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Quaternionic Hermitian spinor systems and compatibility conditions

Alberto Damiano,* David Eelbode† and Irene Sabadini

(Communicated by G. Gentili)

Abstract. In this paper we show that the systems introduced in [12] and [22] are equivalent, both giving the notion of quaternionic Hermitian monogenic functions. This makes it possible to prove that the free resolution associated to the system is linear in any dimension, and that the first cohomology module is nontrivial, thus generalizing the results in [22]. Furthermore, exploiting the decomposition of the spinor space into $\mathfrak{sp}(m)$-irreducibles, we find a certain number of "algebraic" compatibility conditions for the system, suggesting that the usual spinor reduction is not applicable.

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1 Introduction

Classical Clifford analysis can be seen as an elegant tool for studying (elliptic) first-order differential operators which are invariant with respect to a suitable action of the orthogonal Lie algebra $\mathfrak{so}(m)$. Since this branch of classical analysis is nowadays to be considered as an independent field of research, we here only mention the generally accepted standard references [3, 11, 16] or [8] for an approach through algebraic analysis methods. Whereas most of the classical work is centered around the Dirac operator $\underline{\partial}_x$ in $\mathbb{R}^m$ acting on functions taking values in either the whole Clifford algebra $\mathbb{R}_m$ or the spinor spaces $\mathbb{S}$, Clifford analysis techniques have also successfully been used to investigate higher spin Dirac operators such as Rarita–Schwinger operators, see [5, 26], and operators between more general representation spaces. From a more general point of view, all these differential operators arise as projections of the Stein–Weiss generalized gradient ∇ acting on sections φ of suitable (higher) spin bundles on a Riemannian spin manifold $\mathcal{M}$, see e.g. [25].

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Recently however, the notions of complex and of quaternionic *Hermitian* Clifford analysis were introduced. These are refinements of the classical orthogonal framework in which two complex Dirac operators (respectively four quaternionic Dirac operators), commuting with the action of the unitary algebra $\mathfrak{u}(m)$ (respectively the real symplectic algebra $\mathfrak{sp}(m)$) are studied. The key point is that the Lie algebras underlying the symmetry of the system can be realized inside the Lie algebra $\mathbb{C}_{2m}^{(2)}$, allowing one to formulate the resulting function system in terms of standard objects from the orthogonal setting such as the Dirac operator $\underline{\partial}_x$ on $\mathbb{R}^{2m}$ and spinor spaces. It also means that complex (respectively quaternionic) Hermitian Clifford analysis allows the study of irreducible $\mathfrak{sl}(m)$-modules within a function theoretical context (respectively $\mathfrak{sp}_{2m}(\mathbb{C})$-modules). This theory is still under full development, we refer e.g. to [1, 2, 4, 14, 24] for the complex setting and [12, 13, 22] for the quaternionic setting. In this paper, we extend the analysis started in our paper [10] with the complex Hermitian case to the quaternionic Hermitian case. In particular, we investigate some algebraic properties of the complex associated to the Hermitian system which entails some analytic consequences. To this purpose, we will combine techniques coming from algebraic analysis using computational techniques, see reference [8], with the language of representation theory, see [15].

The results we obtain show that, despite the fact that quaternionic Hermitian monogenic functions in one variable are described by a system of equations (and not by a unique equation like classical monogenic functions, which are nullsolutions of the Dirac operator) and have a nontrivial resolution, they possess important properties of functions in one variable, like the existence of non-removable compact singularities.

The system describing quaternionic Hermitian monogenic functions consists of four equations which, translated into real components, have several redundancies. In the last section we describe, at least in low dimensions, the nature of these redundancies.

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2 Background material

We begin this section by recalling the basic facts which lead to the Hermitian setting. For a more complete treatment we refer the reader to [14]. Let us consider the orthogonal vector space $\mathbb{R}^{4m}$ endowed with the symmetric real bilinear form $\mathcal{B}_{\mathbb{R}}(\,\cdot\,,\,\cdot\,)$ of signature $(0, 4m)$. The group $\mathrm{SO}_{\mathbb{R}}(4m)$ is the real Lie group of automorphisms of $\mathbb{R}^{4m}$ of unit determinant which preserve the inner product associated to $\mathcal{B}_{\mathbb{R}}(\,\cdot\,,\,\cdot\,)$. The choice of a specific element $I_{4m} \in \mathrm{SO}_{\mathbb{R}}(4m)$ such that $I_{4m}^2 = -\mathbf{1}_{4m}$ allows to define a complex structure on the vector space $\mathbb{R}^{4m}$. Let us introduce the standard orthonormal basis $\{e_p\}$ for which $\mathcal{B}_{\mathbb{R}}(e_p, e_q) = -\delta_{pq}$ and let us define the complex structure

$$I_{4m} = I = \mathrm{diag}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

where the 2×2 matrices are repeated $2m$ times. The so-called Witt basis in $\mathbb{C}^{4m} = \mathbb{R}^{4m} \otimes \mathbb{C}$ is defined by

$$\mathfrak{f}_p = +\frac{1}{2}(e_{2p-1} - ie_{2p}) \quad \text{and} \quad \mathfrak{f}_p^\dagger = -\frac{1}{2}(e_{2p-1} + ie_{2p}), \quad p = 1, \dots, 2m.$$

If we set W^+ and W^- the subspaces of $\mathbb{C}^{4m}$ generated by $\{\mathfrak{f}_p\}$ and $\{\mathfrak{f}_p^\dagger\}$ respectively, we have that $\mathbb{C}^{4m} = W^+ \oplus W^-$. Indeed, the operators

$$\pi^+ = \frac{1}{2}(1 + iI) \quad \text{and} \quad \pi^- = \frac{1}{2}(1 - iI)$$

are mutually annihilating projection operators with $\mathfrak{f}_p = \pi^+(e_{2p-1})$, $\mathfrak{f}_p^\dagger = -\pi^-(e_{2p-1})$ for all $p = 1, \dots, 2m$.

For $X \in \mathbb{R}^{4m}$, we have:

$$\pi^+(X) = \underline{Z} = \sum_{p=1}^{2m} \mathfrak{f}_p z_p \in \mathbb{C}^{2m}, \qquad z_p = x_{2p-1} + ix_{2p}$$

$$\pi^-(X) = -\underline{Z}^\dagger = \sum_{p=1}^{2m} \mathfrak{f}_p^\dagger z_p^c \in \mathbb{C}^{2m}, \quad z_p^c = x_{2p-1} - ix_{2p}.$$

Remark 1. The subspaces $W^\pm$ are eigenspaces of the complex linear map $I_\mathbb{C}$ with respect to the eigenvalues $\pm i$. Defining $\mathcal{B}_\mathbb{C}$ as the complex extension of $\mathcal{B}_\mathbb{R}$ it is immediate to verify that $W^\pm$ are isotropic.

Remark 2. The real Lie algebra $\mathfrak{su}(2m)$ and its complexification $\mathfrak{sl}(2m)$ can be realized inside the Lie algebra $\mathbb{C}_{4m}^{(2)}$ of bivectors in $\mathbb{C}_{4m}$, see [14]. This fact implies the possibility to define, starting from the Dirac operator on $\mathbb{R}^{4m}$, differential operators which are invariant with respect to the action of the special unitary group and which act on functions with values into irreducible $\mathfrak{sl}(2m)$-modules. Those operators are now well known in the case of Hermitian Clifford analysis, see [1], [2], [4], [24].

In order to introduce the quaternionic Hermitian setting, one could consider the standard quaternionic Hermitian form $(\,\cdot\,,\,\cdot\,)_\mathbb{H} : \mathbb{H}^m \times \mathbb{H}^m \to \mathbb{H}$ as follows:

$$(\underline{q}^1, \underline{q}^2)_\mathbb{H} = \sum_{p=1}^{m} \bar{q}_p^1 q_p^2,$$

where $\underline{q}^i = (q_1^i, \dots, q_m^i)$, $i = 1, 2$ and $\bar{q} = x_0 - ix_1 - jx_2 - kx_3$ denotes the conjugate of the quaternion $q = x_0 + ix_1 + jx_2 + kx_3$. The subgroup of $\mathrm{GL}_n(\mathbb{H})$ preserving the above quadratic form gives rise to the symplectic group $\mathrm{Sp}(m)$. It is also possible to introduce this Lie group by defining another complex structure $J_{4m} \in \mathrm{SO}_\mathbb{R}(4m)$ as

$$J_{4m} = J = \mathrm{diag}\left\{ \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}$$

where the 4×4 blocks are repeated m times. With this notation one has

$$\mathrm{Sp}(m) \simeq \mathrm{SO}_{I,J}(4m) := \big\{A \in \mathrm{SO}_{\mathbb{R}}(4m) \mid [A, I] = [A, J] = 0\big\}.$$

A double cover for this group will be denoted by $\mathrm{Spin}_{I,J}(4m)$. Note that I, J can be associated to two bivectors σ_I, $\sigma_J \in \mathbb{R}^{(2)}_{4m}$. It turns out that the product $\sigma_I \sigma_J$ is a third bivector σ_K which can be associated to the element $K = IJ \in \mathrm{SO}_{\mathbb{R}}(4m)$. The vector spaces generated by either $\{1, \sigma_I, \sigma_J, \sigma_K\}$ or $\{\mathbf{1}_{4m}, I, J, K\}$ are both isomorphic, as algebras, to $\mathbb{H}$.

According to [14] we give the following:

Definition 1. Given the complex structures $I, J \in \mathrm{SO}_{\mathbb{R}}(4m)$, we define the projection operators

$$\Pi^{\pm}_{\pm} = \frac{1 \pm jJ}{2} \frac{1 \pm iI}{2} : \mathbb{R}^{4m} \to \mathbb{R}^{4m} \otimes \mathbb{H},$$

where the upper (respectively lower) index refers to $\pm I$ (respectively $\pm J$).

Remark 3. The operators $\Pi^{\pm}_{\pm}$ are mutually annihilating idempotents, thus they are projectors. The subspaces $\Pi^{\pm}_{\pm}[\mathbb{R}^{4m}]$ of $\mathbb{R}^{4m} \otimes \mathbb{H}$ are invariant with respect to the symplectic group $\mathrm{Sp}(m)$ and its Lie algebra $\mathfrak{sp}(m)$.

Definition 2. We define the four quaternionic Hermitian Dirac operators as $\Pi^{\pm}_{\pm}[\partial_X]$, where $\partial_X = \sum_{p=1}^{4m} e_p \partial_{X_p}$ is the standard Dirac operator on $\mathbb{R}^{4m}$.

The definition is a refinement of the notion of complex Hermitian operators on $\mathbb{C}^{2m}$, indeed

$$\Pi^{+}_{+}[\partial_X] + \Pi^{+}_{-}[\partial_X] = \frac{1 + iI}{2}[\partial_X] = +2\underline{\partial}^{\dagger}_{Z}$$
$$\Pi^{-}_{+}[\partial_X] + \Pi^{-}_{-}[\partial_X] = \frac{1 - iI}{2}[\partial_X] = -2\underline{\partial}_{Z}.$$

We have the following proposition:

Proposition 1. *The quaternionic Hermitian Dirac operators $\Pi^{\pm}_{\pm}[\partial_X]$ commute with the L-action of $\mathrm{Spin}_{I,J}(4m)$ defined by $L(s)[f](x) = sf(\bar{s}xs)$, and with the derived dL-action of $\mathfrak{sp}(m)$; in other words, as operators acting on functions with values in $\mathbb{R}_{4m} \otimes \mathbb{H}$ we have that*

$$[\Pi^{\pm}_{\pm}[\partial_X], L(s)] = 0 = [\Pi^{\pm}_{\pm}[\partial_X], dL(B)]$$

for all $s \in \mathrm{Spin}_{I,J}(4m)$ and all $B \in \mathfrak{sp}(m)$.

3 Quaternionic Hermitian monogenic functions

In this section we discuss several equivalent definitions of Hermitian monogenicity. As in [12] we give the following

Definition 3. Let U be an open set in $\mathbb{R}^{4m}$. A real differentiable function $f : U \subseteq \mathbb{R}^{4m} \to \mathbb{R}_{4m} \otimes \mathbb{H}$ is called quaternionic Hermitian monogenic if it satisfies the system

$$\Pi^+_+[\partial_X](f) = \Pi^+_-[\partial_X](f) = \Pi^-_+[\partial_X](f) = \Pi^-_-[\partial_X](f) = 0. \tag{1}$$

Remark 4. Since we can rewrite the system as

$$\begin{aligned}
4\Pi^+_+[\partial_X](f) &= (\partial_X + iI[\partial_X] + jJ[\partial_X] + kK[\partial_X])f = 0\\
4\Pi^+_-[\partial_X](f) &= (\partial_X + iI[\partial_X] - jJ[\partial_X] - kK[\partial_X])f = 0\\
4\Pi^-_+[\partial_X](f) &= (\partial_X - iI[\partial_X] + jJ[\partial_X] - kK[\partial_X])f = 0\\
4\Pi^-_-[\partial_X](f) &= (\partial_X - iI[\partial_X] - jJ[\partial_X] + kK[\partial_X])f = 0
\end{aligned}$$

by taking suitable linear combinations of the equations we have that system (1) is equivalent to

$$\begin{cases}
\partial_X(f) = 0\\
I[\partial_X](f) = 0\\
J[\partial_X](f) = 0\\
K[\partial_X](f) = 0.
\end{cases} \tag{2}$$

In the paper [22] it has been introduced a notion of quaternionic Hermitian monogenic functions which turns out to be equivalent to the one given in Definition 3, as we will see in a while. To this purpose, let us briefly recall some definition from [22]. We first introduce the quaternionic Witt basis.

Definition 4. The quaternionic Witt basis of $\mathbb{H}_{4m} := \mathbb{R}_{4m} \otimes_{\mathbb{R}} \mathbb{H}$, is given by $\{f_\ell, f_\ell^\alpha, f_\ell^\beta, f_\ell^\gamma\}$, where

$$\begin{aligned}
f_\ell &= e_{1+4(\ell-1)} - ie_{2+4(\ell-1)} - je_{3+4(\ell-1)} - ke_{4\ell}\\
f_\ell^\alpha &= e_{1+4(\ell-1)} - ie_{2+4(\ell-1)} + je_{3+4(\ell-1)} + ke_{4\ell}\\
f_\ell^\beta &= e_{1+4(\ell-1)} + ie_{2+4(\ell-1)} - je_{3+4(\ell-1)} + ke_{4\ell}\\
f_\ell^\gamma &= e_{1+4(\ell-1)} + ie_{2+4(\ell-1)} + je_{3+4(\ell-1)} - ke_{4\ell}
\end{aligned}$$

for $\ell = 1, \dots, m$.

We introduce the operator (so-called quaternionic Hermitian vector derivative)

$$\partial_{\underline{q}} = \sum_{\ell=1}^{m} f_\ell \partial_{q_\ell}$$

where

$$\partial_{q_\ell} = \partial_{x_{1+4(\ell-1)}} + i\partial_{x_{2+4(\ell-1)}} + j\partial_{x_{3+4(\ell-1)}} + k\partial_{x_{4\ell}}, \quad \text{for } \ell = 1, \dots, m$$

and its variations

$$\begin{aligned}
(\partial_{\underline{q}})^\alpha &= \sum_{\ell=1}^m (f_\ell \partial_{q_\ell})^\alpha = \sum_{\ell=1}^m f_\ell^\alpha (\partial_{q_\ell})^\alpha \\
&= \sum_{\ell=1}^m (e_{4\ell-3} - ie_{4\ell-2} + je_{4\ell-1} + ke_{4\ell})(\partial_{x_{4\ell-3}} + i\partial_{x_{4\ell-2}} - j\partial_{x_{4\ell-1}} - k\partial_{x_{4\ell}}), \\
(\partial\underline{q})^\beta &= \sum_{\ell=1}^m (f_\ell \partial_{q_\ell})^\beta = \sum_{\ell=1}^m f_\ell^\beta (\partial_{q_\ell})^\beta \\
&= \sum_{\ell=1}^m (e_{4\ell-3} + ie_{4\ell-2} - je_{4\ell-1} + ke_{4\ell})(\partial_{x_{4\ell-3}} - i\partial_{x_{4\ell-2}} + j\partial_{x_{4\ell-1}} - k\partial_{x_{4\ell}}), \\
(\partial_{\underline{q}})^\gamma &= \sum_{\ell=1}^m (f_\ell \partial_{q_\ell})^\gamma = \sum_{\ell=1}^m f_\ell^\gamma (\partial_{q_\ell})^\gamma \\
&= \sum_{\ell=1}^m (e_{4\ell-3} + ie_{4\ell-2} + je_{4\ell-1} - ke_{4\ell})(\partial_{x_{4\ell-3}} - i\partial_{x_{4\ell-2}} - j\partial_{x_{4\ell-1}} + k\partial_{x_{4\ell}}).
\end{aligned}$$

We can introduce another notation. Let us write

$$\partial_{\underline{q}} =^0 \partial_{\underline{q}} + i^1\partial_{\underline{q}} + j^2\partial_{\underline{q}} + k^3\partial_{\underline{q}},$$

where

$$\begin{aligned}
{}^0\partial_{\underline{q}} &= \sum_{l=1}^m (e_{4l-3}\partial_{x_{4l-3}} + e_{4l-2}\partial_{x_{4l-2}} + e_{4l-1}\partial_{x_{4l-1}} + e_{4l}\partial_{x_{4l}}) \\
{}^1\partial_{\underline{q}} &= \sum_{l=1}^m (e_{4l-3}\partial_{x_{4l-2}} - e_{4l-2}\partial_{x_{4l-3}} - e_{4l-1}\partial_{x_{4l}} + e_{4l}\partial_{x_{4l-1}}) \\
{}^2\partial_{\underline{q}} &= \sum_{l=1}^m (e_{4l-3}\partial_{x_{4l-1}} + e_{4l-2}\partial_{x_{4l}} - e_{4l-1}\partial_{x_{4l-3}} - e_{4l}\partial_{x_{4l-2}}) \qquad (3) \\
{}^3\partial_{\underline{q}} &= \sum_{l=1}^m (e_{4l-3}\partial_{x_{4l}} - e_{4l-2}\partial_{x_{4l-1}} + e_{4l-1}\partial_{x_{4l-2}} - e_{4l}\partial_{x_{4l-3}}).
\end{aligned}$$

It is immediate to verify that

$$\begin{aligned}
{}^0\partial_{\underline{q}} &= \frac{1}{4}\left(\partial_{\underline{q}} + \partial_{\underline{q}}^\alpha + \partial_{\underline{q}}^\beta + \partial_{\underline{q}}^\gamma\right) \\
{}^1\partial_{\underline{q}} &= -\frac{i}{4}\left(\partial_{\underline{q}} + \partial_{\underline{q}}^\alpha - \partial_{\underline{q}}^\beta - \partial_{\underline{q}}^\gamma\right)
\end{aligned}$$

$$
{}^2\partial_{\underline{q}} = -\frac{j}{4}\left(\partial_{\underline{q}} - \partial^{\alpha}_{\underline{q}} + \partial^{\beta}_{\underline{q}} - \partial^{\gamma}_{\underline{q}}\right)
$$

$$
{}^3\partial_{\underline{q}} = -\frac{k}{4}\left(\partial_{\underline{q}} - \partial^{\alpha}_{\underline{q}} - \partial^{\beta}_{\underline{q}} + \partial^{\gamma}_{\underline{q}}\right).
$$

Definition 5. Let U be an open set in $\mathbb{R}^{4m}$. A real differentiable function $f : U \subseteq \mathbb{R}^{4m} \to \mathbb{H}_{4m}$ is called q-Hermitian monogenic if it satisfies all the four equations

$$
\partial_{\underline{q}} f = \partial^{\alpha}_{\underline{q}} f = \partial^{\beta}_{\underline{q}} f = \partial^{\gamma}_{\underline{q}} f = 0,
$$

or, equivalently,

$$
{}^0\partial_{\underline{q}} f = {}^1\partial_{\underline{q}} f = {}^2\partial_{\underline{q}} f = {}^3\partial_{\underline{q}} f = 0. \tag{4}
$$

Proposition 2. *By choosing suitable complex structures, Definitions* 3 *and* 5 *are equivalent.*

Proof. If we choose the complex structures $I', J' \in \mathrm{SO}(4m)$ defined by

$$
I'_{4m} = I' = \mathrm{diag}\left\{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right\}
$$

$$
J'_{4m} = J' = \mathrm{diag}\left\{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\right\}
$$

and $K' = I'J'$, the corresponding system (2) written with I', J', K' coincides with (4). □

Another useful characterization of quaternionic Hermitian functions (in short q-Hermitian functions) can be given by introducing in $\mathbb{C}_{4m}$ the Witt basis $\{\mathfrak{f}_p, \mathfrak{f}_p^{\dagger}\}$ with $p = 1, \dots, 2m$. We define the operators (see [13]):

$$
\begin{aligned}
\underline{\partial}_Z &= \sum_{p=1}^{2m} \mathfrak{f}_p^{\dagger}\partial_{z_p} \qquad & J[\underline{\partial}_Z] &= i\sum_{p=1}^{2m} \mathfrak{f}_{2p-1}\partial_{z_{2p}} - \mathfrak{f}_{2p}\partial_{z_{2p-1}} \\
\underline{\partial}_Z^{\dagger} &= \sum_{p=1}^{2m} \mathfrak{f}_p\partial_{z_p^c} \qquad & J[\underline{\partial}_Z^{\dagger}] &= i\sum_{p=1}^{2m} \mathfrak{f}_{2p}^{\dagger}\partial_{z_{2p-1}^c} - \mathfrak{f}_{2p-1}^{\dagger}\partial_{z_{2p}^c}
\end{aligned} \tag{5}
$$

where $z_p = x_{2p-1} + ix_{2p}$ and z^c denotes its conjugate. We have (see [13]):

Proposition 3. *A $\mathbb{C}_{4m}$-valued function f is q-Hermitian monogenic if and only if it satisfies the system*

$$
\underline{\partial}_Z f = \underline{\partial}_Z^{\dagger} f = J[\underline{\partial}_Z] f = J[\underline{\partial}_Z^{\dagger}] f = 0. \tag{6}
$$

Remark 5. Note that the four operators in (6) are invariant with respect to the action of the complex symplectic algebra $\mathfrak{sp}_{2m}(\mathbb{C})$.

Let us now introduce the following notation:

Definition 6. For all $0 \leq j \leq 2m$, we define the space $\mathbb{S}_j$ of homogeneous spinors of degree j as $\mathbb{S}_j = \Lambda_j^\dagger \mathbb{I}$, where $\Lambda_j^\dagger$ is the subspace generated by the products of j-vectors $\mathfrak{f}_k^\dagger \in W^-$ and $\mathbb{I} = \mathfrak{f}_1 \mathfrak{f}_1^\dagger \ldots \mathfrak{f}_{2m} \mathfrak{f}_{2m}^\dagger$.

We have the following:

Proposition 4. *With respect to the decomposition* $\mathbb{S} = \bigoplus_{j=0}^{2m} \mathbb{S}_j$ *we can represent the Dirac operators appearing in system* (6) *as the following block matrices with entries in* $\mathbb{C}[\underline{Z}, \underline{Z}^\dagger]$

$$\underline{\partial}_Z = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{2m} & 0 \end{bmatrix} \tag{7}$$

where the i, j*-th block is a* $\binom{2m}{i}$ *times* $\binom{2m}{j}$ *matrix of either zeroes or given by the restriction of the Dirac operator* $d_i := \pi_{\mathbb{S}_i} \circ [\underline{\partial}_Z]_{|\mathbb{S}_{i-1}} : \mathbb{S}_{i-1} \to \mathbb{S}_i$*, and*

$$\underline{\partial}_Z^\dagger = \begin{bmatrix} 0 & \delta_0 & 0 & \cdots & 0 \\ 0 & 0 & \delta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \delta_{2m-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{8}$$

where $\delta_i := \pi_{\mathbb{S}_i} \circ [\underline{\partial}_Z^\dagger]_{|\mathbb{S}_{i+1}} : \mathbb{S}_{i+1} \to \mathbb{S}_i$*. Similarly we have*

$$J[\underline{\partial}_Z^\dagger] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ d_1' & 0 & \cdots & 0 & 0 \\ 0 & d_2' & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{2m}' & 0 \end{bmatrix} \tag{9}$$

and

$$J[\underline{\partial}_Z] = \begin{bmatrix} 0 & \delta_0' & 0 & \cdots & 0 \\ 0 & 0 & \delta_1' & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \delta_{2m-1}' \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{10}$$

in which the blocks d_i', δ_i' can be obtained from d_i, δ_i by the substitutions $z_{2p-1} \to -z_{2p}^c$, $z_{2p} \to z_{2p-1}^c$, $z_{2p-1}^c \to z_{2p}$, $z_{2p}^c \to -z_{2p-1}$.

Proof. It is an immediate consequence of the preceding discussion. □

Proposition 5. *The operators $\underline{\partial}_Z$, $\underline{\partial}_Z^\dagger$, $J[\underline{\partial}_Z]$, $J[\underline{\partial}_Z^\dagger]$ satisfy the relations*

$$
\begin{gathered}
(\underline{\partial}_Z)^2 = (\underline{\partial}_Z^\dagger)^2 = (J[\underline{\partial}_Z])^2 = (J[\underline{\partial}_Z^\dagger])^2 = 0 \\
\underline{\partial}_Z J[\underline{\partial}_Z] + J[\underline{\partial}_Z]\, \underline{\partial}_Z = 0 \\
\underline{\partial}_Z^\dagger J[\underline{\partial}_Z] + J[\underline{\partial}_Z]\, \underline{\partial}_Z^\dagger = 0 \\
\underline{\partial}_Z\, J[\underline{\partial}_Z^\dagger] + J[\underline{\partial}_Z^\dagger]\, \underline{\partial}_Z = 0 \\
\underline{\partial}_Z^\dagger\, J[\underline{\partial}_Z^\dagger] + J[\underline{\partial}_Z^\dagger]\, \underline{\partial}_Z^\dagger = 0 \\
4(\underline{\partial}_Z\, \underline{\partial}_Z^\dagger + \underline{\partial}_Z^\dagger\, \underline{\partial}_Z) = 4(J[\underline{\partial}_Z]\, J[\underline{\partial}_Z^\dagger] + J[\underline{\partial}_Z^\dagger]\, J[\underline{\partial}_Z]) = \Delta.
\end{gathered}
$$

Proof. The first and the last relations are well known (see e.g. [10]). The remaining relations can be proved by direct computation by writing the various operators as in (5). □

Proposition 6. *The relations of the previous proposition translate into the following*

$$
\begin{aligned}
d_{i+1} d_i &= 0, \quad & i &= 1, \dots, 2m-1 \\
\delta_i \delta_{i+1} &= 0, & i &= 0, \dots, 2m-2 \\
d_{i+1} \delta_i' + \delta_{i+1}' d_{i+2} &= 0, & i &= 0, \dots, 2m-2, \\
\delta_0' d_1 &= 0, \\
d_{2m} \delta_{2m-1}' &= 0, \\
\delta_i \delta_{i+1}' + \delta_i' \delta_{i+1} &= 0, & i &= 0, \dots, 2m-2 \\
d_i d_{i-1}' + d_i' d_{i-1} &= 0, & i &= 2, \dots, 2m \\
\delta_{i+1} d_{i+2}' + d_{i+1}' \delta_i &= 0, & i &= 0, \dots, 2m-2, \\
\delta_0 d_1' &= 0, \\
d_{2m}' \delta_{2m-1} &= 0.
\end{aligned}
$$

Furthermore, setting a and b to be any two symbols within the set $\{d, \delta, d', \delta'\}$, and choosing any two indices i, j such that $j \neq i \pm 1$, we have $a_i b_j = 0$.

Proof. The first set of relations immediately follows from the previous proposition and by the description of the operators in matrix form given in Proposition 4. The last set is a consequence of the fact that the two matrices representing a_i and b_j are incompatible. □

Remark 6. It is in fact the case that some extra conditions hold for the operators in the q-Hermitian system, for instance in the case $m = 1$ we have $d_2' = \delta_0$ and $\delta_0' = d_2$. Such conditions do not really constitute differential conditions, but rather they reflect the fact that some of the rows of P_{4m} are linearly dependent. In Section 5 we will also give an interpretation of such “degree zero" relations, which we also refer to as “algebraic constraints” as opposed to “differential constraints”, from the point of view of symplectic invariance.

4 Algebraic analysis of the operators

We begin by the description of the characteristic variety of the module associated to the q-Hermitian system. In the following, R will denote the polynomial ring $\mathbb{C}[x_1, \ldots, x_{4m}]$.

Definition 7. Let $P(D)$ be the matrix associated to the q-Hermitian system and let P be the polynomial matrix symbol of $P(D)$ whose rows form a set of minimal generators. We define $\mathcal{M}_{4m}$ to be R-module given by the cokernel of the map P^t.

Theorem 1. *The characteristic variety of the R-module $\mathcal{M}_{4m}$ is given by*

$$\left\{ X = (x_1, \ldots, x_{4m}) \in \mathbb{C}^{4m} \;\middle|\; \sum_{i=1}^{4m} x_i^2 = 0 \right\}.$$

Proof. To show the result we will consider the q-Hermitian system (4). Written in this form, the system is given by the Dirac operator $\partial_X = \partial_{X_1}$ and the variations ∂_{X_i}, $i = 2, 3, 4$ obtained by suitable changes of variables (see (3)). Let us denote the Fourier transform of the four blocks of the matrix associated to the system by $M_1, \ldots, M_4$. Note that M_1 is the matrix whose columns correspond to the vector variable $X = X_1$ and its multiplication by a unit $e_{i_1 \ldots i_r}$. The matrices M_2, M_3, M_4 are obtained in a similar way, through the vector variables X_2, X_3, X_4 which are obviously obtained by applying suitable variable substitutions on X_1. Note that $X_i \cdot X_j = 0$ if $i \neq j$. The determinant of each block is given by a power of the polynomial $\sum_{i=1}^{4m} x_i^2$, so the equation $\sum_{i=1}^{4m} x_i^2 = 0$ is one of the defining equations of the characteristic variety. We now consider a maximal minor M whose columns are taken from the various blocks. The columns of M will be $Ze, Z'e', Z''e'', \ldots$ where $Z, Z', Z'', \ldots$ are X_i, $i = 1, \ldots, 4$ and $e, e', e'', \ldots$ are units. Modulo a right multiplication by e, so that the first column of M is Z, we can compute $M^T M$:

$$M^T M = \begin{bmatrix} Z \\ Z'e' \\ \ldots \\ Z''e'' \end{bmatrix} \begin{bmatrix} Z & Z'e' & \ldots & Z''e'' \end{bmatrix} = \begin{bmatrix} Z^2 & ZZ'e' & \ldots & ZZ''e'' \\ \ldots & & \ldots & \end{bmatrix}.$$

The elements of the first row are either zero, by the orthogonality relations among the X_i's, or they are equal to $\sum_{i=1}^{4m} x_i^2$. Thus for any X_i satisfying $\sum_{i=1}^{4m} x_i^2 = 0$ the determinant of $M^T M$ is already zero and so is $\det M$. □

Remark 7. In general, when dealing with several Dirac operators, the orthogonality conditions are required for the description of the characteristic variety (see [7]). Here these conditions are automatically satisfied.

Corollary 1. *For the module $\mathcal{M}_{4m}$ we have* $\mathrm{Ext}^1(\mathcal{M}_{4m}, R) \neq 0$.

Proof. By contradiction, if $\mathrm{Ext}^1(\mathcal{M}_{4m}, R) = 0$, then by Corollary 1, p. 377 in [21], the dimension of the characteristic variety would be less than $4m - 1$. □

Remark 8. In the complex Hermitian case all the relevant Ext-modules are nonzero. In the q-Hermitian case some Ext-modules can be zero, and some other can be nonzero depending on the dimension. For example, in the case $m = 1$ only the first and the last Ext-module are nonzero, while the remaining ones vanish identically.

A consequence of this discussion and in particular of the fact that $\mathrm{Ext}^1(\mathcal{M}_{4m}, R) \neq 0$ is the following result which we have already encountered in the case of the complex Hermitian monogenic functions, see [9], [10], and which has already been noticed in [22] in the low dimensional cases:

Corollary 2. *Let K be a compact convex subset of an open set $U \subseteq \mathbb{R}^{4m}$, $m \geq 1$ and let f be a q-Hermitian monogenic function on $U \setminus K$. Then f cannot in general be extended to a q-Hermitian monogenic function on U.*

Example 1. To provide an example of a q-Hermitian monogenic function with a compact non-removable singularity, we cannot use the kernel functions appearing in the integral representation formulas (see [22]) since these functions are not q-Hermitian monogenic. Note that this is the case even when considering Hermitian monogenic functions in the complex setting: it is wellknown that the kernel used in the Cauchy integral formula is not Hermitian monogenic itself. In view of the fact that both classical Clifford analysis as well as the complex and the quaternionic Hermitian Clifford analysis refine the theory of harmonic functions, it seems natural to start from the fundamental solution for the Laplace operator Δ_m on $\mathbb{R}^m$ i.e. (up to a constant):

$$E_m(r) := r^{2-m}, \quad m \geq 4,\ m \text{ even},$$

where $r^{-2} = \{\underline{Z}, \underline{Z}^\dagger\}^{-1}$. Let us consider the simplest case, i.e. $m = 4$ and let us consider the $\mathbb{S}_1$-valued function defined by

$$E_4^{(1)}(r) := E_4(r)(\mathfrak{f}_1^\dagger + \mathfrak{f}_2^\dagger)\mathbb{I}, \quad \text{where} \quad \mathbb{I} = \mathfrak{f}_1\mathfrak{f}_1^\dagger\mathfrak{f}_2\mathfrak{f}_2^\dagger.$$

By recalling Proposition 4, it is easily verified that also the function

$$J[\underline{\partial}_Z^\dagger]J[\underline{\partial}_Z]\underline{\partial}_Z^\dagger\underline{\partial}_Z E_4^{(1)}(r)$$

is $\mathbb{S}_1$-valued. By recalling that $\partial_{z_i} E_4(r) = -z_i\{\underline{Z}, \underline{Z}^\dagger\}^{-2}$ and $\partial_{z_i^c} E_4(r) = -z_i^c\{\underline{Z}, \underline{Z}^\dagger\}^{-2}$ it is then possible to show that

$$\begin{aligned}
\underline{\partial}_Z^\dagger\underline{\partial}_Z E_4^{(1)}(r) &= -\underline{\partial}_Z^\dagger\Big((\mathfrak{f}_1^\dagger z_1 + \mathfrak{f}_2^\dagger z_2)\{\underline{Z}, \underline{Z}^\dagger\}^{-2}(\mathfrak{f}_1^\dagger + \mathfrak{f}_2^\dagger)\mathbb{I}\Big) \\
&= 2(\mathfrak{f}_1 z_1^c + \mathfrak{f}_2 z_2^c)(\mathfrak{f}_1^\dagger z_1 + \mathfrak{f}_2^\dagger z_2)\{\underline{Z}, \underline{Z}^\dagger\}^{-3}(\mathfrak{f}_1^\dagger + \mathfrak{f}_2^\dagger)\mathbb{I} \\
&\quad - (\mathfrak{f}_1\mathfrak{f}_1^\dagger + \mathfrak{f}_2\mathfrak{f}_2^\dagger)\{\underline{Z}, \underline{Z}^\dagger\}^{-2}(\mathfrak{f}_1^\dagger + \mathfrak{f}_2^\dagger)\mathbb{I} \\
&= 2(z_1 z_1^c\mathfrak{f}_1\mathfrak{f}_1^\dagger + z_2 z_1^c\mathfrak{f}_1\mathfrak{f}_2^\dagger + z_1 z_2^c\mathfrak{f}_2\mathfrak{f}_1^\dagger + z_2 z_2^c\mathfrak{f}_2\mathfrak{f}_2^\dagger)\{\underline{Z}, \underline{Z}^\dagger\}^{-3}(\mathfrak{f}_1^\dagger + \mathfrak{f}_2^\dagger)\mathbb{I} \\
&\quad - \{\underline{Z}, \underline{Z}^\dagger\}^{-2}(\mathfrak{f}_1^\dagger + \mathfrak{f}_2^\dagger)\mathbb{I} \\
&= \{\underline{Z}, \underline{Z}^\dagger\}^{-2}(\mathfrak{f}_1^\dagger + \mathfrak{f}_2^\dagger)\mathbb{I} - 2(z_2 z_1^c\mathfrak{f}_2^\dagger + z_1 z_2^c\mathfrak{f}_1^\dagger)\{\underline{Z}, \underline{Z}^\dagger\}^{-3}\mathbb{I}.
\end{aligned}$$

In a similar way, invoking rather lengthy computations, we can prove that

$$J[\underline{\partial}_Z^\dagger]J[\underline{\partial}_Z]\underline{\partial}_Z^\dagger\underline{\partial}_Z E_4^{(1)}(r)$$

is not identically zero and it clearly has a compact singularity at the origin. This function is q-Hermitian monogenic, as can be seen using Proposition 3 and the relations in Proposition 5. Note that it is necessary to consider the $\mathbb{S}_1$-valued function $E_4^{(1)}(r)$ since the $\mathbb{S}_0$ or $\mathbb{S}_2$-valued parts of a function come from a holomorphic or anti-holomorphic system of equations, whose compact singularities can be removed.

Using the description of the characteristic variety for the Dirac system in four vector variables, we can describe the free resolution associated to the q-Hermitian Dirac operator in any dimension, thus generalizing the results in [22].

Theorem 2. *The free resolution of the module $\mathcal{M}_n$ associated to the system* (2) *in real dimension $4m$ is linear of length $4m$.*

Proof. The system of Equations (4) can be written in the form

$$\partial_{\underline{X}_1} f = \partial_{\underline{X}_2} f = \partial_{\underline{X}_3} f = \partial_{\underline{X}_4} = 0, \tag{11}$$

where $\partial_{\underline{X}_1}$ denotes the standard Dirac operator $\partial_{\underline{X}}$ and the operators $\partial_{\underline{X}_i}$, $i = 2, 3, 4$ are obtained from $\partial_{\underline{X}}$ via a suitable change of coordinates. By Theorem 1, the characteristic variety of the module $\mathcal{M}_{4m}$ is given by $\sum_{i=1}^{4m} x_i^2 = 0$. This means that the principal ideal $\mathcal{A} := (\sum_{i=1}^{4m} x_i^2)$ is an associated prime for the module $\mathcal{M}_{4m}$. The depth of $\mathcal{A}$ is $4m$ because it is a free module over $R := \mathbb{C}[x_1, \dots, x_{4m}]$. A wellknown result in commutative algebra states that $\mathrm{pd}(\mathcal{M}_{4m})$, the projective dimension of $\mathcal{M}_{4m}$, is greater than or equal to $\mathrm{depth}(\mathcal{P})$ for every associate prime $\mathcal{P}$, and since the number of variables is an upper bound, we have in this case $\mathrm{pd}(\mathcal{M}_{4m}) = \mathrm{depth}(\mathcal{A}) = 4m$.

Linearity of the resolution follows from Proposition 6. Indeed, one can consider the q-Hermitian Dirac system as a $16m \times 4m$ matrix P_{4m} with entries in $\mathcal{D} := \mathbb{C}\langle d_i, d_i', \delta_i, \delta_i' \mid i = 0 \dots m\rangle$. This is the free associative non-commutative algebra over the set of symbols $\Sigma = \{d_i, d_i', \delta_i, \delta_i'\}$, and one can take the quotient of this algebra with the two sided ideal $\mathcal{I}$ generated by the relations of Proposition 6. It is easily seen that $\mathcal{D}/\mathcal{I}$ is a GR-algebra (or Gröbner-ready algebra, according to the terminology of [20]), i.e. it is a quotient of the algebra generated, as a vector space, by the standard monomials in Σ, and satisfying relations of the type

$$ts = C_{ts} \cdot st + D_{ts}, \quad s, t \in \Sigma, \quad s < t,$$

where C_{ts} is an upper triangular matrix of complex numbers indexed over Σ, $<$ is an order on the symbols of Σ, and D_{ts} are polynomials containing terms that are strictly smaller than st with respect, for example, to the lexicographic ordering on monomials induced by $<$. One can then construct the free resolution of the row span of the matrix P_{4m} within $\mathcal{D}/\mathcal{I}$, using one of the methods outlined in [20]. We will follow the standard Schreyer's method with S-polynomials, which naturally extends to GR-algebras. Our aim is to prove that at each step, only linear relations will appear. Looking at the matrices (7)÷(10), we need to look for syzygies of their rows. However, since in each row there is at most one nonzero element, it is sufficient to calculate the syzygies column by column. Let us fix one of the columns. There appear only four nonzero elements $a_i^1, b_j^1, a_i^2, b_j^2$, using the notation of Proposition 6, with $j \neq i+1$. Therefore, once we calculate their S-polynomials vanish identically. This means that the only way to annihilate them is to use the relations $a_{i\pm1}^k a_i^k = b_{j\pm1}^k b_j^k = 0$. The only two cases we did not include are the first and the last column which can be treated in a similar way. This proves linearity at the first step. Since the syzygies obtained this way are again of the same type, with at most one nonzero element in each row, we can conclude that all matrices in the resolution will be of degree one by recursion. □

Despite the fact that other systems in hypercomplex analysis correspond to Cohen–Macaulay modules, e.g. the Cauchy–Fueter system and the Moisil–Theodorescu system, the q-Hermitian system is different as shown in the next corollary.

Corollary 3. *The module $\mathcal{M}_{4m}$ is not Cohen–Macaulay.*

Proof. We have $\dim \mathcal{M}_{4m} = 4m-1$ as it equals the dimension of the characteristic variety. Moreover, if $\wp_{4m}$ denotes the ideal of the variables in R, by the Auslander–Buchsbaum formula we have $\operatorname{depth}(\wp_{4m}, \mathcal{M}_{4m}) = 4m - \operatorname{pd}(\mathcal{M}_{4m}) = 0$. Thus $\operatorname{depth}(\wp_{4m}, \mathcal{M}_{4m}) \neq \dim(\mathcal{M}_{4m})$ and the statement follows. □

In order to describe explicitly the free resolution associated to the q-Hermitian system, we consider the Fourier transform of the matrices (7)÷(10), which we will denote by the same symbol. The context will make clear if we are consider matrices of differential operators or of polynomials. Let Q_{4m} be the matrix obtained by putting the four blocks $\underline{\partial}_Z, \underline{\partial}_Z^\dagger, J[\underline{\partial}_Z^\dagger], J[\underline{\partial}_Z]$ in a column. The rows of the matrix Q_{4m} generate the same module as the rows of the matrix P_{4m}, however, as we already noted, there are redundancies. This means that at the level of syzygies there will be some degree zero relations. However, this description is convenient in order to describe the matrix of the first syzygies in terms of the original operators. We obtain:

Proposition 7. *The map of the degree one first syzygies of* $\operatorname{coker}(Q_{4m})$ *is given by*

$$\begin{bmatrix} \underline{\partial}_Z & 0 & 0 & 0 \\ 0 & \underline{\partial}_Z^\dagger & 0 & 0 \\ 0 & 0 & J[\underline{\partial}_Z^\dagger] & 0 \\ 0 & 0 & 0 & J[\underline{\partial}_Z] \\ \underline{\partial}_Z & 0 & J[\underline{\partial}_Z^\dagger] & 0 \\ 0 & \underline{\partial}_Z^\dagger & 0 & J[\underline{\partial}_Z] \\ J[\underline{\partial}_Z] & 0 & 0 & \underline{\partial}_Z \\ 0 & J[\underline{\partial}_Z^\dagger] & \underline{\partial}_Z^\dagger & 0 \end{bmatrix}. \tag{12}$$

Proof. The matrix is obtained using the relations of Proposition 6. Note that the relations involving the Laplace operator do not generate syzygies because, if one writes the relations

$$\Delta \cdot s = s \cdot \Delta, \quad s \in \Sigma,$$

these are identically satisfied given the fact that all elements of Σ are idempotents. □

As a corollary we now prove the vanishing of $\mathrm{Ext}^0(\mathcal{M}_{4m}, R)$ which means that when a q-Hermitian function on the complement of a compact set K admits an extension inside K, then this extension is unique.

Corollary 4. *For the module* $\mathcal{M}_{4m}$ *we have* $\mathrm{Ext}^0(\mathcal{M}_{4m}, R) = 0$.

Proof. The module $\mathcal{M}_{4m}$ is $R^{2^{4m}}/\langle P^t_{4m}\rangle$. A minimal system of generators for the denominator is given by the rows of P_{4m}, since they are independent homogeneous vectors of degree one. This fact is sufficient to conclude that $\mathrm{Ext}^0(\mathcal{M}_{4m}, R) = 0$. □

5 Algebraic constraints clarified

In this section we illustrate how one can interpret the degree zero compatibility conditions for the q-Hermitian system on the level of representation theory. To do so we decompose the space of spinors, where the functions on which the operators act take values, into irreducible $\mathfrak{sp}(m)$-modules, and we study how the operators split according to this decomposition. Our aim is to study compatibility conditions for the following inhomogeneous system of equations, involving the operators that are used to introduce the notion of q-monogenicity:

$$\begin{cases} \underline{\partial}_Z f(\underline{z}, \underline{z}^\dagger) = g(\underline{z}, \underline{z}^\dagger) \\ \underline{\partial}_Z^\dagger f(\underline{z}, \underline{z}^\dagger) = h(\underline{z}, \underline{z}^\dagger) \\ J[\underline{\partial}_Z] f(\underline{z}, \underline{z}^\dagger) = p(\underline{z}, \underline{z}^\dagger) \\ J[\underline{\partial}_Z^\dagger] f(\underline{z}, \underline{z}^\dagger) = r(\underline{z}, \underline{z}^\dagger), \end{cases} \tag{13}$$

where all functions take values in the spinor space $\mathbb{S}$ and depend on the complex variables $(\underline{z}, \underline{z}^\dagger) \in \mathbb{R}^{4m}$. As a notation, given a spinor valued function f, we will write it as $f = \sum_{i=0}^{2m} f_i$ where each f_i is its homogeneous component with values in $\mathbb{S}_i$. We will often make use of the following relations:

$$\begin{aligned} [Q, \underline{\partial}_Z] &= 0 & [P, \underline{\partial}_Z^\dagger] &= 0 \\ [Q, J(\underline{\partial}_Z^\dagger)] &= 0 & [P, J(\underline{\partial}_Z)] &= 0 \\ [Q, \underline{\partial}_Z^\dagger] &= iJ[\underline{\partial}_Z^\dagger] & [P, \underline{\partial}_Z] &= iJ[\underline{\partial}_Z] \\ [Q, J(\underline{\partial}_Z)] &= -i\underline{\partial}_Z & [P, J(\underline{\partial}_Z^\dagger)] &= -i\underline{\partial}_Z^\dagger, \end{aligned}$$

for which we refer to our earlier paper [13].

5.1 The case $m = 1$.

Proposition 8. *The algebraic constraints for the system* (13) *for* $m = 1$ *are*

$$\begin{cases} g_2 = \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger p_0 \\ r_2 = \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger h_0. \end{cases}$$

Proof. Writing down the explicit form of the operator, restricted to $\mathbb{S}_1$, one immediately observes that the two operators d_2 and δ_0' have the same form, in the sense that they both act on 1-homogeneous spinors and give the same values under the natural identification of $\mathbb{S}_0$ with $\mathbb{S}_2$. Explicitly, we have $g_2 = \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger p_0$ and $r_2 = \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger h_0$, the last one being the dual relation of the first, coming from the comparison of d_2' and δ_0. □

Remark 9. Technically speaking, at this point one could also find a relation between d_1 and δ_1'. They both end up in the space $\mathbb{S}_1$, but act on functions taking values in spinor spaces that can be identified. This, however, does not appear in the calculation of syzygies with computational methods, since the operators appear in different *columns* in the matrices (7) and (10).

As a general principle, we will look for relations between operators ending up in isomorphic spaces, as they correspond to different *rows* of the symbol matrix, since compatibility conditions for the system (13) are syzygies of these rows.

5.2 The case $m = 2$.

Proposition 9. *The algebraic constraints for the system* (13) *for* $m = 2$ *are*

$$\begin{cases} 2G_0 = ip_0 \\ 2R_0 = -ih_0 \\ 2H_4 = ir_4 \\ 2P_4 = -ig_4. \end{cases}$$

Proof. We start from the complex vector space $\mathbb{C}^8$ and its Clifford algebra $\mathbb{C}_8$, which after multiplication with the idempotent $I_1I_2I_3I_4$ gives rise to the spinor space $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- = \mathbb{S}_0 \oplus \cdots \oplus \mathbb{S}_4$. The so-called homogeneous spinor spaces $\mathbb{S}_j$ are all $\mathfrak{sl}(4)$-irreducible, but they are not necessarily $\mathfrak{sp}_4(\mathbb{C})$-irreducible. Indeed, the module $\mathbb{S}_2$ splits into the direct sum of two irreducibles:

$$\mathbb{S}_2 = \mathbb{S}_2^Q \oplus P\mathbb{S}_4 \cong \mathbb{S}_2^P \oplus Q\mathbb{S}_0,$$

where P and Q denote the $\mathfrak{sp}_4(\mathbb{C})$-invariant multiplication operators

$$P = \mathfrak{f}_2\mathfrak{f}_1 + \mathfrak{f}_4\mathfrak{f}_3 \quad \text{and} \quad Q = \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger + \mathfrak{f}_3^\dagger \mathfrak{f}_4^\dagger,$$

and the superscript P or Q refers to the fact that we intersect with the kernel for these operators. For example, $\mathbb{S}_2^P = \mathbb{S}_2 \cap \ker(P)$. The other homogeneous spinor spaces *are* irreducible with respect to $\mathfrak{sp}_4(\mathbb{C})$, so that we omit the subscripts P or Q. In this particular example, we have the following explicit expression:

$$\begin{aligned} \mathbb{S}_2^Q &= \mathrm{span}_{\mathbb{C}}(\mathfrak{f}_1^\dagger\mathfrak{f}_3^\dagger I, \mathfrak{f}_1^\dagger\mathfrak{f}_4^\dagger I, \mathfrak{f}_2^\dagger\mathfrak{f}_3^\dagger I, \mathfrak{f}_2^\dagger\mathfrak{f}_4^\dagger I) \oplus \mathbb{C}(\mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger - \mathfrak{f}_3^\dagger\mathfrak{f}_4^\dagger)I \\ P\mathbb{S}_4 &= \mathbb{C}(\mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger + \mathfrak{f}_3^\dagger\mathfrak{f}_4^\dagger). \end{aligned}$$

Due to the fact that our function system is related to invariance with respect to $\mathfrak{sp}_4(\mathbb{C})$, which is reflected in the decomposition of the space $\mathbb{S}$ in which our functions take values into irreducibles $\mathbb{S}_j^P$ and $\mathbb{S}_j^Q$, we are expecting compatibility conditions which reflect this refinement. Let us then introduce the following notation: $f = (f_0, f_1, f_2 \oplus PF_4, f_3, f_4)$, where $f_2 \in \mathbb{S}_2^Q$ and F_4 takes its values in the remaining (one-dimensional) piece. We distinguish the following three steps:

Step 1. Consider first of all the function $f_1 \in \mathcal{C}^\infty(\mathbb{C}^8, \mathbb{S}_1)$. By definition, we have that

$$\underline{\partial}_Z f_1 := \varphi_2 \oplus Q\Phi_0 \in \mathbb{S}_2 = \mathbb{S}_2^P \oplus Q\mathbb{S}_0.$$

In order to find the components of $\underline{\partial}_Z f_1$ in each of these summands, we have to consider the action of the operator P. On the one hand we have $P\underline{\partial}_Z f_1 = PQ\Phi_0 = 2\Phi_0$, whereas on the other hand, using the commutator relations above, we also have

$$P\underline{\partial}_Z f_1 = \underline{\partial}_Z P f_1 + iJ[\underline{\partial}_Z]f_1 = iJ[\underline{\partial}_Z]f_1,$$

so that we conclude

$$\underline{\partial}_Z f_1 = \Big(\underline{\partial}_Z - \frac{i}{2}\, QJ[\underline{\partial}_Z]\Big) f_1 \oplus \frac{i}{2}\, QJ[\underline{\partial}_Z]f_1.$$

From the system, we get $\underline{\partial}_Z f = g$, hence $\underline{\partial}_Z f_1 = g_2 \oplus QG_0$, which means that the initial data G_0 have to be compared with the initial data for $J[\underline{\partial}_Z]f = p$. Indeed, $J[\underline{\partial}_Z]f$ has a component in $\mathbb{S}_0$ given by p_0 and this leads to the following compatibility condition:

$$2G_0 = P(g_2 \oplus QG_0) = P\Big(\frac{i}{2}\, QJ[\underline{\partial}_Z]f_1\Big) = ip_0.$$

Step 2. Similar calculations can now be done for the other raising operator:

$$J[\underline{\partial}_Z^\dagger]f_1 := \psi_2 \oplus Q\Psi_0 \in \mathbb{S}_2 = \mathbb{S}_2^P \oplus Q\mathbb{S}_0.$$

On the one hand, the projection on each of these irreducible summands gives

$$J[\underline{\partial}_Z^\dagger]f_1 = \Big(J[\underline{\partial}_Z^\dagger] + \frac{i}{2}\, Q\underline{\partial}_Z^\dagger\Big) f_1 \oplus -\frac{i}{2}\, Q\underline{\partial}_Z^\dagger f_1,$$

whereas from the system we get that $J[\underline{\partial}_Z^\dagger]f = r \Rightarrow J[\underline{\partial}_Z^\dagger]f_1 = r_2 \oplus QR_0$. This means that the initial data R_0 have to be compared with the initial data for $\underline{\partial}_Z^\dagger f = h$. Indeed, $\underline{\partial}_Z^\dagger f$ has a component in $\mathbb{S}_0$ given by h_0 and this leads to the following compatibility condition:

$$2R_0 = P(r_2 \oplus QR_0) = P\Big(-\frac{i}{2}\, Q\underline{\partial}_Z^\dagger f_1\Big) = -ih_0.$$

Step 3. Next, consider the function $f_3 \in \mathcal{C}^\infty(\mathbb{C}^8, \mathbb{S}_3)$. A completely analogous argument can now be applied to the lowering operators $\underline{\partial}_Z^\dagger$ and $J[\underline{\partial}_Z]$, and this will eventually lead to the following conditions:

$$2H_4 = Q(h_2 \oplus PH_4) = Q\Big(\frac{i}{2}\, PJ[\underline{\partial}_Z^\dagger]f_3\Big) = ir_4.$$

and

$$2P_4 = Q(p_2 \oplus PP_4) = Q\Big(-\frac{i}{2}\, P\underline{\partial}_Z f_3\Big) = -ig_4. \qquad \square$$

Remark 10. A priori, one could look for relations coming from operators starting from the space $\mathbb{S}_2$, since they end up in either $\mathbb{S}_1$ or $\mathbb{S}_3$ which are clearly isomorphic. However, the above four complex relations exhaust the situation since, thanks to computations with CoCoA, we know the exact number of real algebraic constraints, which is 8 in this case.

5.3 The case $m = 3$.

Proposition 10. *The algebraic constraints for the system* (13) *for* $m = 3$ *are*

$$\begin{cases} 3G_0 = ip_0 \\ 3R_0 = -ih_0 \\ 3H_6 = ir_6 \\ 3P_6 = -ig_6 \\ Q^3 P_0 = -6iG_6 \\ Q^3 H_0 = -6iR_6. \end{cases}$$

Proof. This time we will start from the complex vector space $\mathbb{C}^{12}$ and its associated Clifford algebra $\mathbb{C}_{12}$, which after multiplication with the idempotent $I_1I_2I_3I_4I_5I_6$ gives rise to the spinor space $\mathbb{S} = \mathbb{S}_+ \oplus \mathbb{S}_- = \mathbb{S}_0 \oplus \ldots \mathbb{S}_6$. Again, these homogeneous spinor spaces are $\mathfrak{sl}(6)$-irreducible, but they are not necessarily $\mathfrak{sp}_6(\mathbb{C})$-irreducible. Indeed, the following modules split into the direct sum of two irreducible summands:

$$\begin{aligned} \mathbb{S}_2 &= \mathbb{S}_2^P \oplus Q\mathbb{S}_0 \\ \mathbb{S}_3 &= \mathbb{S}_3^P \oplus Q\mathbb{S}_1 \cong \mathbb{S}_3^Q \oplus P\mathbb{S}_5 \\ \mathbb{S}_4 &= \mathbb{S}_4^Q \oplus P\mathbb{S}_6, \end{aligned}$$

where P and Q denote the multiplication operators

$$P = \mathfrak{f}_2\mathfrak{f}_1 + \mathfrak{f}_4\mathfrak{f}_3 + \mathfrak{f}_6\mathfrak{f}_5 \quad \text{and} \quad Q = \mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger + \mathfrak{f}_3^\dagger\mathfrak{f}_4^\dagger + \mathfrak{f}_5^\dagger\mathfrak{f}_6^\dagger.$$

The other homogeneous spinor spaces *are* irreducible with respect to $\mathfrak{sp}_6(\mathbb{C})$, whence we again omit the subscripts P or Q. Let us then again consider the compatibility conditions for the inhomogeneous system (13), where all functions take values in the spinor space $\mathbb{S}$ and depend on the complex variables $(\underline{z}, \underline{z}^\dagger) \in \mathbb{R}^{12}$. In order to derive conditions, we first need to introduce notation to label each of the components of a function taking values in one of the irreducible slots. For our initial function $f \in \mathcal{C}^\infty(\mathbb{R}^{12}, \mathbb{S})$ for example, we put

$$f = f_0 + f_1 + \begin{pmatrix} f_2 \\ QF_0 \end{pmatrix} + \begin{pmatrix} f_3 \\ QF_1 \end{pmatrix} + \begin{pmatrix} f_4 \\ PF_6 \end{pmatrix} + f_5 + f_6,$$

where each column as a whole will be represented by means of $\widetilde{f_i}$; e.g. $\widetilde{f_4} = f_4 + PF_6$. In order to illustrate how the algebraic information coming from the representation theory will reflect on the syzygies, we proceed as follows:

Step 1. Consider the function $f_1 \in \mathcal{C}^\infty(\mathbb{R}^{12}, \mathbb{S}_1)$. By definition, we have that

$$\underline{\partial}_Z f_1 := \varphi_2 \oplus Q\Phi_0 \in \mathbb{S}_2 = \mathbb{S}_2^P \oplus Q\mathbb{S}_0.$$

In order to find the components of $\underline{\partial}_Z f_1$ in each of these summands, we have to consider the action of the operator P. We have that $P\underline{\partial}_Z f_1 = PQ\Phi_0 = 3\Phi_0$, where we have used the fact that the restriction of the operator PQ to $\mathbb{S}_0$ acts as the constant 3. On the other hand, invoking the commutation relations, we have:

$$P\underline{\partial}_Z f_1 = \underline{\partial}_Z P f_1 + iJ[\underline{\partial}_Z] f_1 = iJ[\underline{\partial}_Z] f_1,$$

so that we conclude

$$\underline{\partial}_Z f_1 = \Big(\underline{\partial}_Z - \frac{i}{3}\, QJ[\underline{\partial}_Z]\Big) f_1 \oplus \frac{i}{3}\, QJ[\underline{\partial}_Z] f_1.$$

From the system, we get $\underline{\partial}_Z f = g$, hence $\underline{\partial}_Z f_1 = g_2 \oplus QG_0$, which means that the initial data G_0 have to be compared with the initial data for $J[\underline{\partial}_Z] f = p$. Indeed, $J[\underline{\partial}_Z] f$ has a component in $\mathbb{S}_0$ given by p_0 and this leads to the following compatibility condition:

$$3G_0 = P(g_2 \oplus QG_0) = P\Big(\frac{i}{3}\, QJ[\underline{\partial}_Z] f_1\Big) = ip_0.$$

Along completely similar lines, one can consider the effect of the operator $J[\underline{\partial}_Z^\dagger]$. This leads to a similar condition, explicitly given by $3R_0 = -ih_0$.

Step 2. One the other hand, one can of course consider the 'dual' result starting from the function $f_5 \in \mathcal{C}^\infty(\mathbb{R}^{12}, \mathbb{S}_5)$. Again using completely similar arguments, involving the commutation relations, one is easily led to the conditions

$$3H_6 = +ir_6 \quad \text{and} \quad 3P_6 = -ig_6.$$

Step 3. Finally, we will use the fact that starting from a function $f_3 \in \mathcal{C}^\infty(\mathbb{R}^{12}, \mathbb{S}_3)$, it is possible to reach a component taking values in the spaces $\mathbb{S}_0$ and $\mathbb{S}_6$ (as an irreducible subspace of the spaces $\mathbb{S}_2$ and $\mathbb{S}_4$ respectively). Moreover, there is a natural identification between these extremal (one-dimensional) spaces so that it is natural to compare those functions. To do so, we first of all note that the natural isomorphism can be expressed by means of $Q^3 : \mathbb{S}_0 \mapsto \mathbb{S}_6$. Let us then prove the following result:

Lemma 1. *For any function* $f_3 \in \mathcal{C}^\infty(\mathbb{R}^{12}, \mathbb{S}_3)$*, the following result holds:*

$$Q^3\big[PJ[\underline{\partial}_Z]f_3\big] = -6iQ\underline{\partial}_Z f_3 \,.$$

Proof: It suffices to note that $J[\underline{\partial}_Z]f_3$ is $\mathbb{S}_2$-valued, which means that the operator QP acts as a constant on the summand $Q\mathbb{S}_0 \subset \mathbb{S}_2$ (since the summand $\mathbb{S}_2^P$ will be annihilated). It is then easily seen that $QP\mathbb{S}_2 = QPQ\mathbb{S}_0 = 3Q\mathbb{S}_0$. Together with $[Q, J[\underline{\partial}_Z]] = -i\underline{\partial}_Z$, we thus get:

$$Q^3\big[PJ[\underline{\partial}_Z]f_3\big] = 3Q^2\big[J[\underline{\partial}_Z]f_3\big] = 3Q\big[-i\underline{\partial}_Z f_3 + J[\underline{\partial}_Z]Qf_3\big] = -6iQ\underline{\partial}_Z f_3. \quad \square$$

On the level of data then, this gives rise to the compatibility condition $Q^3P_0 = -6iG_6$. In a completely similar way, we are also led to $Q^3H_0 = -6iR_6$. Considering that CoCoA returns only 6 syzygies of degree zero, we have the statement. $\square$

5.4 Conclusions. Based on the calculations of the previous paragraphs, we can now make some general considerations on the algebraic constraints of the q-Hermitian system. In the low dimensional cases we presented, it is clear that such constraints appear only when there is a copy of $\mathbb{S}_0$ inside one of the homogeneous spinor spaces. Since such copies only appear in the decomposition of $\mathbb{S}_{2j}$ into irreducibles, for $1 \leq j \leq m$, we have a total of $2m$ algebraic conditions. Each pair of such conditions comes from considering a spinor in $\mathbb{S}_{2j-1}$, $1 \leq j \leq m$ and applying either $\underline{\partial}_Z$ and $J[\underline{\partial}_Z^\dagger]$, or $\underline{\partial}_Z^\dagger$, $J[\underline{\partial}_Z]$. A priori, one could expect more constraints to come from the various copies of other homogeneous spinor spaces, which occur in higher dimensions. However, experiments with CoCoA suggest that this is never the case. Therefore, we conjecture that all the algebraic constraints are concentrated in even homogeneous pieces of the data functions, these being the only spaces in which there is a copy of $\mathbb{S}_0$. As a consequence of this fact, it is not possible to split the system into two equivalent parts using the classical decomposition $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. We already noticed in [10] that this was not always possible for the complex Hermitian Dirac system. Indeed only for some values of the dimension the reduction yielded two equivalent systems. Also in the quaternionic case, we expect that the reduction of, for example, system (4) into even and odd part will never be possible, since the algebraic constraints would only appear in one of the two systems.

References

[1] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen, V. Souček, Fundaments of Hermitean Clifford analysis. I. Complex structure. *Complex Anal. Oper. Theory* **1** (2007), 341–365. MR2336028 (2008g:30041) Zbl 1131.30019

[2] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen, V. Souček, Fundaments of Hermitean Clifford analysis. II. Splitting of h-monogenic equations. *Complex Var. Elliptic Equ.* **52** (2007), 1063–1079. MR2374972 (2009b:30102) Zbl 1144.30019

[3] F. Brackx, R. Delanghe, F. Sommen, *Clifford analysis*. Pitman 1982. MR697564 (85j:30103) Zbl 0529.30001

[4] F. Brackx, H. De Schepper, F. Sommen, A Hermitian setting for wavelet analysis: the basics, *Proceedings of the 4th International Conference on Wavelet Analysis and its Applications*, University of Macau, China, 2005.

[5] J. Bureš, F. Sommen, V. Souček, P. Van Lancker, Symmetric analogues of Rarita–Schwinger equations. *Ann. Global Anal. Geom.* **21** (2002), 215–240. MR1896475 (2003d:58054) Zbl 1025.58013

[6] CoCoA Team, CoCoA: a system for doing Computations in Commutative Algebra. cocoa.dima.unige.it

[7] F. Colombo, A. Damiano, I. Sabadini, D. C. Struppa, Quaternionic hyperfunctions on five-dimensional varieties in $\mathbb{H}^2$. *J. Geom. Anal.* **17** (2007), 435–454. MR2358765 (2008j:30064) Zbl 1135.30021

[8] F. Colombo, I. Sabadini, F. Sommen, D. C. Struppa, *Analysis of Dirac systems and computational algebra*, volume 39 of *Progress in Mathematical Physics*. Birkhäuser 2004. MR2089988 (2005m:30052) Zbl 1064.30049

[9] A. Damiano, D. Eelbode, I. Sabadini, Algebraic analysis of Hermitian monogenic functions. *C. R. Math. Acad. Sci. Paris* **346** (2008), 139–142. MR2393630 (2008m:58076) Zbl 1135.30022

[10] A. Damiano, D. Eelbode, I. Sabadini, Invariant syzygies for the Hermitian Dirac operator. *Math. Z.* **262** (2009), 929–945. MR2511757 (2010b:58043) Zbl 1175.58014

[11] R. Delanghe, F. Sommen, V. Souček, *Clifford algebra and spinor-valued functions*, volume 53 of *Mathematics and its Applications*. Kluwer 1992. MR1169463 (94d:30084) Zbl 0747.53001

[12] D. Eelbode, A Clifford algebraic framework for $\mathfrak{sp}(m)$-invariant differential operators. *Adv. Appl. Clifford Algebr.* **17** (2007), 635–649. MR2356263 (2008k:58074) Zbl 1127.58035

[13] D. Eelbode, Quaternionic monogenic function systems. Preprint, 2007.

[14] D. Eelbode, Irreducible $\mathrm{sl}(m)$-modules of Hermitean monogenics. *Complex Var. Elliptic Equ.* **53** (2008), 975–987. MR2453891 (2009k:30057) Zbl 1159.30029

[15] W. Fulton, J. Harris, *Representation theory*. Springer 1991. MR1153249 (93a:20069) Zbl 0744.22001

[16] J. E. Gilbert, M. A. M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*. Cambridge Univ. Press 1991. MR1130821 (93e:42027) Zbl 0733.43001

[17] G.-M. Greuel, V. Levandovskyy, H. Schönemann, SINGULAR::PLURAL 2.1. A Computer Algebra System for Noncommutative Polynomial Algebras. Centre for Computer Algebra, University of Kaiserslautern, 2003. www.singular.uni-kl.de/plural

[18] G.-M. Greuel, V. Levandovskyy, H. Schönemann, SINGULAR 3.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern, 2007. www.singular.uni-kl.de

[19] M. Kreuzer, L. Robbiano, *Computational commutative algebra.* 1. Springer 2000. MR1790326 (2001j:13027) Zbl 0956.13008

[20] V. Levandovskyy, Non-commutative Computer Algebra for polynomial algebras: Gröbner bases, applications and implementation. Ph.D. thesis, 2005.

[21] V. P. Palamodov, *Linear differential operators with constant coefficients*. Springer 1970. MR0264197 (41 #8793) Zbl 0191.43401

[22] D. Peña-Peña, I. Sabadini, F. Sommen, Quaternionic Clifford analysis: the Hermitian setting. *Complex Anal. Oper. Theory* **1** (2007), 97–113. MR2276734 (2007k:30095) Zbl pre05177135

[23] R. Rocha-Chávez, M. Shapiro, F. Sommen, *Integral theorems for functions and differential forms in* $\mathbb{C}^m$. Chapman & Hall/CRC, Boca Raton, FL 2002. MR1889406 (2003a:32004) Zbl 0991.32002

[24] I. Sabadini, F. Sommen, Hermitian Clifford analysis and resolutions. *Math. Methods Appl. Sci.* **25** (2002), 1395–1413. MR1949504 (2004k:30127) Zbl 1013.30033

[25] E. M. Stein, G. Weiss, Generalization of the Cauchy-Riemann equations and representations of the rotation group. *Amer. J. Math.* **90** (1968), 163–196. MR0223492 (36 #6540) Zbl 0157.18303

[26] P. Van Lancker, Rarita–Schwinger fields in the half space. *Complex Var. Elliptic Equ.* **51** (2006), 563–579. MR2230266 (2007e:30077) Zbl 1117.30041

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