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Recommended Citation

Schneider, M.A. & Nunez, M.A. (2020). A capital asset pricing model with idiosyncratic risk and the sources of the beta anomaly. *ESI Working Paper 20-06* . https://digitalcommons.chapman.edu/esi_working_papers/300/

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A Capital Asset Pricing Model with Idiosyncratic Risk and the Sources of the Beta Anomaly

Comments

ESI Working Paper 20-06

Previously titled "A Capital Asset Pricing Model with Market Sentiment".

A Capital Asset Pricing Model with Idiosyncratic Risk and the Sources of the Beta Anomaly

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December 16, 2020

Abstract

We introduce a generalization of the classical capital asset pricing model in which market uncertainty, market sentiment, and forms of idiosyncratic volatility and idiosyncratic skewness are priced in equilibrium. We derive two versions of the model, one based on a representative agent who cares about three criteria (risk, robustness, and expected returns), and the other with a microfoundation based on three types of investors (speculators, hedgers, and arbitrageurs). We apply the resulting capital asset pricing model with idiosyncratic risk (IR-CAPM) to provide a new theoretical account of the beta anomaly, one of the most fundamental and widely studied empirical limitations of the CAPM. We show that the IR-CAPM explains the main conditional relationships involving the beta anomaly in the literature including the time variation of the beta anomaly across optimistic and pessimistic periods and across high and low uncertainty periods, the relationship between the beta anomaly and the correlation between a stock's beta and its idiosyncratic volatility, and the concentration of the beta anomaly among stocks with high idiosyncratic maximum returns.

JEL classification: D81.

Keywords: beta anomaly; idiosyncratic skewness; market sentiment; market uncertainty

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1 Introduction

In the capital asset markets studied by Sharpe (1964) all investors know the true distribution of asset returns and trade to diversify their risks. Consequently, only systematic (market) risk is priced in equilibrium. In practice, however, some agents have different levels of information and may trade with other motives. Some agents may trade primarily to speculate in hopes of making a short-term profit, while others may trade to hedge against market downturns. Such a market with speculators and hedgers is consistent with the historical folklore of financial markets that has witnessed persistent waves of optimism and pessimism.

In this paper, we consider a market with three types of agents. One type consists of standard agents who have correct beliefs about mean asset returns, determined in equilibrium. Since these agents are informed and their trade effectively serves to correct mispricing, these agents serve to some degree as arbitrageurs. The other two types of agents are uninformed agents, some with optimistic beliefs who trade primarily to speculate and others with pessimistic beliefs who trade to hedge against market downturns. We drop the assumption of a rational expectations equilibrium. We also do not require that heterogeneous beliefs are, on average, accurate, as assumed by previous authors (e.g., Levy et al., 2006). The resulting Capital Asset Pricing Model with idiosyncratic risk (IR-CAPM) generalizes the standard CAPM by allowing for market uncertainty, market sentiment, idiosyncratic volatility, and idiosyncratic skewness to affect equilibrium expected returns.

Before considering the heterogeneous agent market in Section 3, we first show in Section 2 that a version of the IR-CAPM can be derived in a representative agent framework in which the representative agent cares about three features of asset returns: (i) expected returns with respect to the agent’s subjective prior distribution, (ii) risk or dispersion of returns with respect to the agent’s prior, and (iii) robustness of returns to different specifications of the agent’s prior, reflecting the model uncertainty risk that the agent’s prior may be mis-specified. We measure risk based on the variance of returns as in classical portfolio theory, and we measure robustness based on the Hurwicz criterion which includes as a special case Wald’s (1950) maximin criterion that is widely used in robust optimization.

Both versions of the IR-CAPM predict that an asset’s idiosyncratic skewness and idiosyncratic volatility are priced in equilibrium. In contrast, standard generalizations of the CAPM do not include a role for the pricing of idiosyncratic risk. Prior generalizations of the CAPM have considered different measures of systematic risk that are not captured by the traditional CAPM beta. These include the downside beta (Ang et al., 2006), the tail beta (Kelly and Jiang, 2014), and the bear beta (Lu and Murray, 2019), for measuring covariance between an asset and the market in bad market states, as well as models which incorporate co-skewness (e.g., Kraus and Litzenberger, 1976; Harvey and Siddique, 2000; Schneider et al., 2020) to account for investor preferences for extreme positive returns when the market performs well. Hong and Sraer (2016) break from the traditional generalizations of the CAPM by developing a model in which an asset’s own standard deviation is priced. However, their model does not price an asset’s idiosyncratic skewness. While some prior work has argued that idiosyncratic skewness is priced (Brunnermeier et al., 2007; Mit-

ton and Vorkink, 2007; Barberis and Huang, 2008), these ideas have not been developed into a generalization of the classical CAPM in which idiosyncratic skewness is priced in equilibrium.

Our first contribution is to introduce and derive two versions of the IR-CAPM. In both versions, an asset’s idiosyncratic skewness is priced, as documented empirically by Bali et al. (2011) and Cheon and Lee (2018), and an asset’s idiosyncratic volatility is priced, as documented by Ang et al. (2006). The IR-CAPM also predicts that systematic market uncertainty and systematic market sentiment are priced in equilibrium, consistent with Bali and Zhou (2016), and Baker and Wurgler (2006).

Our second main contribution is to show that the IR-CAPM predicts the beta anomaly (Black et al., 1972; Frazzini and Pedersen, 2014) and its conditional relationships documented in the literature. The beta anomaly in which stocks with high market beta have relatively low expected returns compared to low beta stocks is one of the most prominent and fundamental empirical violations of the CAPM. Several conditional relationships involving the beta anomaly have also been established: The beta anomaly is concentrated in stocks with high maximum returns (Bali et al., 2017); it is stronger in periods of high market sentiment (Antonioni et al., 2016), in particular, when the correlation between a stock’s beta and its idiosyncratic volatility are simultaneously high (Liu et al., 2018); and the beta anomaly is stronger in periods of high market uncertainty (Hong and Sraer, 2016). These findings constitute basic puzzles pertaining to both the cross-section and the time series of the beta anomaly that no model in the literature yet comprehensively explains.

Among the leading theoretical explanations, Frazzini and Pedersen (2014) explain the beta anomaly based on a model of leverage constraints, while Hong and Sraer (2016) explain the beta anomaly based on the higher sensitivity of high beta assets to investor disagreement. However, these approaches do not consider the dependence of the beta anomaly on market sentiment or its concentration among stocks with high idiosyncratic skewness. Schneider et al. (2020) show that the abnormal profits from the beta anomaly in the cross-section of returns can be largely explained by co-skewness. However, they do not explore the time series variation in these anomalies across periods of high versus low sentiment or high versus low uncertainty. In addition, Kumar et al. (2019) and Jiang et al. (2020) find that variation in a variety of anomaly returns is explained by idiosyncratic skewness but is not explained by co-skewness.

2 A Representative Agent Approach to Idiosyncratic Risk

We develop a capital asset pricing model in which forms of an asset’s own idiosyncratic skewness and idiosyncratic volatility are priced. The pricing of idiosyncratic risk emerges from the interaction between the structure of the market state space, the market short-sale constraint, and the preferences of the representative agent.

Our representative agent exhibits two robust behavioral biases for choice under risk and uncertainty: ambiguity aversion (Ellsberg, 1961) and positive skewness preference. One form of ambiguity aversion is a preference for making decisions that are robust to mis-specified proba-

bility models (Wald, 1950; Hurwicz, 1951). Ambiguity aversion and skewness preference are not only laboratory phenomena – they provide explanations for failures of economic theory in markets: ambiguity aversion provides an explanation for buying-selling price gaps in markets (Dow and da Costa Werlang, 1992) and for the equity premium puzzle (e.g., Ju and Miao, 2012). Skewness preference can explain much of the risk-seeking behavior observed in markets such as the simultaneous purchasing of lottery tickets and insurance (Friedman and Savage, 1948) and the overpricing of long-shots in betting markets (Weitzman, 1965).

Given the strong behavioral support for ambiguity aversion and skewness preference and their relevance in applications, it seems desirable for a model of choice under risk and uncertainty to predict both behaviors. We observe that both behaviors are naturally accommodated by a model of robust decision making. Robust decision models typically focus on worst-case scenarios. A less conservative approach pioneered by Hurwicz (1951) maximizes the convex combination of the worst-case and best-case scenarios. The Hurwicz α -criterion provides a simple approach to incorporating a preference for robustness toward model uncertainty (by overweighting the worst outcome), and a preference for positively skewed returns (by overweighting the best outcome).

Formally, let there be a non-empty finite set of states S , a non-empty convex set of possible outcomes X in \mathbb{R} , and a set of ambiguous acts \mathcal{F} , where an act $f \in \mathcal{F}$ is a mapping $f : S \rightarrow X$. Denote by $f(s)$ the outcome that occurs if act f is chosen and state s occurs. Let $\Delta(S)$ denote the set of all possible probability distributions on S with generic (vector) element π .

We consider a representative investor who cares primarily about three features of asset returns: (i) the expected return on an asset with respect to the investor’s prior π ; (ii) risk or dispersion of returns with respect to π ; and (iii) robustness of returns to different specifications of π . The investor is contemplating investment in n risky assets. Denote by $R_j \in \mathcal{F}$, for $j = 1, \dots, n$, the act obtained by investing in the j -th asset, where $r_{js} := R_j(s) \in X$ is the return of asset j in state s . Further, if the investor has subjective probability distribution $\pi \in \Delta(S)$ across states, denote by $r_j := E_\pi(R_j) = \sum_{s \in S} \pi_s r_{js}$ the expected return on asset j across states. We assume that there exist states \bar{s} and \underline{s} in S that are the “common best” and “common worst” states for all the assets, that is,

$$\begin{aligned} \bar{r}_j := r_{j\bar{s}} &= \max_{s \in S} r_{js}, \\ \underline{r}_j := r_{j\underline{s}} &= \min_{s \in S} r_{js}, \end{aligned}$$

for all j . There is an additional safe asset with return $r_0 > 0$ in every state, corresponding to the constant act R_0 that maps every state to r_0 .

Instead of considering all possible acts, the investor concentrates on comparing acts from this subset of \mathcal{F} :

$$\mathcal{P} := \left\{ R \in \mathcal{F} : \exists (w_0, w) \in \mathbb{R}^{n+1}, w_0 + \sum_{j=1}^n w_j = 1, R(s) = \sum_{j=0}^n w_j R_j(s), \forall s \in S \right\}.$$

We call an element R in \mathcal{P} a *portfolio* with corresponding *holdings vector* (w_0, w) . The investor evaluates portfolios according to risk-return-robustness preferences given by

$$V(R) = \theta (\mu(R) - \rho\sigma^2(R)) + (1 - \theta)\psi(R). \quad (1)$$

for $R \in \mathcal{P}$, where

$$\mu(R) := E_\pi(R) = \sum_{j=0}^n w_j E_\pi(R_j) = w_0 r_0 + \sum_{j=1}^n w_j r_j$$

is the expected return of the portfolio,

$$\sigma^2(R) := \sum_{s \in S} \pi_s (R(s) - \mu(R))^2 = \sum_{s \in S} \pi_s \left(\sum_{j=1}^n w_j (r_{js} - r_j) \right)^2$$

is the variance of the portfolio, and

$$\begin{aligned} \psi(R) &:= \alpha \max_{s \in S} \{R(s)\} + (1 - \alpha) \min_{s \in S} \{R(s)\} \\ &= \alpha \max_{s \in S} \left\{ w_0 r_0 + \sum_{j=1}^n w_j r_{js} \right\} + (1 - \alpha) \min_{s \in S} \left\{ w_0 r_0 + \sum_{j=1}^n w_j r_{js} \right\} \end{aligned}$$

is the robust (Hurwicz) value of the risky assets in the portfolio. The parameter $\rho \geq 0$ represents the agent's degree of risk aversion, $\alpha \in [0, 1]$ represents the agent's degree of optimism (the degree to which the agent overweights the best-case scenario), and $\theta \in (0, 1]$ represents the agent's confidence in his beliefs regarding the true probability distribution over states. When $\theta = 1$, the agent is fully confident in his beliefs about the true distribution (he knows the means and covariances exactly), and equation (1) reduces to the standard mean-variance preferences under objective risk. When θ is close to zero, the agent is completely uncertain about the true distribution, and engages in robust optimization, based on the Hurwicz criterion, which does not depend on π .

The preference function (1) spans three prominent decision models: risk-neutral subjective expected utility ($\rho = 0, \theta = 1$), mean-variance analysis ($\rho > 0, \theta = 1$), and a special case of prospect theory (Tversky and Kahneman, 1992) with a textbook prospect theory probability weighting function (Wakker, 2014) that overweights the tails of the distribution ($\rho = 0, \theta \in (0, 1)$). Further, it yields a separation of the investor's beliefs (represented by π), the investor's uncertainty (represented by θ), and the investor's ambiguity attitudes (represented by α). A generalization of the consumption CAPM in which the representative agent overweights the best and worst-case scenarios is given by Zimper (2012). Our approach extends the classical CAPM to the more general domain of uncertainty where the probability model is not precisely known, in which case a concern for robustness becomes important.

To set up the investor's portfolio problem, let r denote the vector $[r_1, \dots, r_n]^T$, r^s denote the vector $[r_{1s}, \dots, r_{ns}]^T$ for $s \in S$, and Σ denote the matrix $\sum_{s \in S} \pi_s (r^s - r)(r^s - r)^T$. Matrix Σ is

the covariance matrix of the risky assets and is assumed to be positive definite. Function V can be re-written as

$$V(R) = \theta (r_0 w_0 + r^T w - \rho(w^T \Sigma w)) + (1 - \theta) \left(\alpha \max_s \{w_0 r_0 + w^T r^s\} + (1 - \alpha) \min_s \{w_0 r_0 + w^T r^s\} \right),$$

for all $R \in \mathcal{P}$. The investor wishes to maximize V on \mathcal{P} . We assume that the maximum can be achieved at an optimal solution $(w_0^*, w^*) \geq 0$, that is, short selling is not optimal in this market.

Since feasible vector holdings must satisfy $w_0 + e^T w = 1$, where e denotes the n -dimensional all-ones vector, under our assumptions maximizing $V(R)$ is equivalent to maximizing the unrestricted function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as

$$g(w) := \theta (r_0 + (r - r_0 e)^T w - \rho(w^T \Sigma w)) + (1 - \theta) \left(\alpha \max_s \{r_0 + (r^s - r_0 e)^T w\} + (1 - \alpha) \min_s \{r_0 + (r^s - r_0 e)^T w\} \right). \quad (2)$$

Notice that in general g is a continuous function, but because of the Hurwicz term, it is not necessarily differentiable at points w where there are ties for the maximum or the minimum of $w^T r^s$ across S . On the other hand, when restricted to nonnegative vectors ($w \geq 0$), because of our common best state and common worst state assumptions, we obtain

$$g(w) = \theta (r_0 + (r - r_0 e)^T w - \rho(w^T \Sigma w)) + (1 - \theta) (\alpha (r_0 + (\bar{r} - r_0 e)^T w) + (1 - \alpha) (r_0 + (\underline{r} - r_0 e)^T w)),$$

where $\bar{r} := [\bar{r}_1, \dots, \bar{r}_n]^T$ and $\underline{r} := [\underline{r}_1, \dots, \underline{r}_n]^T$. Therefore, g is differentiable and strictly concave in the region defined by nonnegative holdings vectors, and hence the equilibrium is guaranteed to exist in this case.

Proposition 1 *Under the assumptions in this section, it follows that at optimality we have*

$$r_j - r_0 = \beta_j (r_m - r_0 + \gamma (\alpha \bar{r}_m + (1 - \alpha) \underline{r}_m - r_0)) + \gamma (\alpha (r_0 - \bar{r}_j) + (1 - \alpha) (r_0 - \underline{r}_j)) \quad (3)$$

for all $j = 1, \dots, n$, where

$$\begin{aligned} \gamma &:= \frac{1 - \theta}{\theta}, \\ r_m &:= r_0 w_0^* + r^T w^*, \\ \bar{r}_m &:= r_0 w_0^* + \bar{r}^T w^*, \\ \underline{r}_m &:= r_0 w_0^* + \underline{r}^T w^*, \end{aligned}$$

and

$$\beta_j := \frac{\text{Cov}(R_j, R^*)}{\sigma^2(R^*)}. \quad (4)$$

We refer to expression (3) as the IR-CAPM identity as it generalizes the CAPM to incorporate a role for idiosyncratic risk. In addition to systematic volatility β_j , the IR-CAPM accounts for market optimism α and market uncertainty $1 - \theta$. From (3), we see that the IR-CAPM also prices extreme idiosyncratic positive returns \bar{r}_j , idiosyncratic disaster risk r_j , and consequently, idiosyncratic volatility (which depends on both \bar{r}_j and r_j).

3 A Heterogeneous Agent Approach to Idiosyncratic Risk

We next derive another version of the IR-CAPM from a setting with three types of market participants: informed agents who know the true means of asset returns determined in equilibrium and effectively serve as arbitrageurs by leveraging their information, uninformed agents who are optimistic about the market and overestimate the returns on all assets and uninformed pessimistic agents who underestimate the returns on all assets. The setup roughly corresponds to a market with three different motives for trading: agents who serve as arbitrageurs, speculators who trade hoping to make a short-term profit (optimists), and hedgers who trade to insure against a market-downturn by selling some assets short (pessimists). Hence, the market consists of agents who are neutral, bullish, or bearish on the stock market.

Although classical finance theory predicts that arbitrage will correct all mispricing, behavioral finance has argued that there are limits to arbitrage (e.g., De Long et al., 1990). Pontiff (2006) and McLean and Pontiff (2016) note that the greatest limit to arbitrage is idiosyncratic risk, which is the largest arbitrage cost facing informed traders. This observation is further developed by Bégin et al. (2020) who find empirically that “the normal component of idiosyncratic risk, which is easily diversifiable, is not priced after accounting for other sources of risk. Firm-specific jump risk, however, is priced” (p. 199). They conclude that “Tail risk thus plays a central role in the pricing of idiosyncratic risk” (p.155). Idiosyncratic tail risk may then be priced by the market since it cannot be fully diversified away. This observation provides a motivation for the pricing of extreme idiosyncratic returns in the IR-CAPM.

In this section, we show that when agents have heterogeneous beliefs the pricing of idiosyncratic risk directly follows when it is not assumed a priori that the average belief is “unbiased”. Historically, a main purpose of heterogeneous agent CAPM models (e.g., Lintner, 1969; Huang and Litzenberger, 1998; Levy et al., 2006; Chiarella et al., 2010), has been to provide a foundation for the CAPM pricing relationship when agents have heterogeneous beliefs. Consequently, it is usually assumed explicitly or implicitly that the heterogeneous expectations are, on average, unbiased, so that the standard CAPM pricing relationship holds with respect to the “true” probability distribution¹. In contrast, we allow for a systematic bias in the aggregate beliefs of the heterogeneous agents. We observe that any such bias leads to the pricing of idiosyncratic risk.

We consider an economy where there is one risk-free asset with return r_0 and n risky assets. Like in Section 2, we denote by R_j a random variable representing the stochastic return on asset

¹Atmaz and Basak (2018) consider a market where the average belief is biased. However, they do not develop their approach as a generalization of the CAPM and they do not consider the beta anomaly.

j . As in the Hong and Sraer (2016) model, we assume that the investors agree in that the return vector $R := (R_1, \dots, R_n)$ of the risky assets has a multivariate normal distribution with a positive definite covariance matrix Σ , but they disagree on the mean vector of asset returns (location) of the distribution. Concretely, we assume that the market consist of a finite set I of investors which is partitioned into three mutually exclusive subgroups I_e , I_o , and I_p . All investors within any of these subgroups agree in the mean vector of asset returns. Investors in set I_e are informed agents with correct beliefs about expected returns that are determined in equilibrium and constitute a fraction θ of the investor population. Investors in set I_o are uninformed optimistic agents who overestimate the expected returns on all assets and constitute a fraction $\bar{\alpha} := \alpha(1 - \theta)$ of the investor population. Similarly, Investors in set I_p are uninformed pessimistic agents who underestimate the expected returns on all assets and constitute a fraction $\underline{\alpha} := (1 - \alpha)(1 - \theta)$ of the investor population. Observe that $\theta = |I_e|/|I|$, $\bar{\alpha} = |I_o|/|I|$, and $\underline{\alpha} = |I_p|/|I|$. We assume that these subgroups are nonempty, so that $0 < \alpha, \theta < 1$.

Corresponding to the investor partition into three groups, we let r_j denote the true expected return on asset j upon which all investors in set I_e agree. Similarly, we denote by \bar{r}_j and \underline{r}_j the agreed perceived optimistic and pessimistic expected returns on asset j for all investors in sets I_o and I_p , respectively, with $\underline{r}_j \leq r_j \leq \bar{r}_j$. We also use notation r , \bar{r} , and \underline{r} to refer to the corresponding n -dimensional vectors with coordinates r_j , \bar{r}_j , and \underline{r}_j , $1 \leq j \leq n$, respectively.

We apply the general CAPM with heterogeneous beliefs from Chiarella et al. (2010) to our stylized market structure. Let v_0^i denote the initial wealth of agent $i \in I$ and let $w_{ij} \geq 0$ be the proportion of agent i 's initial wealth invested in asset j . The wealth W_i of agent i 's portfolio is a random variable given by

$$W_i = v_0^i (1 + r_0 + w_i^T (R - r_0 e)), \quad (5)$$

where w_i is the n -dimensional vector with coordinates w_{ij} , $j = 1, \dots, n$. We assume that agent i has a utility function u_i that is twice differentiable, concave, and strictly increasing. The investors solve the following choice problem:

$$\max_{\omega} E_i[u_i(W_i)], \quad (6)$$

where E_i represents the conditional expectation with respect to the believed risky-assets probability distribution of investor i , for all $i \in I$.

We denote by w_i^* an optimal solution to (6), that is, w_i^* is the vector of proportions of the wealth allocations invested in risky assets of investor i when the market is in equilibrium. Similarly to Huang and Litzenberger (1998), let $v_m := \sum_{i \in I} v_0^i$ denote the total wealth in the economy at the beginning of the investing period, and $W_m := \sum_{i \in I} W_i$ be the random variable representing the end-of-period wealth in the economy, where W_i is computed using (5) evaluated on w_i^* . Let R_m denote the return on the aggregate market wealth when in equilibrium. These variables are related by the following equation:

$$R_m = \frac{W_m - v_m}{v_m},$$

or

$$W_m = v_m (1 + R_m).$$

Following Huang and Litzenberger (1998), if we replace W_i by the expression on the right-hand-side of (5) and add with respect to $i \in I$ to compute W_m , we obtain

$$R_m = r_0 + w_a^T (R - r_0 e),$$

where

$$w_a := \frac{1}{v_m} \sum_{i \in I} v_0^i w_i^*. \quad (7)$$

From this, it follows that the conditional expectation of the aggregate market return satisfies

$$E_i[R_m] = \begin{cases} r_m := r_0 + w_a^T (r - r_0 e) & \text{if } i \in I_e, \\ \bar{r}_m := r_0 + w_a^T (\bar{r} - r_0 e) & \text{if } i \in I_o, \\ \underline{r}_m := r_0 + w_a^T (\underline{r} - r_0 e) & \text{if } i \in I_p, \end{cases}$$

for all $i \in I$, and hence, the unconditional expectation of the aggregate market return across investors is

$$E[R_m] = \theta r_m + \bar{\alpha} \bar{r}_m + \underline{\alpha} \underline{r}_m = r_0 + w_a^T (r_a - r_0 e), \quad (8)$$

where

$$r_a := \theta r + \bar{\alpha} \bar{r} + \underline{\alpha} \underline{r}. \quad (9)$$

From (8), it follows that $E[R_m]$ is the aggregate belief about the unconditional mean market return, which is based on the correct beliefs of the informed agents and the optimistic and pessimistic expectations of the uninformed agents. In general, the bias in the aggregate market belief is systematic: it overestimates risk premia in optimistic periods and it underestimates risk premia in pessimistic periods.

As usual, the global absolute risk aversion for agent i is defined as:

$$\lambda_i := -\frac{E_i[u_i''(W_i)]}{E_i[u_i'(W_i)]},$$

for all $i \in I$. We denote the aggregate risk aversion in the market by $\lambda_a = (\sum_{i \in I} \lambda_i^{-1})^{-1}$. This setup imposes minimal restrictions on investor preferences. We build on this approach to construct a market with asymmetric information and heterogeneity in optimism and pessimism. To derive a more specific asset pricing formula, we assume that all investors have the same global (constant) absolute risk aversion, i.e., $\lambda_i = \lambda > 0$ for all $i \in I$, so that $\lambda_a = \lambda/|I|$. In the following proposition, we present a new characterization of equilibrium risk premia that generalizes the CAPM.

Proposition 2 *Under the assumptions in this section and when the market is in equilibrium:*

$$r_a - r_0 e = \beta (E[R_m] - r_0), \quad (10)$$

where

$$\beta := \frac{\Sigma w_a}{w_a^T \Sigma w_a} = \frac{\text{cov}(R, R_m)}{\text{var}(R_m)}, \quad (11)$$

and so, the risk premium on any asset j for $j = 1, \dots, n$ is given by

$$r_j - r_0 = \beta_j \left(\frac{E[R_m] - r_0}{\theta} \right) + \bar{\alpha} \left(\frac{r_0 - \bar{r}_j}{\theta} \right) + \underline{\alpha} \left(\frac{r_0 - \underline{r}_j}{\theta} \right). \quad (12)$$

Formula (12) allows for the pricing of systematic volatility (β_j), market sentiment (α) and market uncertainty ($1 - \theta$), as well as the market expectations about positive idiosyncratic skewness (\bar{r}_j), negative idiosyncratic skewness (\underline{r}_j) and idiosyncratic volatility (which depends in part on \bar{r}_j and \underline{r}_j). Market sentiment is quantified in (12) as the fraction of optimistic agents in the market as compared to the fraction of pessimistic agents, while market uncertainty is quantified as the fraction of uninformed agents in the market as compared to the fraction of informed agents.

4 The Beta Anomaly

One of the most fundamental empirical limitations of the CAPM is the beta anomaly (Black et al., 1972), in which assets with high systematic volatility (beta) do not necessarily earn higher expected returns. Bali et al. (2017) remark “The positive (negative) abnormal returns of portfolios composed of low-beta (high-beta) stocks, which we refer to as the beta anomaly, is one of the most persistent and widely studied anomalies in empirical research of security returns” (p. 2370). The beta anomaly challenges the central empirical prediction of the CAPM that investors demand higher expected returns for assets with higher systematic risk. Corollary 1 provides necessary and sufficient conditions for the beta anomaly and predicts its main conditional relationships. For a high beta stock, H , and a low beta stock, L , we say the beta anomaly holds if $r_H < r_L$.

Corollary 1 (Beta Anomaly) *Consider two stocks H and L with $\beta_H > \beta_L$. Then,*

(i) *(Beta anomaly) Under (12), for all $\alpha, \theta \in (0, 1)$, $r_H < r_L$ if and only if $\alpha(1 - \theta)(\bar{r}_H - \bar{r}_L) > (\beta_H - \beta_L)E[R_m] + (1 - \alpha)(1 - \theta)(\underline{r}_L - \underline{r}_H)$.*

(ii) *(Beta and high MAX stocks) The beta anomaly holds if and only if \bar{r}_H is sufficiently high².*

(iii) *(Beta, IVOL, and market sentiment) If the Beta-IVOL correlation is sufficiently strong such that $\bar{r}_H > \bar{r}_L + (\beta_H - \beta_L)\bar{r}_m$ and $\underline{r}_H < \underline{r}_L + (\beta_H - \beta_L)\underline{r}_m$, then $r_H - r_L$ is decreasing in α .*

(iv) *(Beta and market uncertainty) If the beta anomaly holds, then $r_H - r_L$ is increasing in θ .*

²In particular, the beta anomaly holds if and only if $\bar{r}_H > \frac{(\beta_H - \beta_L)E[R_m] + (1 - \alpha)(1 - \theta)(\underline{r}_L - \underline{r}_H)}{\alpha(1 - \theta)} + \bar{r}_L$.

(v) Let $E[R_m] > 0$ and $r_L \geq r_H$. Then the positive relation between beta and expected return ($r_H > r_L$) holds if α is sufficiently close to 0, or if θ is sufficiently close to 1, or if \bar{r}_H is sufficiently close to \bar{r}_L .

In Corollary 1, the beta anomaly occurs if a high beta stock earns lower expected returns than a low beta stock. This definition is consistent with the empirical finding by Frazzini and Pedersen (2014) that “a betting against beta (BAB) factor, which is long leveraged low-beta assets and short high-beta assets, produces significant positive risk-adjusted returns” (p. 1). Corollary 1 (ii) predicts that the beta anomaly will be concentrated in high beta stocks with high maximum returns. Consistent with this prediction, Bali et al. (2017) find that the beta anomaly is concentrated in high beta stocks with high maximum returns.

Bali et al. (2011) finds correlations between high maximum returns and idiosyncratic volatility and between low minimum returns and idiosyncratic volatility to each be approximately 0.75. Since high maximum returns and low minimum returns are highly correlated with idiosyncratic volatility, Corollary 1 (iii) predicts that the beta anomaly holds in high sentiment (i.e., high α) periods when the correlation between beta and idiosyncratic volatility is high (i.e., when high beta stocks have high maximum returns and low minimum returns). Consistent with this prediction, Liu et al. (2018) find that the beta anomaly holds in high sentiment periods when the correlation between beta and idiosyncratic volatility are simultaneously high, but does not hold in low sentiment periods or in periods where the beta-IVOL correlation is low.

Corollary 1 (iv) predicts that the beta anomaly will be stronger in periods of higher market uncertainty. Consistent with this prediction, Hong and Sraer (2016) find that the beta anomaly is stronger when there is greater market uncertainty.

Corollary 1 (v) predicts that the traditional positive relation between beta and expected return will hold in pessimistic periods (as observed by Antoniou et al. (2016)), or in periods with low uncertainty (as observed by Hong and Sraer (2016)) or among stocks with low maximum returns (as observed by Bali et al. (2017)).

5 Conclusion

We derived a generalization of the capital asset pricing model that accounts for model uncertainty, positive skewness, disaster risk, and market sentiment, thereby linking four strands of the asset pricing literature. The resulting IR-CAPM can be expressed as a three-factor asset pricing model for the cross-section of returns.

We applied the IR-CAPM to provide a unified theoretical explanation of the beta anomaly and its main conditional effects documented in the literature. More broadly, our approach provides a theoretical foundation for the pricing of idiosyncratic risk in the cross-section of returns.

Appendix

Proof of Proposition 1: Since g from (2) is differentiable at $w^* \geq 0$ and attains its maximum at that point, it satisfies the first order necessary condition $\nabla g(w^*) = 0$. Hence, w^* satisfies the following equation

$$\theta(r - r_0e) + (1 - \theta)(\alpha(\bar{r} - r_0e) + (1 - \alpha)(\underline{r} - r_0e)) = 2\theta\rho Cw. \quad (13)$$

Multiplying by w^* on both sides of (13), we obtain

$$\theta(r_m - r_0) + (1 - \theta)(\alpha(\bar{r}_m - r_0) + (1 - \alpha)(\underline{r}_m - r_0)) = 2\theta\rho w^{*T} Cw^*. \quad (14)$$

Multiplying by e_j on both sides of (13), where e_j is the j -th canonical vector in \mathbb{R}^n , we obtain

$$\theta(r_j - r_0) + (1 - \theta)(\alpha(\bar{r}_j - r_0) + (1 - \alpha)(\underline{r}_j - r_0)) = 2\theta\rho e_j^T Cw^*. \quad (15)$$

Dividing the left-hand side of (15) by the left-hand side of (14), we obtain

$$\frac{\theta(r_j - r_0) + (1 - \theta)(\alpha(\bar{r}_j - r_0) + (1 - \alpha)(\underline{r}_j - r_0))}{\theta(r_m - r_0) + (1 - \theta)(\alpha(\bar{r}_m - r_0) + (1 - \alpha)(\underline{r}_m - r_0))} = \frac{w^* C e_j}{w^{*T} C w^*} = \beta_j,$$

from which

$$r_j - r_0 + \frac{1 - \theta}{\theta} (\alpha\bar{r}_j + (1 - \alpha)\underline{r}_j - r_0) = \beta_j \left(r_m - r_0 + \frac{1 - \theta}{\theta} (\alpha\bar{r}_m + (1 - \alpha)\underline{r}_m - r_0) \right),$$

and (3) follows. ■

Proof of Proposition 2: Following Huang and Litzenger (1998), in equilibrium we have

$$w_i^* = \frac{1}{\lambda v_0^i} \Sigma^{-1} E_i [R - r_0e].$$

It follows from (7) and (9) that

$$w_a = \frac{1}{v_m} \sum_{i \in I} v_0^i w_i^* = \frac{1}{\lambda v_m} \Sigma^{-1} \sum_{i \in I} E_i [R - r_0e] = \frac{1}{\lambda_a v_m} \Sigma^{-1} (r_a - r_0e).$$

This implies

$$\begin{aligned} \Sigma w_a &= \frac{1}{\lambda_a v_m} (r_a - r_0e), \\ w_a^T \Sigma w_a &= \frac{1}{\lambda_a v_m} w_a^T (r_a - r_0e). \end{aligned}$$

Using (8), we obtain

$$E[R_m] = r_0 + \lambda_a v_m w_a^T \Sigma w_a,$$

and so

$$\lambda_a v_m = \frac{1}{w_a^T \Sigma w_a} (E[R_m] - r_0).$$

Therefore, using (11), we get

$$r_a - r_0 e = \lambda_a v_m \Sigma w_a = \frac{1}{w_a^T \Sigma w_a} (E[R_m] - r_0) \Sigma w_a = \beta (E[R_m] - r_0),$$

and (10) follows. By re-arranging terms in (10), we also obtain (12). ■

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