2019

A Full Characterization of Best-Response Functions in the Lottery Colonel Blotto Game

Dan Kovenock
Chapman University, kovenock@chapman.edu

David Rojo Arjona
Chapman University, rojoarjo@chapman.edu

A Full Characterization of Best-Response Functions in the Lottery Colonel Blotto Game

Comments
Working Paper 19-05

This paper was later published as:

Kovenock, D., & Arjona, D. R. (2019). A full characterization of best-response functions in the lottery Colonel Blotto game. *Economics Letters, 182*, 33-36. doi: 10.1016/j.econlet.2019.05.040

A Full Characterization of Best-Response Functions in the Lottery Colonel Blotto Game*

Dan Kovenock[a], David Rojo Arjona[b]

[a] *Economic Science Institute, Chapman University, 1 University Dr, Orange, CA 92866, USA.*
[b] *Smith Institute for Political Economy and Philosophy, Chapman University, 1 University Dr, Orange, CA 92866, USA.*

Abstract

We fully characterize best-response functions in Colonel Blotto games with lottery contest success functions.

Keywords: Multi-Battle contest, Colonel Blotto game, Contest success function, Best-response, Conflict.

JEL codes: C61, C72

---

1. Introduction

Friedman [2] derives the unique equilibrium of a two-player Colonel Blotto game with a lottery success function [4] in each battlefield. He also offers a characterization of a player's best-response function in the game. However, for some values of the primitives, his proposed characterization results in infeasible negative allocations to battlefields.

This article completely characterizes a player's best response function in the game, thereby correcting Friedman. This facilitates the analysis of Stackelberg models of attack and defense, where a complete account of subgame behavior is required for equilibrium analysis. It also aids in behavioral analyses of Blotto games, where systematic deviations from optimal behavior are analyzed.[1]

*Corresponding authors: Dan Kovenock: kovenock@chapman.edu; David Rojo Arjona: rojoarjo@chapman.edu.

[1] See, for example, [1].

2. Model

Suppose two players, $i = 1, 2$, are endowed with budgets, $X_i \in \mathbb{R}_{++}$. The two players simultaneously divide their respective budgets into (sunk) allocations across $n \geq 2$ independent battlefields. The finite set of battlefields is denoted by B and player i's value of winning battlefield $j \in B$ is $v_{ij} \in \mathbb{R}_{++}$. Any allocation to battlefield j by player i needs to be non-negative, $x_{ij} \in \mathbb{R}_{+}$. Player i's pure strategy space is the set of non-negative n-tuples $\mathbf{x_i} = (x_{i1}, x_{i2}, ..., x_{in})$ satisfying $\sum_{j=1}^{n} x_{ij} \leq X_i$, $i = 1, 2$.

Each player's objective function is given by $\Pi_i = \sum_{j=1}^{n} p_{ij} v_{ij}$ where p_{ij} is the probability of player i winning battlefield j, determined by a lottery contest success function:

$$p_{ij}(x_{ij}, x_{-ij}) = \begin{cases} \frac{x_{ij}}{(x_{ij}+x_{-ij})} & \text{if } x_{ij} + x_{-ij} \neq 0 \\ \frac{1}{2} & \text{if } x_{ij} = x_{-ij} = 0 \end{cases}$$

2.1. The Maximization Problem and Kuhn-Tucker Conditions

Suppose player $-i$ selects an allocation $\mathbf{x_{-i}} \gg \mathbf{0}$. In this case, p_{ij} and Π_i are continuous in x_{ij} $\forall j$.[2] Then, the constrained optimization problem faced

[2]There are two other cases. One alternative case involves x_{-ij} such that $x_{-ij} > 0$ $\forall j \in C_{-i} \neq \emptyset$ and $x_{-ij} = 0$ $\forall j \in B \setminus C_{-i} \neq \emptyset$. In this case, $p_{ij}v_{ij}$ is continuous in x_{ij} for $j \in C_{-i}$. For $j \in B \setminus C_{-i}$, p_{ij} (and the associated $p_{ij}v_{ij}$) is discontinuous at $x_{ij} = 0$, taking the value $\frac{1}{2}$ at $x_{ij} = 0$ and 1 for all $x_{ij} > 0$. Because $p_{ij}v_{ij}$ is strictly increasing in x_{ij} for $j \in C_{-i}$ and there is no smallest x_{ij} strictly greater than 0 for $j \in B \setminus C_{-i}$, there is no best response to $\mathbf{x_{-i}}$. The final case involves $\mathbf{x_{-i}}$ such that $x_{-ij} = 0$ $\forall j$. In this case, any feasible allocation $\mathbf{x_i}$ such that $x_{ij} > 0$ $\forall j$ is a best response.

by player i can be expressed as

$$\begin{aligned} \underset{\mathbf{x_i}}{\text{maximize}} \quad & \sum_{j=1}^{n} \frac{x_{ij}}{x_{ij}+x_{-ij}} v_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij} \leq X_i \ \ \text{(budget constraint)}, \\ & x_{ij} \geq 0 \ \ \forall j \ \ \text{(non-negativity constraint)}. \end{aligned}$$

The corresponding Lagrangian is

$$\mathcal{L} = \sum_{j=1}^{n} \frac{x_{ij}}{x_{ij}+x_{-ij}} v_{ij} + \mu_0 \left(X_i - \sum_{j=1}^{n} x_{ij} \right) + \sum_{j=1}^{n} \mu_j x_{ij}.$$

Taking the derivative with respect to x_{ij} and equating it to 0,

$$\frac{\partial \mathcal{L}}{\partial x_{ij}} = \frac{x_{-ij} v_{ij}}{(x_{ij}+x_{-ij})^2} - \mu_0 + \mu_j = 0 \quad \forall j. \tag{1}$$

In addition, we have $n+1$ dual feasibility conditions (see expression (2)) and $n+1$ complementary slackness conditions (see expressions (3) and (4)):

$$\mu_0, \mu_j \geq 0 \quad \forall j, \tag{2}$$

$$\mu_0 \left(X_i - \sum_{j=1}^{n} x_{ij} \right) = 0, \tag{3}$$

$$\mu_j x_{ij} = 0 \quad \forall j. \tag{4}$$

3. Best-Response Functions

Without loss of generality, we assume that battlefields are ordered such that

$$\frac{v_{i1}}{x_{-i1}} \geq \frac{v_{i2}}{x_{-i2}} \geq ... \geq \frac{v_{in}}{x_{-in}}. \tag{5}$$

Define

$$k^* = max\{k \in B : \frac{v_{ij}}{x_{-ij}} > \frac{\left(\sum_{l=1}^{j} (x_{-il} v_{il})^{\frac{1}{2}}\right)^2}{\left(X_i + \sum_{l=1}^{j} x_{-il}\right)^2} \quad \forall j \leq k\}. \tag{6}$$

Note that the bracketed inequality holds if and only if

$$\left(\frac{v_{ij}}{x_{-ij}}\right)\left(X_i+\sum_{l=1}^{j}x_{-il}\right)^2>\left(\sum_{l=1}^{j}(x_{-il}v_{il})^{\frac{1}{2}}\right)^2$$

$$\Leftrightarrow\left(\frac{v_{ij}}{x_{-ij}}\right)^{\frac{1}{2}}\left(X_i+\sum_{l=1}^{j}x_{-il}\right)>\sum_{l=1}^{j}x_{-il}\left(\frac{v_{il}}{x_{-il}}\right)^{\frac{1}{2}}$$

$$\Leftrightarrow\left(\frac{v_{ij}}{x_{-ij}}\right)^{\frac{1}{2}}X_i>\sum_{l=1}^{j}x_{-il}\left[\left(\frac{v_{il}}{x_{-il}}\right)^{\frac{1}{2}}-\left(\frac{v_{ij}}{x_{-ij}}\right)^{\frac{1}{2}}\right] \tag{7}$$

Note first that (5) implies that the left hand side (LHS) of condition (7) is non-increasing in j and strictly decreasing from j to $j+1$ if and only if $\frac{v_{ij}}{x_{-ij}}>\frac{v_{i(j+1)}}{x_{-i(j+1)}}$. Moreover, the right hand side (RHS) is non-decreasing in j and is strictly increasing from j to $j+1$ if and only if $\frac{v_{ij}}{x_{-ij}}>\frac{v_{i(j+1)}}{x_{-i(j+1)}}$.

One consequence is that if $\frac{v_{is}}{x_{-is}}=\frac{v_{ij}}{x_{-ij}}$ for $s\neq j$, then, the inequality holds for j if and only if it holds for s.

Clearly, for $j=1$, the inequality in condition (6) reduces to $\frac{v_{il}}{x_{-i1}}>\frac{x_{-i1}v_{i1}}{(X_i+x_{-i1})^2}$, which clearly holds because for $X_i>0$, $(X_i+x_{-i1})^2>x_{-i1}^2$.

As a consequence, there is a unique set $K=\{1,...,k^*\}$ such that $\forall j\in\{1,...,k^*\}$ the inequality in condition (6) holds and $\forall j>k^*$ the inequality does not hold.

We now claim.

Proposition 1 (Best Response Function)**.** *For given primitives (X_i and v_{ij}, $j=1,...,n$) and a given allocation of the other player $\mathbf{x_{-i}}\gg 0$, the unique optimal allocation of player i, $\mathbf{x_i}=(x_{i1},...,x_{in})$, to battlefields $b\in B$ is characterized as follows:*

$$x_{ib}=\begin{cases}\frac{(x_{-ib}v_{ib})^{\frac{1}{2}}}{\sum_{j\in K}(x_{-ij}v_{ij})^{\frac{1}{2}}}\left(X_i+\sum_{j\in K}x_{-ij}\right)-x_{-ib} & \text{if } b\in K\\ 0 & \text{if } b\in B\setminus K\end{cases} \tag{8}$$

where $K=\{1,...,k^*\}$ *is such that* $k^*=max\{k\in B:\frac{v_{ij}}{x_{-ij}}>\frac{\left(\sum_{l=1}^{j}(x_{-il}v_{il})^{\frac{1}{2}}\right)^2}{\left(X_i+\sum_{l=1}^{j}x_{-il}\right)^2}\ \forall j\leq k\}$

Note that the corresponding expression for x_{ib} when $b \in K$ will be strictly positive if

$$\frac{v_{ib}}{x_{-ib}} > \frac{\left(\sum_{j\in K}\left(x_{-ij}v_{ij}\right)^{\frac{1}{2}}\right)^2}{\left(X_i + \sum_{j\in K} x_{-ij}\right)^2}. \tag{9}$$

Since (9) holds by definition for $b = k^*$, (5) implies that all battlefields $b \in K$ receive a strictly positive allocation.

Proof. The maximization problem in section 2.1 satisfies the linearity constraint qualifications and the Kuhn-Tucker conditions are necessary. In addition, the conditions for sufficiency also hold: concavity of the value function and inequalities described by continuously differentiable convex functions. Consequently, there is a unique global constrained maximizer (see, for example, Theorem M.K.4 in [3]).

In what follows, we verify that the candidate solution of Proposition 1 satisfies the Kuhn-Tucker conditions, which are necessary and sufficient and conclude that the candidate solution must be the unique global constrained maximizer of the problem. Proposition 1 suggests an optimizer in which positive allocations $x_{ip} > 0$ are assigned to battlefields $p \in K$ and a zero allocation, $x_{iz} = 0$, to battlefields $z \in B \setminus K$.

To verify the claim for the first case, note that, for a positive allocation, the complementary slackness condition (4) requires $\mu_p = 0 \quad \forall p \in K$. Therefore, condition (1) for battlefields $p \in K$ is reduced to:

$$\frac{x_{-ip}v_{ip}}{\left(x_{ip} + x_{-ip}\right)^2} = \mu_0 \quad \forall p \in K. \tag{10}$$

Note that $\mu_0 > 0$ and $\mu_p = 0$ satisfy the dual feasibility conditions (2). Rearranging and taking the square root,

$$x_{ip} + x_{-ip} = \left(\frac{x_{-ip}v_{ip}}{\mu_0}\right)^{\frac{1}{2}} \quad \forall p \in K.$$

Summing both sides over the elements $j \in K$,

$$\sum_{j\in K} x_{ij} + \sum_{j\in K} x_{-ij} = \frac{\sum_{j\in K}\left(x_{-ij}v_{ij}\right)^{\frac{1}{2}}}{\mu_0^{\frac{1}{2}}}.$$

Given that $\mu_0 > 0$, the complementary slackness condition (3) implies $X_i - \sum_{j=1}^{n} x_{ij} = 0$. Because (8) implies that $x_{ip} > 0$ if and only if $p \in K$, it follows that $\sum_{j \in K} x_{ij} = X_i$. Therefore,

$$X_i + \sum_{j \in K} x_{-ij} = \frac{\sum_{j \in K} (x_{-ij} v_{ij})^{\frac{1}{2}}}{\mu_0^{\frac{1}{2}}}.$$

Solving for μ_0,

$$\mu_0 = \frac{\left(\sum_{j \in K} (x_{-ij} v_{ij})^{\frac{1}{2}}\right)^2}{\left(X_i + \sum_{j \in K} x_{-ij}\right)^2}. \tag{11}$$

Substituting from (10) for μ_0 in (11) and rearranging,

$$x_{ip} = \frac{(x_{-ip} v_{ip})^{\frac{1}{2}}}{\sum_{j \in K} (x_{-ij} v_{ij})^{\frac{1}{2}}} \left(X_i + \sum_{j \in K} x_{-ij} \right) - x_{-ip}. \tag{12}$$

Thus, we have confirmed that the necessary and sufficient conditions hold for the specification of x_{ib}, $b \in K$, in Proposition 1. To verify the claim for $z \in B \setminus K$, note that, when $x_{iz} = 0$ for $z \in B \setminus K$, (1) reduces to

$$\frac{v_{iz}}{x_{-iz}} = \mu_0 - \mu_z \quad \forall z \in B \setminus K. \tag{13}$$

Substituting the RHS of expression (11) for μ_0 in expression (13) and solving for μ_z,

$$\mu_z = \frac{\left(\sum_{j \in K} (x_{-ij} v_{ij})^{\frac{1}{2}}\right)^2}{\left(X_i + \sum_{j \in K} x_{-ij}\right)^2} - \frac{v_{iz}}{x_{-iz}} \quad \forall z \in B \setminus K.$$

The dual feasibility condition (2) $\mu_z \geq 0$ holds for $z \in B \setminus K$ if and only if the RHS of the expression above is non-negative.

Rearranging,

$$\frac{v_{iz}}{x_{-iz}} \leq \frac{\left(\sum_{j \in K} (x_{-ij} v_{ij})^{\frac{1}{2}}\right)^2}{\left(X_i + \sum_{j \in K} x_{-ij}\right)^2} \quad \forall z \in B \setminus K. \tag{14}$$

Note that the remaining complementary slackness condition (4) is trivially satisfied. Condition (14) and the condition defining elements of K in (6) are mutually exclusive and exhaust all possible cases. Thus, our proposed characterization of x_{ib} is the unique solution. □

4. Revisiting Friedman's Best-Response Function

In a version where $v_{ij} = v_{-ij}$, Friedman [2] provides the following characterization for the best response function (see equation (12) of [2]).

$$x_{ij} = \frac{(x_{-ij}v_{ij})^{\frac{1}{2}}}{\sum_{j=1}^{n}(x_{-ij}v_{ij})^{\frac{1}{2}}}(X_i + X_{-i}) - x_{-ij} \tag{15}$$

This proposed expression coincides with expression (12) when player $-i$ exhausts his budget and when $K = B$. However, $K = B$ is not generally true. Thus, Friedman's expression can result in negative allocations under certain combinations of admissible primitives. For example, when $n = 2$, $v_{11} = v_{12} = 1$, $X_1 = 5$, $X_2 = 50$, $x_{21} = 10$ and $x_{22} = 40$, his proposed expression results in $x_{12} = -\frac{10}{3} < 0$. This is not feasible.

Proposition 1 puts this problem into perspective. A negative allocation to battlefield 2 in Friedman's proposed solution implies that condition (14) is satisfied with strict inequality. Thus, battlefield $2 \notin K$. Consequently, $x_{12} = 0$. Given the primitives of the example and $\mathbf{x_2} = (10, 40)$, it can be easily verified that, according to Proposition 1, the optimal allocations to battlefields $1 \in K$ and $2 \notin K$ are $x_{11} = X_1 = 5$ and $x_{12} = 0$, respectively.

Acknowledgments

We thank an anonymous referee of [1] for suggesting that we examine this issue and Sebastian Cortes-Corrales for helpful comments.

References

[1] S. M. Chowdhury, D. Kovenock, D. Rojo Arjona, and N. T. Wilcox. Focality and asymmetry in multi-battle contests. Working Paper 16-16. Economic Science Institute, 2016.

[2] L. Friedman. Game-theory models in the allocation of advertising expenditures. *Operations Research*, 6(5):699–709, 1958.

[3] A. Mas-Colell, M. Whinston, and J. Green. *Microeconomic Theory.* Oxford University Press, 1995.

[4] G. Tullock. Efficient rent seeking. In J. M. Buchanan, R. D. Tollison, and G. Tullock, editors, *Toward a theory of the rent-seeking society*, pages 97–113. College Station, TX: Texas A&M University Press, 1980.