu(\Delta c)}{\sigma(\Delta c)}$ to calibrate $\{\underline{\xi}, \overline{\xi}\}$. The aggregate consumption series is the sum of non-durables and services from the National Income and Product Account (NIPA) monthly series DNDGRC and DSERRC. Population is the U.S. civilian non-institutional population age 16 and over, FRED[2] series CNP16OV. Finally, we use the consumer price index (CPI) from CRSP[3] U.S. Treasury and Inflation Indexes. The per capita real consumption growth is the growth rate of consumption divided by CPI times the population. Table 1 shows the summary statistics for the log of the monthly real per capita consumption growth. Thus, we have $\frac{\mu(\Delta c)-\min(\Delta c)}{\sigma(\Delta c)} = 4.69$ and $\frac{\max(\Delta c)-\mu(\Delta c)}{\sigma(\Delta c)} = 3.27$, that we round up to the closest integer, that is, $\underline{\xi} = 5$ and $\overline{\xi} = 4$. Table 2 shows the moments and the calibration that we use to determine the set of parameters that satisfy both the equity premium sentiment (9) and the risk-free rate sentiment (11). The upper panel of figure 1 depicts equation (12).

[1] The Euler equation of the risk-free rate in the EU framework under assumption 3 is $rf = \rho + \theta\mu - \frac{1}{2}\sigma^2$ and recall that we dropped the second order terms.

[2] Federal Reserve Bank of St. Louis: `https://www.stlouisfed.org`.

[3] The Center for Research in Security Prices, whose data is available at Wharton Research Data Services: `https://wrds-www.wharton.upenn.edu`.

Table 1: Summary statistics of log real per capita consumption growth.

Variable	**Mean**	**St. Dev.**	**Min**	**Max**	**N**
Δc	0.00125	0.00435	-0.0192	0.0155	719

Monthly data (1959m1-2018m12).

Table 2: Moments in the historical equity premium and risk-free rate Euler equations.

xr	q	rf	μ	σ	$\underline{\xi}$	$\overline{\xi}$
0.08	0.16	0.01	0.01	0.01	5.0	4.0

Notice that although θ is larger than θ_{EU}, the maximum θ is not much higher than 1. Of course, one can relax this constraint if one assumes that the same α need not satisfy both the equity-premium and the risk-free rate for any value of γ. The lower panel of figure 1 depicts equation (9), which coincides with equation (11) if the parameters (θ, ρ) are on the locus in the upper panel. Thus, any point on the upper and lower panel is consistent with the set of parameters that satisfy both the equity-premium and the risk-free rate.

The Sentiment CAPM matches the full magnitudes of the historical equity premium and the risk-free rate even with a small degree of risk aversion (e.g., log utility), and a moderate degree of ambiguity aversion (e.g., even with an α above 0.4). Leading alternatives to the CCAPM generate uniform risk aversion and so cannot explain the systematic risk-seeking behavior toward positively skewed prospects that has been documented in both laboratory experiments and financial markets.

An advantage of our approach is that it can match both the equity premium and risk-free rate with low risk aversion (e.g., even with θ between 0 and 1). To match the historical equity premium, asset pricing models often assume an implausibly large degree of risk aversion. For instance, the long-run risk model (Bansal and Yaron, 2004) assumes a risk-aversion coeficient of 10 to match the equity premium, and the classical consumption CAPM cannot even generate ten percent of the historical equity premium with a risk-aversion coefficient of 10 (Mehra and Prescott, 1985).

Models of non-expected utility preferences also struggle to match the observed equity

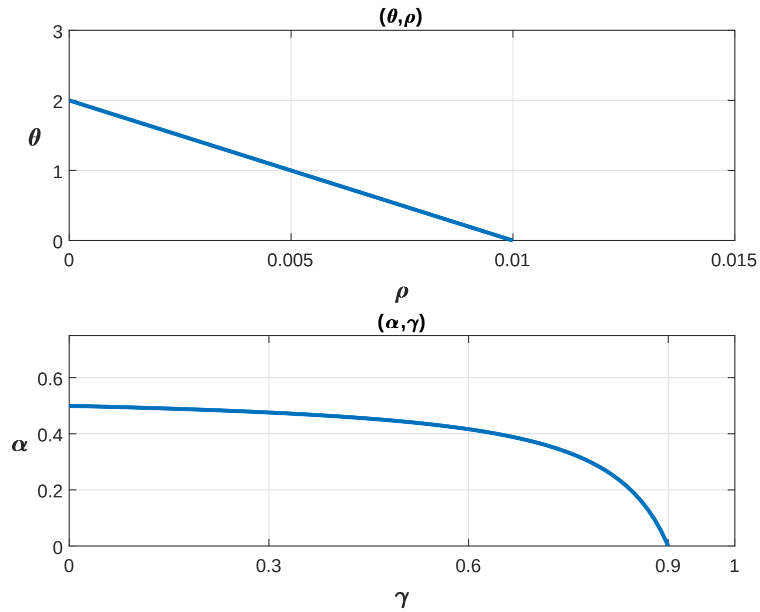

Figure 1: The upper panel shows the set of (θ, ρ) that make any point (α, γ) on the lower panel satisfy both the equity-premium and the risk-free rate.

premium. Chapman and Polkovnichenko (2009) show that representative agent economies for a wide range of first-order risk-averse preferences including risk-averse rank-dependent utility (Quiggin, 1982), disappointment aversion (Gul, 1991; Routledge and Zin, 2010), loss aversion (Kahneman and Tversky, 1979), and ambiguity aversion (Gilboa and Schmeidler, 1989) do not generate an equity premium greater than 4 percent under any of their calibrations. Pagel (2016) shows that while the Kőszegi and Rabin (2006, 2007) model of reference-dependent preferences can match the equity premium, it generates counterfactually large volatility in the risk-free rate.

Models of ambiguity aversion do not require substantial risk aversion but instead require a large degree of ambiguity aversion. For instance, the robust control approach to asset pricing pioneered by Hansen and Sargent (2001) focuses on the worst-case scenarios, as does the maxmin multiple priors model of Gilboa and Schmeidler, thereby substituting a large amount of risk aversion with a large amount of ambiguity aversion. Ju and Miao (2012) study the smooth model of ambiguity aversion (Klibanoff et al., 2005) in an asset pricing context which allows in principle for less extreme ambiguity aversion. However, they require a relative ambiguity aversion coefficient of more than 8 to match the equity premium.

5 Empirical Study of Sentiment and the Equity Premium

In section 4, we defined the sentiment in equation (8) as the residual that satisfies the equity premium equation in the form of $\alpha_t = f(x_t, q_t, rf_t)$. Then, we showed that on average, given that the parameters (θ, ρ) satisfy (12), the same (average) α and γ that satisfy the equity premium also satisfy the risk-free rate equation. The next natural step is to evaluate the empirical support for the theory. In particular, if we specify a statistical model for the log stock-returns, we can construct the time series for x_t and q_t, using which we can directly construct α_t from (8). Once we construct α_t, we can compare it to the monthly survey data indexes for consumer and investor sentiment. Of course, we do not maintain that there must be a linear relation between α_t and survey indexes, however, the existence of a strong correlation provides empirical support for the model.

Sentiment indexes: The most used consumer sentiment indexes are the Consumer Confidence Index (CCI) from the Conference Board[4] and the Consumer Sentiment Index (ICS) from the University of Michigan[5]. The most widely used investor sentiment data is constructed by Baker and Wurgler (2006), and consists of two indexes: the Baker-Wurgler (BW) index and the Orthogonal Baker-Wurgler ($\mathrm{BW}^{\perp}$) index, where the second index is constructed similarly to the first but the business cycle variations are removed.[6] We also use the market bullish (Bull) and bearish (Bear) sentiment indexes from the American Association of Individual Investors[7], where the Bull index can be treated as an index of investor optimism.

Our sample period spans more than three decades of monthly data from November 1987 through December 2018. It contains the dot-com bubble of the late 1990's, the housing bubble of the early 2000's the great recession of 2007 to 2009, and two of the longest U.S.

[4] `https://www.conference-board.org/data/consumerconfidence.cfm`
[5] `http://www.sca.isr.umich.edu/`
[6] Data available at: `http://people.stern.nyu.edu/jwurgler/`
[7] `https://www.aaii.com/sentimentsurvey`

business cycle expansions. The sample period starts the month following the October 1987 stock market crash as the post-crash period may reflect a regime-change in how the market prices disaster risk. As Rubinstein (1994) remarks, "One is tempted to hypothesize that the stock market crash of October 1987 changed the way market participants viewed index options." Jackwerth and Rubinstein (1996) similarly note: "it is now well known that since the 1987 crash, Black-Scholes implied volatilities for $S\&P$ 500 Index options have consistently exhibited pronounced smile effects - a fact that can perhaps be best explained by extreme departures from lognormality." Hence, the post-1987 crash period seems particularly appropriate for our setting where the representative agent truncates and overweights the tails of the return distribution, producing a risk-neutral distribution that is asymmetric and fat-tailed.

The risk-aversion parameter that we use to construct α_t is $\theta = 1$. Importantly, the log utility allows us to use the nominal market return, risk-free rate and consumption growth. Henceforth, we do not need to use the inflation data and the series r_t and rf_t are nominal.

In order to construct x_t and q_t from the data, we use a simple Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) model (Bollerslev, 1986). In section 6, we show that having an EU-Hurwicz agent implies return predictability. Specifically, the price-dividend ratio must predict the market return. The GARCH(1,1) model that we use for the log returns is

$$\begin{aligned} r_t &= \theta_0 + \theta_1 pd_{t-1} + \epsilon_t \\ \epsilon_t &= q_t z_t, \quad z_t \sim \mathcal{N}.I.D(0,1) \\ q_t^2 &= \omega_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 q_{t-1}^2, \end{aligned}$$

where we refer to α_1 and β_1 as our ARCH and GARCH terms respectively. This model implies that the current price-dividend ratio gives us the expected value of the next period return ($x_t = \theta_0 + \theta_1 pd_t$) and its conditional standard deviation is q_t. The market return and risk-free rate data are from Kenneth French's web page[8], where we used the monthly data from the file containing the Fama-French three factors, and $r_t = \ln(1 + \text{Mkt}_t)$. We

[8] https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

used Robert Shiller's web page[9] for the data on prices and dividends, and constructed the price-dividend ratio as $pd_t = \frac{P_t}{D_t}$. The GARCH(1,1) model yields significant predictability even up to a year (see table 8), however, the one-month lag model yields marginally more significant results. Table 9 shows that the Akaike and Bayasian information criteria select the GARCH(1,1) model over models with more ARCH and GARCH terms. The estimated expected market return (x_t) and its standards deviation (q_t) based on the GARCH(1,1) model are plotted in figure 6 in Appendix C. Consistent with the literature (Cochrane, 2017), the expected market return and its conditional standard deviation are countercyclical.

Having constructed (x_t, q_t) and given that we have rf_t, we can use (8) to construct α_t. Note that $\theta = 1$ implies $\rho = rf - \mu + \sigma\frac{xr}{q}$ via (12), where everything on the right hand side is a data moment[10] and the monthly data yields $\rho = 0.000032$. Finally, we use $\gamma = 0.8$, which constitutes a relatively small deviation from the EU framework and which satisfies the inequalities associated with basic behavioral anomalies that we discuss in section 7. Importantly, the qualitative results do not depend on the exact value of γ, and any value of γ between 0.7 to 0.9 roughly produces the same results.

High sentiment periods are often associated with historical accounts of speculative bubbles and subsequent crashes. We next consider whether our derived sentiment index can track market indexes during the two most recent bubbles in the United States: the dot-com bubble of the early 2000's and the subsequent housing bubble. Figure 2 shows the estimated theoretical α_t, and the months of salient index values during the two most recent bubble episodes: the dot-com bubble and burst, and the housing bubble and burst. From the figure, we see that the derived sentiment measure is near its peak when the NASDAQ peaked in March 2000, and that the sentiment measure had fallen sharply around the time the NASDAQ reached its trough in October 2002. The sentiment measure also experienced a jump upward after the Dow Jones surpassed 14,000 for the first time in July, 2007, and experienced a sharp decline when the Dow Jones lost 20% of its value, dropping below 12,000 in March, 2008, the same month which marked the first failure of a major investment bank involved in the subprime mortgage crisis (Bear Stearns).

[9] `http://www.econ.yale.edu/~shiller/data/ie_data.xls`

[10] Table 10 reports the summary statistics for the model's variables and estimates.

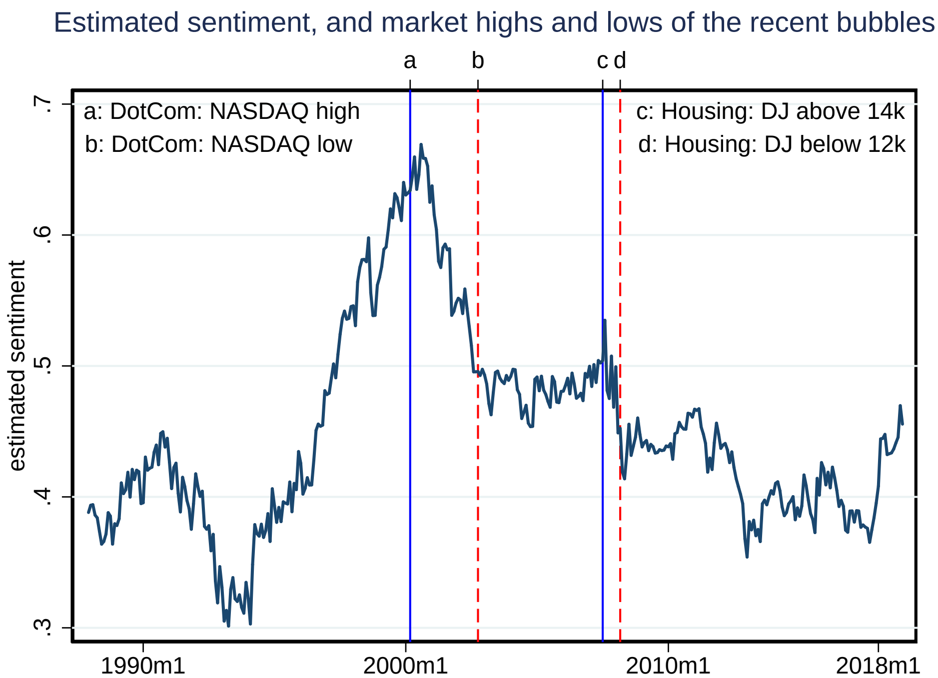

Figure 2: Estimated theoretical sentiment α_t directly from the equity premium equation (8) with maximum and minimum.

Table 3 reports the correlation coefficient between α_t and the survey-based consumer and investor sentiment indexes. The table reveals that the theoretical sentiment index α_t is positively and significantly correlated with the major sentiment indexes in the literature. This provides strong support for the interpretation of α_t as a sentiment index, and for the existence of a link between sentiment and the residual of the Euler equation for the equity premium needed to bridge the gap between the CCAPM and the data.

The consumer sentiment indexes on average have a higher correlation with α_t. Further inspection reveals that their correlation is higher during business cycle expansions. Thus, it seems α_t better describes changes in positive (bullish) sentiment compared to negative (bearish) sentiment. For instance, over the period between the two NBER recessions (1991m4-2001m3) which covers one of the longest U.S. economic expansions, the correlation of α_t with CCI and ICS rises to 0.88 and 0.79 respectively. Table 11 in appendix C contains more information regarding the correlations among α_t, sentiment indexes and model variables.

Table 3: Correlation coefficients between survey based consumer and investor sentiment indexes and the theoretical index α.

sentiment index	$\text{corr}(\alpha, \cdot)$
cci	0.48^{***}
ics	0.40^{***}
bw	0.36^{***}
$bw^{\perp}$	0.35^{***}
$bull$	0.27^{***}
$bear$	-0.09

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

5.1 A Three-Factor Model of Systematic Sentiment

Denoting a survey-based sentiment index by a_t, we expect to have $a_t = A(\alpha_t) + \epsilon_t$, where ϵ_t is the measurement error, and equation (8) in general allows for $a_t = g\big(x_t, q_t, rf_t\big) + \epsilon_t$. As mentioned in the previous section, it is not necessary for $A(\cdot)$ to be linear. Given that x linearly depends on the pd ratio, in this section, we explore the empirical performance of a linear $g\big(pd_t, rf_t, q_t\big)$ to construct a simple three-factor model of systematic sentiment. Without loss of generality, we normalize all the survey-based indexes $a \in \{cci, ics, bw, bw^{\perp}, bull\}$ and variables pd_t, rf_t, q_t to have mean of zero and standard deviation of one. Table 4 reports the performance of regressions

$$a_t = \lambda_1 \, pd_t + \lambda_2 \, rf_t + \lambda_3 \, q_t + \epsilon_t,$$

for the above sentiment indexes, and for comparison the first column reports the same regression for the theoretical α from the previous section. The linear three factor model yields impressive results for the consumer sentiment indexes, explaining 62% and 52% of the variation in CCI and ICS, respectively. Moreover, the size of the coefficients are similar to the theoretical α. The R^2 of the investor sentiment indexes are somewhat lower, yet the t statistics of the pd ratio and r_f are about 9 for the BW indexes. As we saw in the previous section, restriction of the model to economic expansion periods significantly improves its fit for the consumer sentiment indexes. Table 5 shows the R^2 of the three-factor model when

Table 4: Three fundamental factors of the sentiment indexes.

	α	cci	ics	bw	$bw^{\perp}$	$bull$
pd	0.849***	0.602***	0.584***	0.408***	0.437***	0.360***
	(61.31)	(18.36)	(16.02)	(9.10)	(9.90)	(7.29)
r_f	0.353***	0.528***	0.417***	0.403***	0.388***	0.0293
	(25.65)	(16.15)	(11.50)	(9.04)	(8.83)	(0.60)
q	0.336***	-0.291***	-0.325***	-0.0167	-0.126***	-0.0104
	(24.52)	(-8.93)	(-8.95)	(-0.38)	(-2.87)	(-0.21)
R^2	0.93	0.62	0.52	0.28	0.30	0.13

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Monthly data (1987m11-2018m12).

we restrict the time period to economic expansion of the 1990s' (1991m4-2001m3) and the post great recession era (2009m7-2018m12), where the simple three factor model explains between 77% and 62% of the variation in the consumer sentiment indexes. Overall, the fit of the model regarding investor sentiment indexes seem consistent during economic expansion subsamples and the whole sample. Figure 3 shows the performance of the three-factor model by plotting both the sentiment indexes and their predicted values for the whole sample. The same graph for the post great recession period is presented in figure 7 of Appendix B. Overall, the simple three-factor model captures the movements of the consumer sentiment indexes and trends of the investor sentiment indexes very well.

Table 5: The three-factor model of sentiment's fit during economic expansions.

	cci	ics	bw	$bw^{\perp}$	$bull$
	2009m7 – 2018m12				
R^2	0.77	0.62	0.35	0.26	0.098
	1991m4 – 2001m3				
R^2	0.69	0.64	0.28	0.25	0.16

To the extent that empirical measures of sentiment are attempting to identify similar

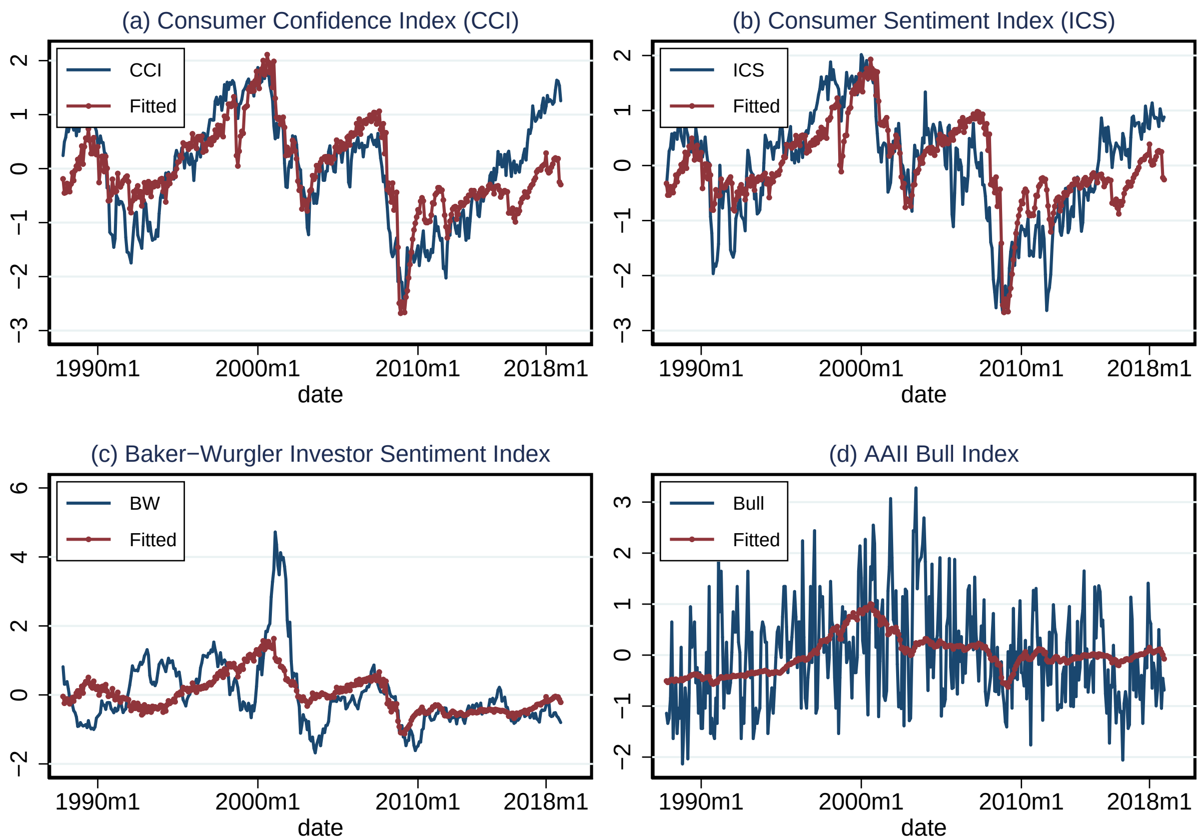

Figure 3: Sentiment indexes and their predicted values using the three-factor model.

underlying constructs, they should (on average) all point in the same direction. That is, they should have significant positive correlations. Importantly, our analysis suggests that once we identify the systematic sentiment factors in the empirical measures and remove them, the residuals need not be correlated anymore. We find that our three factors are systematic in this sense. Table 6 shows the correlations among the sentiment indexes first and then the residuals of sentiment indexes once we remove the three factors. Clearly, there is a high correlation among the consumer sentiment indexes CCI and ICS, and the investor sentiment indexes BW and $\text{BW}^{\perp}$. But there is also a large and significant correlation of about 0.30 among the consumer sentiment indexes and the investor sentiment indexes as shown in the top panel of Table 6. As predicted, these correlations are no longer positive (and in fact become negative and significant) once we remove the three factors, (pd_t, rf_t, and q_t). The correlation of Bull with the other sentiment indexes also becomes negative or insignificant. Thus, after removing the three factors from the sentiment indexes, they point in opposite directions and the residuals can hardly be interpreted as systematic sentiment indexes.

Sibley et al. (2016) show that a set of 13 macroeconomic and financial variables including the 3-month Treasury Bill rate, the dividend yield and stock market volatility collectively explain over 60% of the variation in the Baker-Wurgler sentiment index. Our approach is motivated by theory, which relates sentiment to a narrower set of three fundamental variables, and we find that these three variables are sufficient to remove the systematic component of sentiment from the sentiment indexes.

Table 6: Correlation coefficients among sentiment indexes, and residuals of sentiment indexes once three-factors are removed.

	cci	ics	bw	$bw^{\perp}$	$bull$
cci	1.00				
ics	0.92***	1.00			
bw	0.30***	0.28***	1.00		
$bw \perp$	0.31***	0.28***	0.97***	1.00	
$bull$	0.09*	0.16***	0.12**	0.12**	1.00
	cci_{res}	ics_{res}	bw_{res}	$bw^{\perp}_{res}$	$bull_{res}$
cci_{res}	1.00				
ics_{res}	0.82***	1.00			
bw_{res}	−0.18***	−0.11**	1.00		
$bw^{\perp}_{res}$	−0.23***	−0.18***	0.96***	1.00	
$bull_{res}$	−0.17***	−0.04	−0.03	−0.03	1.00

* p<0.10, ** p<0.05, *** p<0.01

Monthly data (1987m11–2018m12)

6 Sentiment and the Aggregate Stock Market

In addition to providing a resolution to the equity premium puzzle, the Sentiment CAPM predicts other documented effects of sentiment on the aggregate stock market. Here we show that (i) changes in α generate return predictability in the price/dividend ratio, (ii) a positive α produces a non-monotonic (U-shaped) pricing kernel, (iii) an increase in α diminishes the positive relationship between the equity premium and market volatility, (iv) the market Sharpe ratio is higher in low sentiment periods, (v) an increase in α decreases the premium for bearing tail risk, and (vi) a decrease in α produces greater negative skewness in the market's risk-neutral probability density. Return predictability of the price/dividend ratio is a classic finding supported by Campbell and Shiller (1988). U-shaped pricing kernels are supported, for instance, by Sichert (2018). Predictions (iii) and (iv) are supported empirically by Yu and Yuan (2011). Prediction (v) is supported by Chevapatrakul et al. (2019). Prediction (vi) is supported by Han (2007).

6.1 Sentiment and Return Predictability of the P/D Ratio

Consider a Lucas-tree economy with an EU-Hurwicz representative agent that has a time-separable constant relative risk aversion (CRRA) utility of consumption. There is a tree that distributes its dividend every period, and its shares belong to the agent. Assuming that p_t is the ex-dividend price at time t, the representative agent's problem is

$$\max_{s_t, c_t} \tilde{E}_t \sum_{j=0}^{\infty} \beta^j \frac{c_{t+j}^{1-\theta}}{1-\theta}$$

$$c_t + s_t p_t = s_{t-1}(d_t + p_t).$$

Note that the expected value $\tilde{E}_t$ is based on the EU-Hurwicz probability weighting.

The first order condition of the representative agent is

$$\tilde{E}_t \beta \Big(\frac{c_{t+1}}{c_t}\Big)^{-\theta} \Big(\frac{p_{t+1} + d_{t+1}}{p_t}\Big) = 1.$$

The uncertainty that originates from the future dividends d_{t+1}, affects the future return $R_{t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$. We can show that consumption is a fraction of wealth that depends on the expected return and the elasticity of intertemporal substitution (EIS).

Proposition 9. *In the Lucas-tree environment with an EU-Hurwicz representative agent and a CRRA utility function with $\theta \neq 1$, the price-dividend ratio $pd_t \equiv \frac{p_t}{d_t}$ and the expected return $E_t R_{t+1}$ move in opposite directions in response to a permanent or mean-revering change in sentiment.*

Proof. See appendix A. □

Hence, movements in sentiment imply return predictability for pd_t with a negative sign[11].

6.2 Sentiment and the Pricing Kernel Puzzle

The pricing kernel puzzle (see Cuesdeanu and Jackwerth (2018) for a review) is the empirical finding that the pricing kernel or stochastic discount factor is not monotonically decreasing in the market return, but rather increases at the right tail of the distribution, generating a U-shape (e.g., Bakshi et al. (2010), Sichert (2018)). Here we show that the Sentiment CAPM predicts a U-shaped pricing kernel and generates the novel prediction that the pricing kernel should have a more pronounced U shape in high sentiment periods. Consistent with this prediction, Driessen et al. (2019) empirically investigate the time variation of the pricing kernel and report, "the U-shape is stronger in good times than in bad times. The stronger U-shaped pricing kernel indicates that investors are more sensitive towards large negative and positive returns in good times" (p. 3).

Driessen et al. (2019) show that standard asset pricing models including the habit model of Campbell and Cochrane (1999), the long-run risk model of Bansal and Yaron (2004), and the rare disaster models of Gabaix (2012) and Wachter (2013) each produce a monotonically decreasing pricing kernel when projecting the pricing kernel onto the market return, and thus

[11] The restriction $\theta \neq 1$ is a special consequence of the Lucas tree structure. If there are other income sources for the representative agent, predictability can also occur with $\theta = 1$.

cannot explain the U shape. Using the behavioral asset pricing framework of Shefrin (2008), Barone-Adesi et al. (2017) argue for a pricing kernel that has an *inverted* U-shape, arising from an overconfident representative agent that underweights the tails of the distribution. This inverted U-shape is contrary to the U-shape estimated in several recent empirical studies of the pricing kernel (Bakshi et al., 2010; Sichert, 2018; Driessen et al., 2019).

In the Lucas-tree environment, consumption growth approximately equals the wealth growth that in turn approximately equals the market return. As a result, the pricing kernel with the EU-Hurwicz representative agent in terms of the market return has a U-shape. Moreover, the functional form of the stochastic discount factor reveals that the higher the sentiment, the steeper the increasing part of the pricing kernel. Also, the location of the increasing part of the pricing kernel depends on the expected market return and its volatility. Corollary 10 summarizes this result.

Corollary 10. *In the Lucas-tree environment with an EU-Hurwicz representative agent with a CRRA utility function, the pricing kernel has a U-shape in returns. Moreover,*

1. *the higher the sentiment (α_t), the more pronounced the U-shape. That is, the increasing part is steeper.*
2. *the location of the increasing part of the pricing kernel ($\overline{R}_{t+1}$) depends positively on the expected market return (x_t) and the conditional market volatility (q_t).*

Proof. Replacing the consumption growth with the market return in the EU-Hurwicz agent stochastic discount factor and continuing with the assuming that the representative agent truncates the support at $[\underline{R}, \overline{R}]$ yields

$$M_{t+1} = \begin{cases} \gamma\beta \underline{R}_{t+1}^{-\theta} + (1-\gamma)(1-\alpha_t)\beta\underline{R}_{t+1}^{-\theta} & \text{if} \quad R_{t+1} = \underline{R}_{t+1} \\ \gamma\beta R_{t+1}^{-\theta} & \text{if} \quad R_{t+1} \in (\underline{R}_{t+1}, \overline{R}_{t+1}) \\ \gamma\beta\overline{R}_{t+1}^{-\theta} + (1-\gamma)\alpha_t\beta\overline{R}_{t+1}^{-\theta} & \text{if} \quad R_{t+1} = \overline{R}_{t+1}. \end{cases} \tag{13}$$

The probability weights above clearly show that the pricing kernel is U-shaped in R. Moreover, a higher α_t indicates a larger increase on the right side of the pricing kernel. For the second part of the corollary, the conditional log-normality of the market return with $\ln R_{t+1} \sim \mathcal{N}(x_t, q_t^2)$, and the truncation assumption at $\overline{\xi} q_t$, yield $\overline{R}_{t+1} = e^{x_t + \overline{\xi} q_t}$. $\square$

Graph 4 depicts the pricing kernel (13) in which α_t is calculated from (8), with $\overline{\xi} = 4$, and $\underline{\xi} = 5$, and x_t and q_t are from the GARCH(1,1) estimation of the return in the previous section. Specifically, in the right panels $\gamma = 0.8$ (similar to figure 2). In this case, the highest α_t depicted in the top-right panel equals 0.70, at which point $x_t = -0.0044$ and $q_t = 0.045$. In the bottom-right panel, the lowest α_t equals 0.30, at which point $x_t = 0.015$ and $q_t = 0.026$. In the middle-right panel, we used the time average values of α_t, x_t, and q_t, that respectively equal $\alpha = 0.45$, $x = 0.0095$ and $q = 0.041$. In the top-left panel, the highest α_t equals 0.62, and average $\alpha = 0.51$ and the lowest $\alpha_t = 0.43$. Values of x_t and q_t for each left-right pair of panels are the same. Notice that specifying a lower value for γ results in a drop in the variation of α_t as expected.

6.3 Sentiment and the Mean-Variance Relation

French et al. (1987) find that there is a volatility premium for the aggregate stock market: the equity premium is larger in times of higher market volatility. Under the Sentiment CAPM, a decrease in γ increases the effect of stock market volatility and raises the equity premium provided that sentiment is not too high. We establish that the Sentiment CAPM generates a volatility premium in the following corollary:

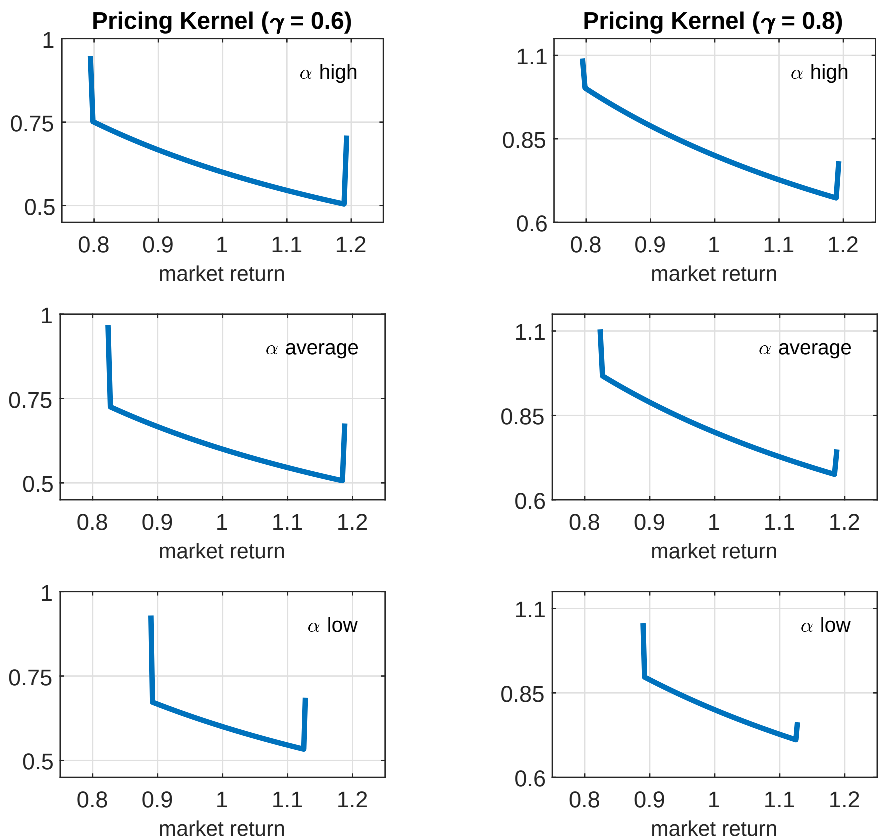

Figure 4: Pricing kernel of the EU-Hurwicz representative agent. The values of x_t and q_t are the estimates from the GARCH(1,1) model, and $\overline{\xi} = 4, \underline{\xi} = 5, \theta = 1$. The right panels are based on $\gamma = 0.8$, and the implied highest, average, and lowest values of α_t. The left panels are based on $\gamma = 0.6$, and the implied highest, average, and lowest values of α_t. Values of x_t and q_t for each left-right pair of panels are the same.

Corollary 11. *(Volatility Premium): With an EH-Hurwicz representative agent and under assumptions 3-5, the equity premium increases with market volatility q_t if the agent is not too optimistic (i.e., $\alpha_t \leq \frac{\underline{\xi}}{\underline{\xi}+\overline{\xi}}$).*

Proof. Under the EU-Hurwicz representative agent and assumptions 3-5, the equity premium is the right-hand side of (7):

$$EP_t = \eta\theta\sigma q_t(1 + x_t + \frac{1}{2}q_t^2) - \\ \frac{1-\gamma}{\gamma}\left[\alpha_t\big(1 - \frac{1}{2}\theta^2\sigma^2 - \theta\overline{\xi}\sigma\big)(xr_t + \overline{\xi}q_t) + (1-\alpha_t)\big(1 - \frac{1}{2}\theta^2\sigma^2 + \theta\underline{\xi}\sigma\big)(xr_t - \underline{\xi}q_t)\right].$$

The change in the equity premium in response to an increase in q_t is:

$$\frac{\partial EP_t}{\partial q_t} = \eta\theta\sigma(1 + x_t + \frac{3}{2}q_t^2) + \\ \left(\frac{1-\gamma}{\gamma}\right)\left(\theta\sigma\big(\underline{\xi}^2 - (\underline{\xi}^2 - \overline{\xi}^2)\alpha_t\big) + \big(1 - \frac{1}{2}\theta^2\sigma^2\big)\big(\underline{\xi} - (\underline{\xi} + \overline{\xi})\alpha_t\big)\right).$$

The first term on the right hand side is always positive. In the second term, since $\frac{1}{2}\theta^2\sigma^2 < 1$ for all the relevant calibrations, it is enough to have $\alpha_t \leq \frac{\underline{\xi}}{\underline{\xi}+\overline{\xi}}$. □

The condition $\alpha_t \leq \frac{\underline{\xi}}{\underline{\xi}+\overline{\xi}}$ is natural and consistent with an overall bias toward ambiguity aversion.

Yu and Yuan (2011) find that the volatility premium is smaller in periods of higher sentiment. This relationship is also predicted by the Sentiment CAPM.

Corollary 12. *With an EH-Hurwicz representative agent and under assumptions 3-5, the volatility premium (the increase in the equity premium in times of greater market volatility) is decreasing in sentiment α_t.*

Proof. See appendix A. □

6.4 Sentiment and the Market Sharpe Ratio

Equation (9) shows that sentiment is (approximately) a function of the market Sharpe ratio with a negative sign. Hence, we have the following immediate corollary:

Corollary 13. *(The Sharpe Ratio and Sentiment): With an EU-Hurwicz representative agent and under assumption 3-5 and up to the first order, the Sharpe ratio is higher in low sentiment periods (low α).*

Corollary 13 is a strong and novel prediction of our analysis. This prediction is also supported empirically. In their empirical study, Yu and Yuan (2011) estimated market volatility with four different volatility models and they consistently observed higher Sharpe ratios in low sentiment periods as predicted by (9). These Sharpe ratios are economically large, ranging from 1.08 to 2.00 for low sentiment periods across their four volatility models for equal-weighted returns and from 0.83 to 2.12 for value-weighted returns. With the current EU-Hurwicz calibration, such Sharpe ratios can be achieved if the sentiment falls to about 50% of its mean value.

6.5 Sentiment and the Tail-Risk Premium

Recent studies have documented a tail risk premium for the aggregate stock market: the equity premium is larger when the market has a fatter left tail, and the tail risk premium is smaller in periods of higher sentiment (Chevapatrakul et al., 2019). In the cross-section, Chabi-Yo et al. (2018) (in their Table 4) find that the Baker-Wurgler sentiment index is significantly negatively related to the premium on stocks with strong lower-tail dependence with the market return. The Sentiment CAPM predicts the existence of a tail risk premium as well as its dependence on sentiment. The following corollary immediately follows from equation (5).

Corollary 14. *(Tail Risk Premium): With an EU-Hurwicz representative agent, the equity premium increases in times of greater tail risk (lower worst-case scenario, $\underline{R}_{t+1}$). Moreover, this tail risk premium is decreasing in the sentiment α_t.*

6.6 Sentiment and the Risk-Neutral Distribution

Intuitively, a smaller α that is consistent with a more pessimistic EU-Hurwicz representative agent, leads to greater negative skewness of the risk-neutral probability density. In other words, the skewness of the risk-neutral probability density is increasing in α. The following corollary shows that this is indeed the case provided that the consumption growth distribution is not too dispersed (as in the log-normal distribution).

Corollary 15. *If the standard deviation of the consumption growth process is sufficiently small, and γ is sufficiently large, then the skewness of the risk-neutral distribution for the EU-Hurwicz representative agent is increasing in sentiment α_t.*

Proof. See appendix A. □

The first condition in Corollary 15 is natural and the second is very plausible. As Freeman (2004) notes, "Aggregate consumption growth has exhibited very low volatility over the past century" (p. 927), consistent with a small volatility of consumption growth assumed in Corollary 15. A value of γ that is sufficiently high is consistent with our finding that only a small deviation from the CCAPM (e.g., $\gamma = 0.8$) is needed to generate the full magnitude of the historical equity premium. As we show in section 7, similarly small deviations from EU (high values of $\gamma < 1$) are sufficient to explain prominent behavioral anomalies from economics laboratory experiments.

The monotonic relationship between sentiment and risk-neutral skewness that is predicted in Corollary 10 has empirical support. In particular, Han (2007) studies whether investor sentiment affects the prices of S&P 500 index options. He observes that "the risk-neutral skewness of the monthly index return is more (less) negative when market sentiment becomes more bearish (bullish)" (p. 387). This observation holds under the Sentiment CAPM since a decrease in α reflects more pessimistic (bearish) sentiment, and Corollary 15 shows that as α decreases, the risk-neutral skewness becomes more negative.

7 Microfoundations

The Sentiment CAPM provides a bridge between behavioral biases in individual choice and aggregate stock market anomalies by linking sentiment to prospect theory. In this section we consider the microfoundations for the Sentiment CAPM.

7.1 The Representative Agent and Microeconomic Data

The representative agent in asset pricing models often bears little resemblance to subjects in economic laboratory experiments. First, the representative agent in asset pricing models is often far more risk-averse than subjects in laboratory studies. Second, the representative agent is often assumed to satisfy the expected utility axioms and to be risk-averse or risk-neutral. Such an agent does not exhibit the commonly observed choice patterns that violate the independence axiom or the assumption of uniform risk aversion. We show that the EU-Hurwicz agent calibrated from Section 4 to match the historical equity premium and the risk-free rate also exhibits systematic violations of the indendence axiom and systematic deviations from risk aversion that are observed in experiments.

Following the application to behavioral anomalies, we further investigate the microfoundations of EU-Hurwicz preferences by considering its implications for expected utiltiy anomalies in individual financial decisions, and establishing an aggregation result in which markets where agents have heterogeneous subjective probability beliefs and heterogeneous ambiguity attitudes can give rise to a representative EU-Hurwicz under certain conditions.

7.1.1 The Allais Paradox over Large and Small Stakes

We next apply the parameter values for EU-Hurwicz preferences that were calibrated in Section 4 to match both the equity premium and the risk-free rate, to three robust findings from economics lab experiments: The Allais paradox (Allais, 1953), the common ratio effect (Kahneman and Tversky, 1979), and the fourfold pattern of risk preferences (Tversky and Kahneman, 1992). We find that the calibrated values from Section 4 resolve these behavioral

Table 7: The Allais Paradox

Choice 1			Choice 2	
A	$(x, p; y, 1-q; 0, q-p)$		**A'**	$(x, p; 0, 1-p)$
B	$(y, 1)$		B'	$(y, q; 0, 1-p)$

anomalies and also predict the observed stake-dependence of the Allais paradox that has been documented in the literature.

The Allais paradox (also known as the common consequence effect) is among the best-known systematic empirical violations of EU. The effect has been documented over large stakes (Allais, 1953; Kahneman and Tversky, 1979), but is not observed over small stakes (Fan, 2002; Huck and Müller, 2012). The Allais paradox involves two choices, each between a pair of lotteries with known probabilities. The form of these choices is shown in Table 7, where $x > y$ and $q > p$: The lottery $(x, p; y, 1-q; 0, q-p)$ offers prize x with probability p, prize y with probability $1-q$, and a payoff of 0 with probability $q-p$ (with analogous notation used for the other lotteries). In the classic version of the paradox, $y = \$1$ million, $x = \$5$ million, $q = 0.11$, and $p = 0.10$. Allais found that many people prefer B in Choice 1 which offers \$1 million with certainty, over lottery A, but prefer A' in Choice 2 which offers a chance at a larger prize. This pattern of preferences violates EU since lotteries A' and B' are constructed from lotteries A and B, respectively, by replacing an 89% chance of \$1 million with an 89% chance of \$0. The observed reversal in preference violates the EU independence axiom.

Kahneman and Tversky (1979) replicated Allais' finding with more modest stakes, setting $(x, y, p, q) = (\$2500, \$2400, 0.33, 0.34)$. They found that most of their experimental subjects preferred B in Choice 1 and A' in Choice 2. Fan (2002) employed smaller stakes, setting $(x, y, p, q) = (\$100, \$20, 0.10, 0.11)$ and found that at such stakes, the Allais paradox largely disappears. Huck and Müller (2012) employed even smaller stakes, setting $(x, y, p, q) = (\$25, \$5, 0.10, 0.11)$. They also found little evidence of the Allais preference pattern. Instead, Fan and Huck and Muller found that people typically chose the two riskier lotteries in both choices. There is a strong intuition for observing the paradox at the large stakes used by Allais and Kahneman and Tversky and for not observing the paradox at small stakes: gambling when a sure \$1 million or even a sure \$2400 is on the table is less attractive than gambling when the sure option is \$5 or even \$20. When no sure option is on the table, the larger difference in payoffs outweighs the 0.01 difference in probabilities, resulting in the selection of lottery A' in Choice 2. A complete explanation of the Allais paradox should then explain the occurrence of the paradox (choosing B and A') at the large stakes used by Allais and Kahneman and Tversky and the selection of A and A' at the small stakes used by Fan and Huck and Muller. The most widely used form of cumulative prospect theory with a power value function defined over gains and losses (Tversky and Kahneman, 1992) cannot explain this full pattern, even allowing for any probability weighting function (Schneider and Day, 2016). In contrast, the EU-Hurwicz model with log utility defined over wealth can simultaneously explain the paradox at large and small stakes.

Figure 5 shows values of γ and levels of wealth for which the EU-Hurwicz model calibration from Section 4 that matches both the historical equity premium and the risk-free rate also simultaneously explains the observed stake dependence of the Allais paradox in the above four experiments, as well as classic experimental versions of the common ratio effect and the fourfold pattern of risk preferences discussed in the following subsections.

7.1.2 The Common Ratio Effect

The EU-Hurwicz agent calibrated to match the equity premium and risk-free rate in section 4 also exhibits the classical form of the common ratio effect, a different violation of independence, in which a person prefers \$3000 with certainty over an 80% chance of \$4000, but prefers a 20% chance of \$4000 over a 25% chance of \$3000 (Kahneman and Tversky, 1979).

7.1.3 The Fourfold Pattern of Risk Attitudes

While the Allais paradox and common ratio effect demonstrate how observed behavior systematically violates the EU independence axiom, the fourfold pattern of risk attitudes (Tversky and Kahneman, 1992) provides a classic demonstration of how observed behavior system-

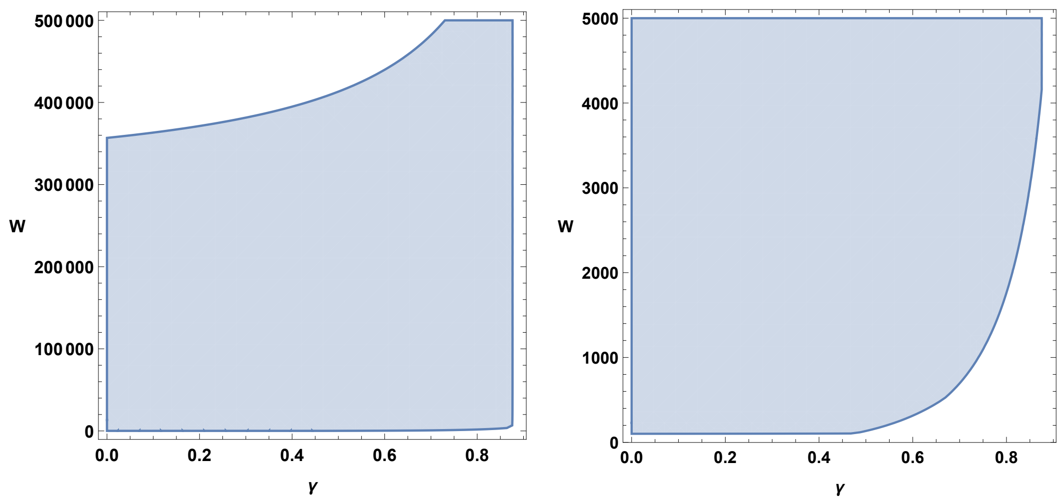

Figure 5: Regions of wealth and parameter γ that simultaneously resolve three behavioral anomalies (the Allais paradox, common ratio effect, and fourfold pattern) and match both the equity premium and the risk-free rate. The left panel emphasizes large wealth levels while the right panel emphasizes very small wealth levels. As the figure shows, higher values of γ are needed for higher values of wealth.

atically deviates from risk aversion. Under the fourfold pattern, a decision maker exhibits risk aversion for gains of high probability and losses of low probability, while exhibiting risk-seeking behavior for gains of low probability and losses of high probability. As an illustration, Tversky and Kahneman found that most of their experimental subjects preferred (i) receiving \$95 with certainty over a 95% chance of winning \$100; (ii) losing \$5 with certainty over a 5% chance of losing \$100; receiving a 5% chance of winning \$100 over \$5 with certainty; (iii) taking a 95% chance of losing \$100 over losing \$95 with certainty. In each case, the complementary probability corresponded to an outcome of \$0. The fourfold pattern is also a robust prediction of the representative EU-Hurwicz agent as shown in Figure 5.

7.2 Expected Utility Violations in Financial Decisions

We next show that the EU-Hurwicz agent also explains empirical violations of EU in investment and insurance decisions. Consider the static problem of an EU-Hurwicz agent who is deciding how much to insure against a stochastic loss, L with maximum loss $L_{\underline{s}}$ and a best-case scenario of no loss. The agent is charged an insurance premium of ky where y is the premium for purchasing full insurance. The agent chooses the level of insurance coverage $k \in [0,1]$ that maximizes (14) where u is strictly increasing, concave, and twice differentiable.

$$\gamma E u(w - ky - (1-k)L_s) + (1-\gamma)\left[\alpha u(w - ky) + (1-\alpha)u(w - ky - (1-k)L_{\underline{s}})\right]. \quad (14)$$

Note that (14) can be written as

$$\gamma E u(m + xr_s) + (1-\gamma)\left[\alpha u(m + xr_{\overline{s}}) + (1-\alpha)u(m + xr_{\underline{s}})\right]. \quad (15)$$

where $m = w - y, x = 1 - k$ and the payoff in state s is $r_s = y - L_s$. As noted by Armantier et al. (2018), this change in notation demonstrates that the coinsurance demand model of Mossin (1968) is equivalent to the portfolio choice model of Pratt (1964). Both of these classical models were developed for the expected utility case ($\gamma = 1$).

In (15), x is the amount invested in a risky asset (such as the market portfolio). The optimal solution to (15) solves (16):

$$\gamma E[u'(m + xr_s)r_s] + (1-\gamma)\left[\alpha u'(m + xr_{\overline{s}})r_{\overline{s}} + (1-\alpha)u'(m + xr_{\underline{s}})r_{\underline{s}}\right] = 0. \quad (16)$$

When $\gamma = 1$, we arrive at the well-known implication of EU that $x > 0$ if and only if the risky asset has a positive expected payoff, regardless of the agent's degree of risk aversion. This general implication of EU is contrary to the empirical finding that many households have limited or no participation in the stock market (Mankiw and Zeldes, 1991), despite the large positive expected return on stocks.

When $\gamma = 1$, we also arrive at the well-known implication of EU that $k < 1$ (it is not optimal to purchase full insurance) regardless of the degree of risk aversion if the premium is not actuarially fair ($y > E(L_s)$), in contrast to the large premiums many households are willing to pay to eliminate risk.

The EU-Hurwicz model provides an explanation for both the limited participation puzzle and the limited insurance puzzle. To illustrate, let $\gamma < 1$ and let $\mathrm{E}[r_s] = 0$. Then, for sufficiently low α, (16) turns negative if $r_{\underline{s}} < 0$ as more weight is shifted to the lowest payoff.

It follows that for a risky asset with sufficiently small but positive expected payoff, and sufficiently small α, an EU-Hurwicz agent will choose not to invest in the asset ($x = 0$). It is also straightforward to show that the amount invested in the risky asset is increasing in α. These predictions are supported by Dimmock et al. (2016) who find a negative relationship between ambiguity aversion and stock market participation, and by Angelini and Cavapozzi (2017) who find that optimism is positively related to both stock ownership and the share of wealth invested in stocks. Equivalently, when α is sufficiently low, an EU-Hurwicz agent will purchase full insurance ($k = 1$) even in cases where the premium is not actuarially fair.

7.3 Aggregation

We next establish an aggregation result in proposition 17 which provides conditions such that a market of EU-Hurwicz agents with heterogeneous subjective probability beliefs and heterogeneous ambiguity attitudes has the same prices as a market with a representative EU-Hurwicz agent. This latter interpretation of proposition 17 links behavioral biases in individual choice to aggregate stock market anomalies.

We also provide a corollary to proposition 17 in which a market with some EU agents and some noise traders (modeled as Hurwicz agents) with heterogeneous ambiguity attitudes has the same prices as a different market with a representative agent that has a prospect theory probability weighting function (in particular, a representative EU-Hurwicz agent).

Consider a Lucas tree economy with a risky asset and a risk-free asset that is of zero aggregate supply. The economy is populated with a unit measure of EU-Hurwicz agents that trade based on their subjective probability distributions and ambiguity attitudes, both of which can vary across agents. Let E_{it} denote the expectation operator for agent $i \in [0, 1]$, let α_{it} denote the ambiguity attitude (degree of optimism for agent $i \in [0, 1]$) and let γ_{it} denote the degree of uncertainty perceived by agent i. Suppose that one share of the risky asset that is valued at P_t bears the stochastic dividend D_{t+1} with the ex-dividend price P_{t+1} next period. The risk-free asset has a payoff of one, and is priced at P_t^b.

Definition 16. A *representative agent* is defined such that if the measure one of traders is substituted with the representative agent, the prices of the risky and riskless assets remain the same.

Proposition 17. *Consider a market of EU-Hurwicz agents with heterogeneous expectations E_{it}, heterogeneous ambiguity attitudes α_{it}, heterogeneous perceptions of uncertainty, γ_{it} and logarithmic utility of consumption. Let there be no short-selling and let agents have the same temporal discount rate β and the same expected value for the inverse of the aggregate dividend growth. Then there exists a representative agent.*

Proof. See appendix A. □

In general, an EU-Hurwicz agent is not equivalent to an EU agent with a specific probability weighting. The reason is that for the EU-Hurwicz agent the best and worst-case scenarios can reverse if the EU-Hurwicz agent takes a short position. In which case the optimism and pessimism weights reverse and cannot be interpreted as probabilities that are assigned to states. However, in the economy we consider, imposing the constraint that agents cannot sell short does not affect the equilibrium since in equilibrium the agents do not sell short. Under this no short-selling constraint, an EU-Hurwicz agent in the economy we consider is formally equivalent to an EU agent that puts additional probability weight on the two extreme outcomes of the distribution.

An *EU-Hurwicz agent* is defined in Definition 1. For this part, we separately define an EU agent and a Hurwicz agent for the cases where $\gamma = 1$ and $\gamma = 0$, respectively. (with logarithmic utility). A Hurwicz agent does not have beliefs represented by a unique subjective probability distribution and instead chooses entirely based on optimism or pessimism by maximizing a convex combination of the best and worst-case utilities across states.

Corollary 18. *Consider a market with measure γ of EU agents and measure $1 - \gamma$ of Hurwicz agents with ambiguity attitude α_t who satisfy the conditions in Proposition 17. If there is no short-selling, then the market has the same equilibrium prices as an economy with a representative EU-Hurwicz agent.*

As an example, consider the case of log-normal returns, where $\ln R_{t+1} = r_{m,t+1}$ has a conditional normal distribution with $r_{m,t+1} \sim \mathcal{N}(\mu, \sigma_t)$. Note that $\frac{D_t}{D_{t+1}} = [\beta R_{t+1}]^{-1}$ because

$R_{t+1} = \frac{P_{t+1}+D_{t+1}}{P_t} = \frac{D_{t+1}}{\beta D_t}$. Thus, a Hurwicz agent who truncates the left and right tails of $\mathcal{N}(\mu, \sigma_t)$ at $\mu - \underline{\xi}\sigma_t$ and $\mu + \overline{\xi}\sigma_t$, assigns the expected value

$$E_{h,t}\frac{D_t}{D_{t+1}} = \frac{1}{\beta}E_{h,t}e^{-r_{m,t+1}} = \frac{1}{\beta}\left(1 - \mu + \big(-\alpha_t\overline{\xi} + (1-\alpha_t)\underline{\xi}\big)\sigma_t\right),$$

where the subjective probability of the right tail is α_t, and the last equation uses the approximation $e^x \approx 1 + x$. On the other hand, an EU agent that uses the true probability has the following expected value

$$E_t\frac{D_t}{D_{t+1}} = \frac{1}{\beta}E_te^{-r_{m,t+1}} = \frac{1}{\beta}\left(1 - \mu + \frac{1}{2}\sigma_t^2\right).$$

Thus, for a representative agent to exist, we need sentiment α_t to satisfy

$$\frac{1}{2}\sigma_t = (1-\alpha_t)\underline{\xi} - \alpha_t\overline{\xi}.$$

Now consider an EU-Hurwicz agent with the following specification

$$\tilde{E}_t\frac{D_t}{D_{t+1}} = \frac{1}{\beta}\tilde{E}_te^{-r_{m,t+1}} = \frac{1}{\beta}\left(\gamma E_te^{-r_{m,t+1}} + (1-\gamma)E_{h,t}e^{-r_{m,t+1}}\right).$$

This specification clearly satisfies the condition of proposition 17, and hence, can be thought of as a representative agent.

8 Conclusion

We conducted the first empirical study of the Sentiment CAPM and find encouraging results: The sentiment parameter, α, in the model is positively and significantly correlated with the leading empirical sentiment indexes. The Sentiment CAPM generates the full magnitude of the historical equity premium and produces a low and stable risk-free rate, even with low risk aversion (*e.g.*, log utility), even with a small deviation from EU (*e.g.*, with $\gamma = 0.8$)

and even with a small degree of ambiguity aversion (*e.g.*, with α above 0.4).[12] The model predicts three fundamental economic quantities to be systematic determinants of sentiment: the risk-free rate, the price/dividend ratio, and the conditional stock market volatility. We used these quantities to construct a simple three-factor model of systematic sentiment and find that these quantities jointly explain roughly 30% to 60% of the variation in the leading sentiment indexes. We also found that while the consumer and investor sentiment indexes are positively correlated, the residuals of these indexes become uncorrelated or negatively correlated after removing these three factors from the indexes. This suggests that the three factors indeed capture the systematic component of sentiment.

We further demonstrated that, in addition to the equity premium puzzle, the Sentiment CAPM provides an explanation for five other anomalies for the aggregate stock market: The return predictability of the price/dividend ratio, the relation between the equity premium and market volatility (and its dependence on sentiment), the relation between the equity premium and market tail risk (and its dependence on sentiment), the effect of sentiment changes on the risk-neutral probability density, and the pricing kernel puzzle.

We also considered the microfoundations of the Sentiment CAPM, demonstrating that the same set of calibrated parameter values that match both the historical equity premium and the risk-free rate also robustly generate three of the most basic behavioral anomalies from economics laboratory experiments. We concluded our analysis with an aggregation result which provided conditions such that a market of EU-Hurwicz agents with heterogeneous subjective probabilities and heterogeneous ambiguity attitudes generates the same equilibrium prices as a market with a representative EU-Hurwicz agent. We observed that the same conditions imply an aggregation result where a market with some EU agents and some noise traders (modeled as Hurwicz agents) who trade on optimism or pessimism generates the same equilibrium prices as a market with a representative EU-Hurwicz agent.

Future research is needed to investigate other implications of the Sentiment CAPM. Of first order importance is the study of the pricing implications of the Sentiment CAPM in bond and option markets, as well as the implications of the Sentiment CAPM for the cross-

[12] Indeed, the average value of the theoretical sentiment index α_t over our sample period that we estimated in section 5 and that enables the model to match the Euler equation for the equity premium is 0.45.

section of returns.

The Sentiment CAPM provides a simple approach to relating the asset pricing literatures on disaster risk, ambiguity aversion, positive skewness, market sentiment, and prospect theory. In doing so, the Sentiment CAPM provides a formal approach to incorporating the qualitative and elusive concept of sentiment into quantitatve models of equilibrium asset pricing. The Sentiment CAPM thereby provides a means to study the effects of animal spirits (Keynes, 1936) or irrational exuberance (Shiller, 2000) on the aggregate stock market.

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Appendix A Proofs

Corollary 7:

Proof. The risk-free rate equation (10) can be re-written as

$$\gamma e^{-\rho-\theta\mu+\frac{1}{2}\theta^2\sigma^2} + (1-\gamma)\left(\alpha_t e^{-\rho-\theta\mu-\theta\bar{\xi}\sigma} + (1-\alpha_t)e^{-\rho-\theta\mu+\theta\underline{\xi}\sigma}\right) = 1 - r_{f,t},$$

where the equality comes from the fact that $\frac{1}{1+r_t^f} = e^{-\log(1+r_t^f)} = e^{-r_{f,t}}$, and that $e^x = 1 + x$, for small x. We continue using the approximation to rewrite the left hand side of the above

$$\gamma\big(1-\rho-\theta\mu+\frac{1}{2}\theta^2\sigma^2\big) + (1-\gamma)\Big(\alpha_t(1-\rho-\theta\mu-\theta\bar{\xi}\sigma) + (1-\alpha_t)(1-\rho-\theta\mu+\theta\underline{\xi}\sigma)\Big)$$
$$= 1 - r_{f,t}.$$

Simplifying the above yields equation (10). □

Corollary 9:

Proof. Consumption equals dividends in the Lucas-tree model, and hence, the Euler equation can be written as the following

$$pd_t = \tilde{E}_t \beta \Big(\frac{d_{t+1}}{d_t} \Big)^{1-\theta} \Big(1 + pd_{t+1} \Big).$$

The solution to the log-linearized version of this equation is

$$\hat{pd}_t = (1-\theta) \sum_{j=0}^{\infty} \left(\beta \Delta d^{1-\theta} \right)^j \tilde{E}_t \hat{\Delta} d_{t+j+1},$$

given that $\beta \Delta d^{1-\theta} < 1$. In this environment, a rise in sentiment increases $\tilde{E}_t \hat{\Delta} d_{t+j+1}$, while the physical distribution of dividend growth has not changed, that is, $E_t \hat{\Delta} d_{t+j+1}$ is unaffected. If $\theta \in (0,1)$, the rise in sentiment increases pd_t. Moreover, we can write the return as

$$R_{t+1} = \frac{(1 + pd_{t+1})}{pd_t} \frac{d_{t+1}}{d_t}.$$

If there is no change in the distribution of future dividend growth, the expected return $E_t R_{t+1}$ drops even if the rise in sentiment is permanent. The drop in expected return is more pronounced if the sentiment series is mean reverting, that is, if tomorrow's sentiment is lower than today's, so that $pd_{t+1} < pd_t$.

For $\theta > 1$, the above argument shows that in response to a permanent or mean-reverting shock to sentiment, pd_t drops and the expected return rises. $\square$

Corollary 12:

Proof. It is enough to show that the effect of q_t on equity premium decreases with sentiment, that is, $\frac{\partial}{\partial \alpha_t} \frac{\partial EP_t}{\partial q_t} < 0$. Formally, we have

$$\frac{\partial^2 EP_t}{\partial q_t \partial \alpha_t} = - \left(\frac{1-\gamma}{\gamma} \right) \left((\underline{\xi}^2 - \overline{\xi}^2) \theta \sigma + (\underline{\xi} + \overline{\xi})(1 - \frac{1}{2} \theta^2 \sigma^2) \right).$$

This expression is negative since we have $\overline{\xi} < \underline{\xi}$ and $\frac{1}{2}\theta^2\sigma^2 < 1$ in all relevant calibrations. Interestingly, this expression does not depend on the value of α_t. $\square$

Corollary 15:

Proof. First, notice that as long as the distribution of the consumption growth has a small standard deviation, we can approximate the physical probability density of the EU-Hurwicz agent with three points: the mean of the distribution (for the EU part), and the two extreme points above and below the mean (for the Hurwicz part). Without loss of generality, we can shift the three-point distribution such that the mean is on zero, and the other two points are at $-\xi$ and ξ (assuming symmetry for simplicity). Moreover, the probabilities of the points $\{-\xi, 0, \xi\}$ are $\{(1-\gamma)(1-\alpha), \gamma, (1-\gamma)\alpha\}$. The Skewness of this distribution is

$$\frac{(2\alpha-1)(1-\gamma)\left(\gamma(2\gamma-1)-8\alpha(1-\alpha)(1-\gamma)^2\right)}{\left((1-\gamma)(4\alpha(1-\alpha)(1-\gamma)+\gamma)\right)^{\frac{3}{2}}}.$$

The derivative of this expression with respect to α is

$$\frac{2\left(\gamma-8\alpha(1-\alpha)(1-\gamma)\right)}{\left((\gamma-1)\left(4\alpha(1-\alpha)(1-\gamma)+\gamma\right)^5\right)^{\frac{1}{2}}}.$$

Thus, $\frac{\partial \text{ Skewness}}{\partial \alpha} > 0$ if

$$\gamma > \frac{8\alpha(1-\alpha)}{1+8\alpha(1-\alpha)}.$$

For the sake of numerical comparison, the right hand side is largest at $\alpha = 0.5$, for which the lower bound on γ is $\frac{2}{3}$. As α approaches 0 or 1, the lower bound on γ approaches zero. More generally, for the asymmetric three point distribution $\{-\underline{\xi}, 0, \overline{\xi}\}$ with weights $\{(1-\gamma)(1-\alpha), \gamma, (1-\gamma)\alpha\}$, the partial derivative condition $\frac{\partial \text{ Skewness}}{\partial \alpha} > 0$ is equivalent to having

$$\gamma > \frac{\alpha(1-\alpha)(\underline{\xi}+\overline{\xi})^3}{-(1-\alpha)^2\underline{\xi}^3+(2+\alpha-3\alpha^2)\underline{\xi}^2\overline{\xi}+\alpha(5-3\alpha)\underline{\xi}\overline{\xi}^2-\alpha^2\overline{\xi}^3},$$

whose behavior is almost identical to the symmetric case for our calibration of $\{\underline{\xi}, \overline{\xi}\}$. $\square$

Proposition 17:

Proof. Under the no short-sale constraint, an EU-Hurwicz agent in the economy we consider is formally equivalent to an EU agent that puts additional probability weight on the two extreme outcomes of the distribution that will be truncated when unbounded relative to an EU agent with correct probability beliefs.

We can then find the general equilibrium solution to the economy populated with a unit measure of agents indexed by $i \in [0,1]$. The maximization problem for agent i is

$$\max_{C_{it}, B_{it}, S_{it}} \log C_{it} + E_t \beta \log C_{it+1} + \cdots$$
$$C_{it} + P_t^b B_{it} + P_t S_{it} = B_{it-1} + (D_t + P_t) S_{it-1},$$

where C_{it} is the consumption and B_{it}, S_{it} are the bond and stock holdings of agent i for the next period. The maximization yields the Euler equations

$$E_{it} \beta \frac{C_{it}}{C_{it+1}} R_{t+1} = 1$$
$$E_{it} \beta \frac{C_{it}}{C_{it+1}} R_t^f = 1,$$

where $R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}$, and $R_t^f = \frac{1}{P_t^b}$. The market clearing conditions for this Lucas tree economy are $\int_0^1 C_{it} di = D_t$, $\int_0^1 B_{it} di = 0$, and $\int_0^1 S_{it} di = 1$. The general equilibrium of this economy is the set of allocations $\{C_{it}, B_{it}, S_{it}\}$ and prices $\{P_t, P_t^b\}$ such that given the prices, the allocations satisfy the Euler equations and the market clearing conditions.

Claim. The general equilibrium solution of the Lucas tree economy is the allocations $C_{it} = (1-\beta) W_{it}$, $B_{it} = 0$, $S_{it} = \frac{\beta W_{it}}{P_t}$ (where $W_{it} = B_{it-1} + (D_t + P_t) S_{it-1}$ is the wealth of agent i at time t, and $W_t = P_t + D_t$ is the aggregate wealth) and the prices $P_t = \frac{\beta}{1-\beta} D_t$, $P_t^b = \left(E_{it} \beta \frac{D_t}{D_{t+1}} \right)^{-1}$.

Proof. It is easy to verify that given the prices, the allocations satisfy the Euler equations and the market clearing conditions. Note that the reason that all agents agree on the bond price is that they all have the same expected value of the inverse of the next period aggregate dividend growth $E_{it} \frac{D_t}{D_{t+1}}$. □

Next, observe that replacing all the agents with the representative agent is the same as removing i in the above equations. Similarly, the general equilibrium consumption is $C_t = (1-\beta)W_t$, with the market clearing condition $B_t = 0$, $S_t = 1$, $C_t = D_t$, and the same stock price. The bond price $P_t^b = \left(E_t \beta \frac{D_t}{D_{t+1}}\right)^{-1}$ agrees with the previous economy if and only if $E_t \frac{D_t}{D_{t+1}} = E_{it} \frac{D_t}{D_{t+1}}$. $\square$

Appendix B Auxiliary Grpahs

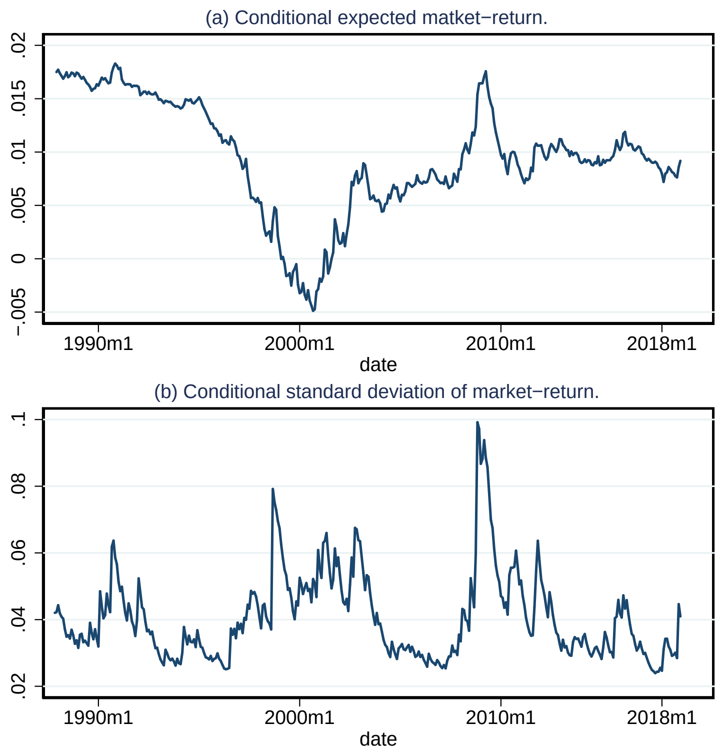

Figure 6: Expected market return and its standard deviation using the GARCH(1,1) model. Monthly data (1987m11-2018m12).

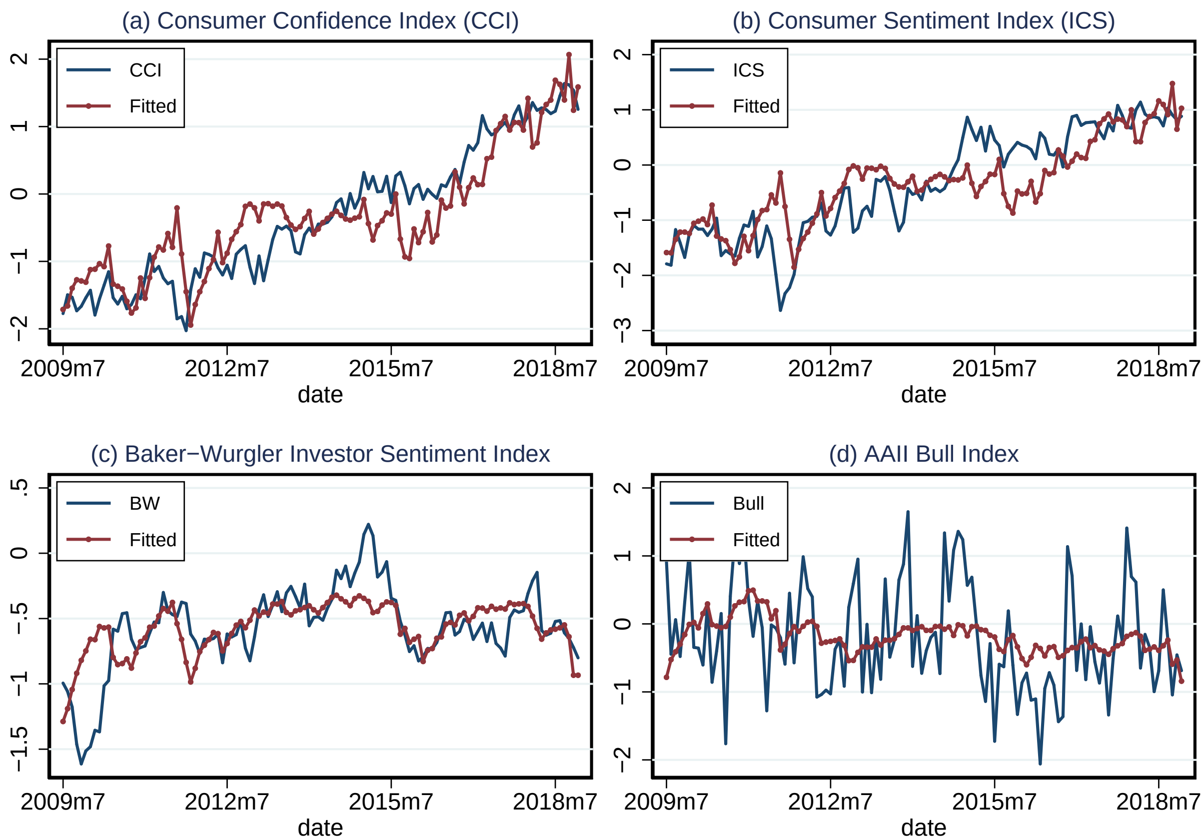

Figure 7: Sentiment indexes and their predicted values using the three-factor model.

Appendix C Auxiliary Tables

Table 8: GARCH(1,1) specifications of market return for up to 12 lags of the *pd* ratio.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
r_t												
L.pd	-3.6e-04**											
	(-2.54)											
L2.pd		-3.4e-04**										
		(-2.38)										
L3.pd			-3.2e-04**									
			(-2.24)									
L4.pd				-3.2e-04**								
				(-2.24)								
L5.pd					-3.1e-04**							
					(-2.16)							
L6.pd						-3.2e-04**						
						(-2.20)						
L7.pd							-3.0e-04**					
							(-2.05)					
L8.pd								-3.0e-04**				
								(-2.06)				
L9.pd									-3.1e-04**			
									(-2.10)			
L10.pd										-3.3e-04**		
										(-2.20)		
L11.pd											-3.1e-04**	
											(-2.07)	
L12.pd												-3.1e-04**
												(-2.11)
Constant	2.8e-02***	2.7e-02***	2.5e-02***	2.5e-02***	2.5e-02***	2.5e-02***	2.4e-02***	2.4e-02***	2.5e-02***	2.6e-02***	2.5e-02***	2.5e-02***
	(3.70)	(3.54)	(3.41)	(3.40)	(3.32)	(3.35)	(3.21)	(3.22)	(3.24)	(3.34)	(3.23)	(3.26)
ARCH												
L.arch	1.7e-01***	1.7e-01***	1.7e-01***	1.7e-01***	1.7e-01***	1.7e-01***	1.7e-01***	1.7e-01***	1.7e-01***	1.7e-01***	1.7e-01***	1.7e-01***
	(3.88)	(3.87)	(3.87)	(3.86)	(3.84)	(3.84)	(3.87)	(3.87)	(3.89)	(3.84)	(3.86)	(3.86)
L.garch	7.9e-01***	7.9e-01***	7.9e-01***	7.9e-01***	7.9e-01***	7.9e-01***	7.9e-01***	7.8e-01***	7.9e-01***	7.9e-01***	7.9e-01***	7.9e-01***
	(14.60)	(14.75)	(14.73)	(14.73)	(14.72)	(14.67)	(14.73)	(14.72)	(15.12)	(14.92)	(15.14)	(15.38)
Constant	9.4e-05**	9.4e-05**	9.4e-05**	9.3e-05**	9.4e-05**	9.4e-05**	9.5e-05**	9.5e-05**	9.4e-05**	9.5e-05**	9.4e-05**	9.3e-05**
	(2.01)	(1.99)	(1.99)	(1.98)	(1.98)	(1.99)	(1.99)	(2.00)	(2.01)	(2.02)	(2.02)	(2.03)
N	373	372	371	370	369	368	367	366	365	364	363	362

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 9: Information criteria for different GARCH specifications of the log market return.

GARCH	LL	df	AIC	BIC
(1,1)	682.06	5	-1354.12	-1334.51
(2,1)	682.54	6	-1353.08	-1329.55
(1,2)	682.56	6	-1353.11	-1329.58
(2,2)	682.57	7	-1351.15	-1323.70

Monthly data (1987m11-2018m12), N=373.

Table 10: Summary statistics of main variables (logged nominal returns).

Variable	**Mean**	**Std. Dev.**	**Min.**	**Max.**	**N**
Δc	0.0032	0.0033	-0.0162	0.0138	373
rf	0.0027	0.0022	-0.0001	0.0084	374
x	0.0095	0.0053	-0.0049	0.0183	373
q	0.0407	0.0131	0.024	0.0992	374
xr	0.0067	0.0055	-0.0101	0.0172	373

Monthly data (1987m11-2018m12).

Table 11: Correlation coefficients among α_t, sentiment indexes and other model variables.

	α	cci	ics	bw	$bw^{\perp}$	$bull$	$bear$	pd	r_f	q	xr	Δc
α	1.00											
cci	0.48***	1.00										
ics	0.40***	0.92***	1.00									
bw	0.36***	0.30***	0.28***	1.00								
$bw^{\perp}$	0.35***	0.31***	0.28***	0.97***	1.00							
$bull$	0.27***	0.09	0.16**	0.12*	0.12*	1.00						
$bear$	−0.09	−0.31***	−0.43***	−0.18***	−0.17**	−0.66***	1.00					
pd	0.84***	0.49***	0.48***	0.35***	0.37***	0.36***	−0.21***	1.00				
r_f	0.22***	0.49***	0.38***	0.34***	0.33***	−0.01	−0.10*	−0.12*	1.00			
q	0.42***	−0.25***	−0.28***	0.00	−0.10	0.03	0.15**	0.13*	−0.09	1.00		
xr	−0.91***	−0.66***	−0.60***	−0.50***	−0.51***	−0.29***	0.21***	−0.91***	−0.28***	−0.09	1.00	
Δc	0.02	0.22***	0.23***	0.00	0.02	0.01	−0.12*	0.02	0.25***	−0.22***	−0.10	1.00

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Monthly data (1987m11-2018m12)