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Axioms for Salience Perception

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Abstract

Models of salience-based choice have become popular in recent years, although there is still no known set of simple conditions or axioms which implies the existence of a salience function. In this paper, we provide simple and natural axioms that characterize the general class of salience functions. As an application we consider a salience-based model of decision making and show that within that setup the fourfold pattern of risk attitudes is a general property of a salience function and that the properties producing that pattern also account for other anomalies involving risky and intertemporal choice.

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1. Introduction

Models of salience-based choice have been developed in recent years for decisions under risk (Bordalo et al., 2012), decisions over time (Koszegi and Szeidl, 2013), decisions under uncertainty (Schneider, Leland, and Wilcox, 2016), decisions across each of these domains (Leland and Schneider, 2016), and for consumer purchase decisions (Bordalo et al., 2013). Each of these approaches typically relies on some notion of a 'salience function'. However, there is no axiomatic foundation for such functions. In this paper, we begin by proposing three simple axioms to characterize the nature of a qualitative salience relation. Aside from basic technical conditions (ordering, continuity, and symmetry), the key requirements are simple monotonicity properties which we term *monotonicity in intervals*, *monotonicity in ratios*, and *monotonicity in differences*. These properties, in turn, imply salience functions will exhibit Ordering, Decreasing Absolute Sensitivity (DAS) and Increasing Proportional Sensitivity (IPS).[1] We then show how, in the context of the model of "salience weighted utility over presentations" (SWUP) proposed in Leland and Schneider (2016), these three properties, individually or in concert, provide a unified explanation for choice anomalies across decision domains. Specifically, we show that under SWUP, DAS is responsible for the tendency to be risk averse for gains and risk seeking for losses at high and moderate probabilities and to appear to discount the future hyperbolically while IPS implies the common ratio effect in risky choice and the magnitude effect in intertemporal choice. In addition, the two properties in concert imply the fourfold pattern of risk preferences.

2. Axioms for Salience Perception

The properties of salience perception have typically been justified on intuitive and empirical grounds (e.g., Bordalo et al., 2012). Here we show that these properties may also be derived from first principles that one might expect to characterize the perceptual system. We consider salience perceptions between pairs of quantities (x, y). These quantities may be pairs of payoffs or probabilities or time delays, for example. Denote the set of quantities being compared by a closed and convex set $X \subset \mathbb{R}_+^2$. Let $\trianglerighteq_s$ be a binary relation called a *salience relation* carrying the interpretation "at least as salient as" over pairs in X, with strictly greater salience and equivalence denoted by $\triangleright_s$ and $\sim_s$. Let $\trianglerighteq_s$ be endowed with the following properties:

[1] The first two of these monotonicity properties also enable us to characterize the entire class of salience functions formalized by Bordalo et al. (2013) while the third enables us to discriminate between specific forms of different salience functions as it is satisfied by the salience function in Bordalo et al. (2012), but not by the salience function in Bordalo et al. (2013).

Axiom 1 (ORDERING AND CONTINUITY). $\trianglerighteq_s$ is a continuous[2] weak order on X.

For $x, y, x', y' \geq 0$., let $\Delta(x, y) := x - y \geq 0$, and let $r(x, y) := x/y \geq 1$. Note that our definitions of differences, $\Delta(x, y)$, and ratios, $r(x, y)$ have, without loss of generality, set $x \geq y$.

Axiom 2 (SYMMETRY). For any $(x, y), (y, x) \in X$, $(x, y) \sim_s (y, x)$.

Axiom 3 (MONOTONICITY IN INTERVALS). For any $(x, y), (x', y') \in X$, if $[y', x']$ is a strict subset of $[y, x]$ then $(x, y) \vartriangleright_s (x', y')$.

Note that if $[y', x']$ is a strict subset of $[y, x]$, then $\Delta(x, y) > \Delta(x', y')$, and $r(x, y) > r(x', y')$. That is, (x, y) has both a larger absolute difference and a larger ratio than (x', y'), in which case (x, y) is more salient than (x', y').

Axiom 4 (MONOTONICITY IN RATIOS). For any $(x, y), (x', y') \in X$, if $\Delta(x, y) = \Delta(x', y')$ with $x' > x$, $y' > y$, then $r(x, y) > r(x', y')$ implies $(x, y) \vartriangleright_s (x', y')$.

Axiom 4 implies that, for a fixed absolute difference, salience increases with larger ratios.

Axiom 5 (MONOTONICITY IN DIFFERENCES). For any $(x, y), (x', y') \in X$, if

$r(x', y') = r(x, y)$, where $x' > x$ and $y' > y$, then $\Delta(x', y') > \Delta(x, y)$ implies $(x', y') \vartriangleright_s (x, y)$.

Axiom 5 implies that, for a fixed ratio, salience increases with larger absolute differences. Axioms 3, 4, and 5 constitute our main assumptions about the perceptual system. As the axioms involve only universal quantifiers, they should be readily testable in experiments.

Building on Bordalo et al. (2012, 2013), we provide the following general definition of a salience function[3]:

Definition 1 (Salience Function 1): A symmetric and continuous function $\sigma: X \to \mathbb{R}$ is a *salience function* if it has the following properties:

1. **Ordering.** For any $(x, y), (x', y') \in X$, if $[y', x'] \subset [y, x]$, then $\sigma(x', y') < \sigma(x, y)$.
2. **Diminishing Absolute Sensitivity:** $\sigma(.)$ exhibits diminishing absolute sensitivity if, for any $(x, y) \in X$, $x, y > 0$ and any $\epsilon > 0$, $\sigma(x + \epsilon, y + \epsilon) < \sigma(x, y)$.
3. **Increasing Proportional Sensitivity:** $\sigma(.)$ exhibits increasing proportional sensitivity if, for any $(x, y) \in X$, $x, y > 0$ and any $\alpha > 1$, $\sigma(\alpha x, \alpha y) > \sigma(x, y)$.

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[2] The notion of continuity invoked here is that used in consumer preference theory in economics: The relation $\trianglerighteq_s$ is continuous if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \trianglerighteq_s y^n$ for all n, $x = \lim_{n\to\infty} x^n$, and $y = \lim_{n\to\infty} y^n$, we have $x \trianglerighteq_s y$ (Mas-Colell, Whinston, and Green (1995), Definition 3.C.1).

[3] Bordalo et al did not require increasing proportional sensitivity and that property appears nowhere in their writings on salience. However, as the axioms indicate, this property seems to be a natural analog to the others, and is assumed elsewhere in the literature, albeit as properties of preferences, by Prelec and Loewenstein (1991).

Definition 2: A function σ *represents* a salience relation $\trianglerighteq_s$ if for all $(x,y),(x',y') \in X$, we have $(x,y) \trianglerighteq_s (x',y')$ if and only if $\sigma(x,y) \geq \sigma(x',y')$.

Proposition 1: *Under Axioms 1-5, there exists a salience function σ that represents $\trianglerighteq_s$.*

Proposition 2: *For any function σ that represents $\trianglerighteq_s$:*

(i) *σ satisfies ordering if and only if $\trianglerighteq_s$ satisfies Axiom 3.*

(ii) *σ satisfies DAS if and only if $\trianglerighteq_s$ satisfies Axiom 4.*

(iii) *σ satisfies IPS if and only if $\trianglerighteq_s$ satisfies Axiom 5.*

Proofs of our propositions are provided in the Appendix. Proposition 1 implies that fundamental quantitative properties of the perceptual system such as ordering, diminishing absolute sensitivity, and increasing proportional sensitivity can be formally derived from basic axioms imposed on a qualitative salience relation. Moreover, Proposition 2 suggests that *any* salience function as defined in Definition 1 must satisfy Axioms 3, 4 and 5.

Similarity-based models of choice proposed by Tversky (1968), Rubinstein (1988, 2003) and Leland (1994, 2002) are precursors to the salience based models mentioned in the introduction. Viewing the salience and the similarity of differences as a continuous spectrum suggests the following definition of a similarity function that can be derived from similar axioms:

Definition 3: A symmetric and continuous function $s: X \to \mathbb{R}$ is a *similarity function* if it satisfies 1 - 3:

1. **Ordering.** If $[x_j, y_j]$ is a subset of $[x_i, y_i]$, then $s(x_j, y_j) > s(x_i, y_i)$.
2. **Increasing Absolute Similarity:** s exhibits increasing absolute similarity if, for any $x_i, y_i > 0$ and any $\epsilon > 0$, $s(x_i + \epsilon, y_i + \epsilon) > s(x_i, y_i)$.
3. **Decreasing Proportional Similarity:** s exhibits decreasing proportional similarity if, for any $x_i, y_i > 0$ and any $\alpha > 1$, $s(\alpha x_i, \alpha y_i) < s(x_i, y_i)$.

Definition 3 implies that two values become more similar as the ratio or absolute difference between them declines.

3. Application: Salience Weighted Utility over Presentations

Now consider the salience-based model of Leland and Schneider (2016). Let there be a finite set, X, of outcomes. A *lottery* is a mapping $p: X \to [0,1]$ such that $\sum_{x \in X} p(x) = 1$. Denote the set of all lotteries by $\Delta(X)$. To represent a lottery, p, we employ a one-dimensional array, $\mathbf{p}$, consisting of $n(\mathbf{p})$ outcomes and $n(\mathbf{p})$ corresponding probabilities. Denote the i^{th} outcome in $\mathbf{p}$ and the i^{th} corresponding probability by $\mathbf{x_i}$ and $\mathbf{p_i}$, respectively. Notice that there could be many arrays representing the same lottery.

Definition 4 (Representation of a lottery): We say that an array $\mathbf{p}$ is a *representation* of lottery p if the following two properties hold:

(i) For i =1,2,...,$n(\mathbf{p})$, $\sum_{\mathbf{i}} \mathbf{p_i} = 1$

(ii) For all i such that $\mathbf{x_i} = x$, $\sum_{\mathbf{i}} \mathbf{p_i} = p(x)$.

Note that a representation $\mathbf{p}$ of lottery p differs from the lottery itself since it permits the same outcome to appear more than once in the array provided that the corresponding probabilities sum to the overall probability of that outcome. Two representations of different lotteries presented jointly, constitute a frame.

Definition 5 (Frame for lotteries): A *presentation* or *frame*, $\mathbf{F}\{\mathbf{p},\mathbf{q}\}$ of two lotteries, p and q, is a matrix containing a representation, $\mathbf{p}$ of p and a representation, $\mathbf{q}$ of q. A visual depiction of a choice presentation is given in Figure 1.

Figure 1. A Choice Presentation for Decisions under Risk

	(x_1,y_1)	(p_1,q_1)	(x_2,y_2)	(p_2,q_2)		(x_i,y_i)	(p_i,q_i)		(x_n,y_n)	(p_n,q_n)
p	x_1	p_1	x_2	p_2	…	x_i	p_i	…	x_n	p_n
q	y_1	q_1	y_2	q_2	…	y_i	q_i	…	y_n	q_n

We consider one-dimensional arrays $\mathbf{p}$ and $\mathbf{q}$ which represent lotteries p and q that offer a finite and equal number of outcomes denoted $\mathbf{x_i}$ and $\mathbf{y_i}$, $i = 1,2, \ldots, n$, where each $\mathbf{x_i}$ occurs with probability, $\mathbf{p_i}$ and each $\mathbf{y_i}$ occurs with probability $\mathbf{q_i}$. We use bold font for attributes in an array and italicized font for attributes in the support of a lottery.

Now consider how a decision maker chooses between arrays like p and q. An extensive body of experimental work in both economics and psychology has demonstrated that even small changes in presentations can have consequential effects on behavior. Moreover, certain representations of choices lead people to make choices that are unambiguously mistakes to the extent they entail preferring less to more. That some observed choices may be mistakes suggests we choose based on an evaluation procedure that is only imperfectly aligned with our true preferences. To explore this possibility, let there be a preference relation, $\succ$, over $\Delta(X)$ reflecting the decision maker's true preferences and assume these preferences conform to expected utility theory. Then for all lotteries $p, q \in \Delta(X)$,

(1) $$p \succ q \text{ if and only if } \sum_{x\in X} p(x)U(x) > \sum_{y\in X} q(y)U(y).$$

To capture the idea that actual choice is based on a judgment related to true preference, we consider a second relation $\gtrsim$ over representations of lotteries. The relation $\gtrsim$ may be viewed as a 'perceptual relation' (rather than a preference relation) which assigns higher rankings to arrays that "look better" given the frame. For the general frame in Figure 1, given (1), an unbiased perceptual relation is:

(2) $$\mathbf{p} \gtrsim \mathbf{q} \text{ if and only if } \sum_{i=1}^{n} \mathbf{p_i} U(\mathbf{x_i}) > \sum_{i=1}^{n} \mathbf{q_i} U(\mathbf{y_i}),$$

for all $\mathbf{p}, \mathbf{q}$, such that $\mathbf{p}$ is a *representation* of p and $\mathbf{q}$ is a *representation* of q and for all $p, q \in \Delta(X)$.

To accommodate the possibility that frames may frustrate the expression of preference, note that equations (1) and (2) provide an *alternative-based evaluation* - one lottery is strictly preferred to another, if and only if it yields a greater expected payoff to the decision maker. Building on Leland and Sileo (1998), note that the alternative-based evaluation in (2) may be written equivalently as an *attribute-based evaluation* such that $\mathbf{p} \hat{\succsim} \mathbf{q}$ if and only if (3) holds:

(3) $$\sum_{i=1}^{n}[(\mathbf{p_i} - \mathbf{q_i})(U(\mathbf{x_i}) + U(\mathbf{y_i}))/2 + (U(\mathbf{x_i}) - U(\mathbf{y_i}))(\mathbf{p_i} + \mathbf{q_i})/2] > 0.$$

The attribute-based evaluation in (3) computes probability differences associated with outcomes weighted by the average utility of those outcomes plus utility differences of outcomes weighted by their average probability of occurrence. A decision maker who chooses among representations according to the attribute-based evaluation in (3) will be indistinguishable from one who chooses according to the alternative-based evaluation in (2) who, in turn, will be indistinguishable from an agent who chooses according to true preferences as in (1). But now suppose that in the process of comparing risky alternatives an agent evaluating options according to (3) notices when the payoff in one alternative is "a lot more money" than the payoff in another and when one alternative offers "a much better chance" of receiving an outcome than the other. In these cases, we will assume that large differences in attribute values across different alternatives are perceived as particularly salient or attract disproportionate attention and are overweighted in the evaluation process. To capture this intuition that larger differences in attributes are over-weighted or attract disproportionate attention, we place salience weights $\phi(\mathbf{p_i}, \mathbf{q_i})$ on probability differences and $\mu(\mathbf{x_i}, \mathbf{y_i})$ on payoff differences, yielding the following model of *salience-weighted evaluation*, in which $\mathbf{p} \hat{\succsim} \mathbf{q}$ if and only if (4) holds:

(4) $$\sum_{i=1}^{n}[\phi(\mathbf{p_i}, \mathbf{q_i})(\mathbf{p_i} - \mathbf{q_i})(U(\mathbf{x_i}) + U(\mathbf{y_i}))/2 + \mu(\mathbf{x_i}, \mathbf{y_i})(U(\mathbf{x_i}) - U(\mathbf{y_i}))(\mathbf{p_i} + \mathbf{q_i})/2] > 0.$$

Leland and Schneider (2016) refer to model (4) as Salience-Weighted Utility over Presentations (SWUP). Note that SWUP has a dual interpretation as a model of *salience-based choice* that overweights large differences or as a model of *similarity-based choice* that underweights small differences. Note also that (3) is clearly a special case of (4) in which all salience weights are equal.

Figure 2. A Choice Presentation for Consumption Plans

	(x_1,y_1)	(r_1,t_1)	(x_2,y_2)	(r_2,t_2)		(x_i,y_i)	(r_i,t_i)		(x_n,y_n)	(r_n,t_n)
a	x_1	r_1	x_2	r_2	…	x_i	r_i	…	x_n	r_n
b	y_1	t_1	y_2	t_2	…	y_i	t_i	…	y_n	t_n

Taking an identical series of steps, Leland and Schneider (2016) adapt the model of discounted utility to make intertemporal choices between presentations **a** and **b** of intertemporal consumption streams as shown in Figure 2. If one chooses **a**, one will receive payoff x_1 in period r_1, x_2 in period r_2 and so forth. Similarly, if one chooses **b** one receives y_1 in period t_1, y_2 in t_2 and so forth.

Let $\hat{\succ}_t$ be a perceptual relation over representations of consumption plans. The intertemporal salience-weighted evaluation function analogous to (4) is as follows where $\mathbf{a} \hat{\succ}_t \mathbf{b}$ if and only if:

$$(5) \quad \sum_i^m [\theta(\mathbf{r_i}, \mathbf{t_i})(\delta^{\mathbf{r_i}} - \delta^{\mathbf{t_i}})(U(\mathbf{x_i}) + U(\mathbf{y_i}))/2 + \mu(\mathbf{x_i}, \mathbf{y_i})\ (U(\mathbf{x_i}) - U(\mathbf{y_i}))(\delta^{\mathbf{r_i}} + \delta^{\mathbf{t_i}})/2] > 0.$$

where $\mu(\mathbf{x_i}, \mathbf{y_i})$ again denotes the salience weight assigned to payoff differences while $\theta(\mathbf{r_i}, \mathbf{t_i})$ denotes the salience weight assigned to differences in the timing of intertemporal payoffs. Weights on payoffs again reflect the idea that the agent notices when the payoff in one alternative is "a lot more money" than the payoff in another while weights of differences in time periods reflect that an agent notices when one outcome will be received "a lot sooner" than another. Large differences on both dimensions are perceived as particularly salient and are overweighted in the evaluation process. When the salience weights in (5) are all equal, a decision maker who chooses as in (5) will be indistinguishable from one who maximizes the discounted utility of intertemporal consumption streams.

3.1 The Nature of Salience Perceptions

The salience functions μ, ϕ, and θ determine the only ways in which the behavior of an agent who focuses disproportionate attention on large differences, a *focal* thinker, differs from a rational agent who chooses according to expected and discounted utility, respectively. In our analysis, we assume a salience function exhibits the properties of the perceptual system in Definition 6 which revises Definition 1 to account for situations involving negative values (e.g., losses).

Definition 6 (Salience Function 2): A *salience function* $\sigma(\mathbf{x}, \mathbf{y})$ is any (non-negative), symmetric[4] and continuous function that satisfies the following three properties:

1. **Ordering:** If $[\mathbf{x}', \mathbf{y}'] \subset [\mathbf{x}, \mathbf{y}]$ then $\sigma(\mathbf{x}', \mathbf{y}') < \sigma(\mathbf{x}, \mathbf{y})$.
2. **Diminishing Absolute Sensitivity (DAS):** σ exhibits diminishing absolute sensitivity if
   (i) For any $\mathbf{x}, \mathbf{y}, \boldsymbol{\epsilon} > 0$, $\sigma(\mathbf{x} + \boldsymbol{\epsilon}, \mathbf{y} + \boldsymbol{\epsilon}) < \sigma(\mathbf{x}, \mathbf{y})$.
   (ii) For any $\mathbf{x}, \mathbf{y}, \boldsymbol{\epsilon} < 0$, $\sigma(\mathbf{x} + \boldsymbol{\epsilon}, \mathbf{y} + \boldsymbol{\epsilon}) < \sigma(\mathbf{x}, \mathbf{y})$.
3. **Increasing Proportional Sensitivity[5] (IPS):** $\sigma(.)$ exhibits increasing proportional sensitivity if for any $\mathbf{xy} > 0$ and any $\boldsymbol{\alpha} > 1$, $\sigma(\boldsymbol{\alpha}\mathbf{x}, \boldsymbol{\alpha}\mathbf{y}) > \sigma(\mathbf{x}, \mathbf{y})$.

[4] A function $f(\mathbf{x}, \mathbf{y})$ is symmetric if $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$.

[5] The definition of IPS includes product $\mathbf{xy} > 0$ and thus also holds if $\mathbf{x}$ and $\mathbf{y}$ are both negative. We let the condition $\mathbf{xy} > 0$ be replaced with $\mathbf{x}, \mathbf{y} > 0$ for functions which are only defined over non-negative values (such as ϕ and θ).

Under the assumption that salient information is readily discerned from the environment, the ordering property is consistent with the “symbolic distance” effect - that it takes adults longer to correctly respond to questions regarding which of two numbers is larger, the smaller their arithmetic difference.[6]

A long tradition in psychology has studied the sensitivity of the perceptual system to changes in the magnitude of a stimulus. Since the Weber-Fechner law was introduced in the 19th century, it has been widely recognized that *diminishing absolute sensitivity* (DAS) is a fundamental property of the perceptual system that applies across a range of sensory modalities including tone, brightness, and distance (Stevens, 1957). Schley and Peters (2014) provide experimental support indicating that a form of diminishing sensitivity also characterizes how the brain maps symbolic numbers onto mental magnitudes.

Support for the assumption that salience perceptions obey *increasing proportional sensitivity* (IPS) can be found in the marketing literature, where IPS is referred to as "the unit effect".[7] Pandelaere et al. (2011), for example, find that the perceived difference between ratings of 7 and 9 on a 0-10 scale appears smaller than the difference between 700 and 900 on a 0-1,000 scale. Similarly, Wertenbroch et al. (2007) report that, even when $1 (in US currency) equals S$1.70 in Singapore currency, “a target price of S$1.70 will appear as less expensive when evaluated against a target budget of S$17.00 than $1 against $10” (p. 3), indicating that the former difference is perceived as bigger than the latter, as implied by IPS.

There is also precedent for assuming these or related properties in the decision making literature. Diminishing sensitivity is assumed as a property of preferences in both original and cumulative prospect theory (Kahneman & Tversky, 1979; Tversky and Kahneman, 1992). In addition, DAS is an analog to the property of *increasing absolute similarity* in Leland (2002) in the context of similarity judgments, and is closely related to Scholten and Read’s (2010) *diminishing absolute sensitivity* assumption for delays and outcomes in intertemporal choice. Bordalo et al. (2012) explicitly assume salience perceptions regarding payoff differences obey Ordering and DAS while Rosa et al. (1993) considered the implications of both DAS and IPS for decisions regarding the introduction of new products. Finally, DAS and IPS are assumed in Prelec and Loewenstein's (1991) model of decision making over time although as properties of preferences. In contrast, our approach does not modify the basic ingredients of the rational economic models – the utility function and discount factor retain their economic interpretation as measuring risk preference and time preference. Instead, these dimensions are weighted by functions that account for the perception of differences in risk, time, and money. Our approach thereby integrates models based on similarity and salience perceptions (e.g., Rubinstein 1988, 2003; Leland 1994, 2002; Bordalo et al. 2012) with the fundamental economic models of preference-based choice.

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[6] For example, consistent with the ordering property, Moyer and Landauer (1967) found that it takes adults longer to answer the question "Which number is larger, 2 or 3?" than to answer the question "Which number is larger, 2 or 7?"
[7] Schley et al (2015) contains a recent summary of this literature.

4. Implications of Diminishing Absolute Sensitivity

4.1 Risk Aversion (Seeking) for Gains (Losses) at High and Moderate Probabilities

A simple behavioral definition of commonly observed risk preferences is given in Definition 7:

Definition 7 (Risk preferences at moderate to high probabilities): Consider the frames in Figure 3 of a choice between a representation $\mathbf{p}$ of a lottery p and its expected value, $\mathbf{E}(\mathbf{p})$, where $\mathbf{x} > \mathbf{y} \geq 0$.

(i) *Risk aversion for gains at moderate to high probabilities* holds if $\mathbf{E}(\mathbf{p}) \gtrsim \mathbf{p}$ for all $\mathbf{p_1} \in$**[0.5,1)**.

(ii) *Risk seeking for losses at moderate to high probabilities* holds if $\mathbf{p}' \gtrsim \mathbf{E}(\mathbf{p}')$ for all $\mathbf{p_1} \in$**[0.5,1)**.

Figure 3. Diminishing Absolute Sensitivity and Attitudes toward Risk

	Risk Aversion for Gains					Risk Seeking for Losses			
(i)	$(\mathbf{x_1,y_1})$	$(\mathbf{p_1,q_1})$	$(\mathbf{x_2,y_2})$	$(\mathbf{p_2,q_2})$	**(ii)**	$(\mathbf{x_1,y_1})$	$(\mathbf{p_1,q_1})$	$(\mathbf{x_2,y_2})$	$(\mathbf{p_2,q_2})$
$\mathbf{E}(\mathbf{p})$	$\mathbf{E}(\mathbf{p})$	$\mathbf{p_1}$	$\mathbf{E}(\mathbf{p})$	$\mathbf{1-p_1}$	$\mathbf{E}(\mathbf{p}')$	$\mathbf{E}(\mathbf{p}')$	$\mathbf{p_1}$	$\mathbf{E}(\mathbf{p}')$	$\mathbf{1-p_1}$
$\mathbf{p}$	$\mathbf{x}$	$\mathbf{p_1}$	$\mathbf{y}$	$\mathbf{1-p_1}$	$\mathbf{p}'$	$\mathbf{-x}$	$\mathbf{p_1}$	$\mathbf{-y}$	$\mathbf{1-p_1}$

To isolate the role of salience perceptions in governing risk attitudes, we state the following proposition for the case where the utility function is linear.

Proposition 3: *Consider the frames in Figure 3 and let* $U(\mathbf{x}) = \mathbf{x}$*. Then for all* $\mathbf{p_1} \in [\mathbf{0.5}, \mathbf{1}]$ *a focal thinker is risk-averse for gains and risk-seeking for losses if μ satisfies ordering and DAS.*

For the case where $\mathbf{p_1} = 0.5$, Proposition 3 requires only DAS. In conjunction with DAS, ordering implies that the result extends to all $\mathbf{p_1} \in [\mathbf{0.5}, \mathbf{1}]$. This stronger result implies that if we observe risk-seeking for gains or risk aversion for losses, such behavior will occur at low probabilities, consistent with the experimental observation of the fourfold pattern of risk attitudes (Tversky and Kahneman, 1992).

4.2 The Common Difference Effect and Hyperbolic Discounting

Just as DAS has implications for attitudes toward risk, it also has implications regarding attitudes toward delay. To illustrate, consider choices between receiving \$100 now (SS) and \$120 in one year (LL), and between \$100 in 10 years (SS′) and \$120 in 11 years (LL′). The stationarity axiom of discounted utility theory requires choices to be either SS and SS′ or LL and LL′. Strotz (1955) conjectured, and subsequent experimental work (as cited in Frederick et al. 2002) has confirmed, that people often choose SS over LL but also select LL′ over SS′. Standard models of hyperbolic and quasi-hyperbolic discounting by Loewenstein and Prelec (1992) and Laibson (1997), respectively, have been proposed for such behavior.

Under SWUP, SS is chosen over LL when the salience weight associated with the difference in delays (now versus in one year) outweighs the weight associated with the difference in payoffs (\$120 versus \$100). However, as a consequence of diminishing absolute sensitivity, the salience weight associated with the time

delays is smaller in the choice between SS$'$ and LL$'$ (10 years versus 11 years), making LL$'$ appear more attractive. More formally, consider choices between a smaller sooner (SS) and a larger later (LL) consumption plan shown in Figure 4 where $y > x > 0$ and $t > r \geq 0$.

Definition 8 (The Common Difference Effect / Hyperbolic Discounting): *The common difference effect* holds if $\mathbf{SS} \precsim_t \mathbf{LL}$ implies $\mathbf{LL}' \succsim_t \mathbf{SS}'$.

Figure 4. Properties of Time Preferences Predicted by SWUP

Simple Consumption Plans

(i)	(x_1, y_1)	Period
SS	$\boldsymbol{x}$	$\boldsymbol{r}$
LL	y	t

Hyperbolic Discounting

(ii)	(x_1, y_1)	Period
SS$'$	x	$r + \Delta$
LL$'$	$\boldsymbol{y}$	$\boldsymbol{t} + \Delta$

The Magnitude Effect

(iii)	(x_1, y_1)	Period
SS$'$	kx	r
LL$'$	$\boldsymbol{ky}$	$\boldsymbol{t}$

Though commonly interpreted as implying that agents discount hyperbolically rather than exponentially as required by discounted utility, the following proposition from Leland and Schneider (2016) suggests instead that the common difference effect and appearance of hyperbolic discounting occur for the same reason people exhibit systematic risk attitudes for gains and losses at high and moderate probabilities.

Proposition 4 (Diminishing Absolute Sensitivity and Hyperbolic Discounting): *A focal thinker exhibits hyperbolic discounting in the frames in Figure 4 if and only if θ satisfies diminishing absolute sensitivity.*

5. Implications of Increasing Proportional Sensitivity

5.1 The General Common Ratio Effect

One of the oldest examples of anomalous behavior under risk is the common ratio effect. A classic example, due to Kahneman and Tversky (1979), is that a decision maker indifferent between a 90% chance of \$3000 and 45% chance of \$6000 will strictly prefer a 0.1% chance of \$6000 over a 0.2% chance of \$3000. The intuition behind these choices is straightforward. In both instances, the decision maker trades-off the difference in payoffs relative to the difference in probabilities of receiving those payoffs. In the first choice, the difference between the probabilities (90%-45%) is more salient than the difference in payoffs resulting in the choice of the safe option. But by IPS, reducing the probabilities holding their ratio constant, reduced the perceived salience of their difference (0.2% -0.1%). As a result, in the second choice, the risky option is selected. More formally:

Definition 9 (General Common Ratio Effect): For frames (i) and (ii) in Figure 5, the *general common ratio effect* holds if for all $\mathbf{y} > \mathbf{x} > \mathbf{0}$; $1 \geq \mathbf{p_1} > \mathbf{q_1} > 0$ and $\boldsymbol{\alpha} \in (0,1)$, $\mathbf{p} \precsim \mathbf{q}$, implies $\mathbf{q}' \succsim \mathbf{p}'$.

Figure 5. The General Common Ratio Effect

(i)	(x_1,y_1)	(p_1,q_1)	(x_2,y_2)	(p_2,q_2)	(ii)	(x_1,y_1)	(p_1,q_1)	(x_2,y_2)	(p_2,q_2)
$\mathbf{p}$	$\mathbf{x}$	$\mathbf{p_1}$	$\mathbf{0}$	$\mathbf{1-p_1}$	$\mathbf{p}'$	$\mathbf{x}$	$\boldsymbol{\alpha}\mathbf{p_1}$	$\mathbf{0}$	$\mathbf{1}-\boldsymbol{\alpha}\mathbf{p_1}$
$\mathbf{q}$	$\mathbf{y}$	$\mathbf{q_1}$	$\mathbf{0}$	$\mathbf{1-q_1}$	$\mathbf{q}'$	$\mathbf{y}$	$\boldsymbol{\alpha}\mathbf{q_1}$	$\mathbf{0}$	$\mathbf{1}-\boldsymbol{\alpha}\mathbf{q_1}$

Proposition 5: *A focal thinker exhibits the general common ratio effect if and only if ϕ satisfies IPS.*

5.2 The Magnitude Effect

The IPS property similarly influences intertemporal choices. In addition to requiring stationarity, discounted utility theory requires choices be invariant to proportional increases in payoffs. However, Thaler (1981) and many others,[8] found such increases in payoffs to yield increasing patience – a phenomenon called the 'magnitude effect' (Loewenstein and Prelec, 1991). For example, an individual indifferent between \$75 today and \$100 in one month will choose \$1000 in one month over \$750 today. This holds under SWUP if the comparison of \$750 and \$1000 is more salient than that between \$75 and \$100 (which is implied by increasing proportional sensitivity). More formally:

Definition 10 (Magnitude Effect): Consider frames (i) and (iii) in Figure 4. The *magnitude effect* holds if for all $k > 1$, $\mathbf{SS} \mathrel{\widehat{\sim}_t} \mathbf{LL}$ implies $\mathbf{LL}' \mathrel{\widehat{\succ}_t} \mathbf{SS}'$.

Proposition 6: *A focal thinker exhibits the magnitude effect if and only if μ satisfies IPS.*

6. Implications following from DAS and IPS - the Fourfold Pattern of Risk Attitudes

As shown above, DAS and ordering have implications for risk preferences at moderate and high probabilities. It turns out IPS has implications for attitudes toward risk involving low probability outcomes. Consider the frames in Figure 6, where we set $\mathbf{x} > 0, \mathbf{k} \geq \mathbf{2}$, and $\mathbf{E(p)} = \mathbf{E(q)}.$ Frames (i) and (ii) each involve a choice between a sure option and a mean-preserving spread. Setting $\mathbf{k} \geq 2$ ensures that the payoff with the largest absolute value occurs at low probabilities. For $\mathbf{k} = 2$, we have $\mathbf{p_1} = 1/3$, independent of $\mathbf{x}$, and $\mathbf{p_1}$ decreases toward 0 as $\mathbf{k}$ increases, holding $\mathbf{x}$ fixed. In the figure, $\mathbf{p_1}$ is the unique value determined by the requirement that $\mathbf{E(p)} = \mathbf{E(q)}.$ For $\mathbf{x} = \$1$, and $\mathbf{k} = 1{,}000{,}000$, the choice in frame (i) resembles the decision of purchasing a lottery ticket ($\mathbf{p_1} \sim 0.000001$), and the choice in frame (ii) resembles the decision to insure against low-likelihood disasters. For such large values of $\mathbf{k}$ relative to $\mathbf{x}$, the payoff $\mathbf{x/k}$ is essentially 0 and its inclusion may be viewed as merely a technical condition. Thus, for large $\mathbf{k}$ and small $\mathbf{x}$, the frames in Figure 6 approximate a choice between a lottery with one non-zero outcome occurring with probability $\mathbf{p_1}$, and its expected value. Formally, we have the following definition:

[8] For an extensive discussion of the literature pertaining to the magnitude effect, see Read (2003).

Definition 11 (Risk Preference at low probabilities): Consider frames (i) and (ii) in Figure 6, where $\mathbf{x} > 0, \mathbf{k} \geq 2$, and $\mathbf{E(p) = E(q)}$:

(i) *Risk-seeking for gains at low probabilities* holds if $\mathbf{q} \succsim \mathbf{p}$.

(ii) *Risk aversion for losses at low probabilities* holds if $\mathbf{p'} \succsim \mathbf{q'}$

Figure 6. Increasing Proportional Sensitivity and Attitudes toward Risk

Lottery Purchase

(i)	**(x_1,y_1)**	**(p_1,q_1)**	**(x_2,y_2)**	**(p_2,q_2)**
$\mathbf{p}$	$\mathbf{x}$	$\mathbf{p_1}$	$\mathbf{x}$	$\mathbf{1-p_1}$
$\mathbf{q}$	$\mathbf{kx}$	$\mathbf{p_1}$	$\mathbf{x/k}$	$\mathbf{1-p_1}$

Insurance Purchase

(ii)	**(x_1,y_1)**	**(p_1,q_1)**	**(x_2,y_2)**	**(p_2,q_2)**
$\mathbf{p'}$	$\mathbf{-x}$	$\mathbf{p_1}$	$\mathbf{-x}$	$\mathbf{1-p_1}$
$\mathbf{q'}$	$\mathbf{-kx}$	$\mathbf{p_1}$	$\mathbf{-x/k}$	$\mathbf{1-p_1}$

Proposition 7: *Let $U(\mathbf{x}) = \mathbf{x}$. Then for frames (i) and (ii) in Figure 3, a focal thinker is risk-seeking for gains at low probabilities and risk-averse for losses at low-probabilities if μ satisfies IPS.*

A general implication of Propositions 3 and 7 is that Ordering, DAS and IPS directly imply the fourfold pattern of risk preferences without any parametric assumptions about the salience functions. A decision maker who exhibits the fourfold pattern displays "risk aversion for gains and risk seeking for losses of high probability and risk seeking for gains and risk aversion for losses of low probability" (Tversky and Kahneman 1992, p. 297). A characteristic example is given in Figure 7. In the figure, note that risk aversion in Frame (i) and risk-seeking behavior in Frame (ii) are directly implied by Proposition 3, while risk-seeking in Frame (iii) and risk aversion in Frame (iv) are directly implied by Proposition 7.

Figure 7. Ordering, DAS, IPS and the Fourfold Pattern of Risk Preferences*

Risk Aversion for high-probability Gains

(i)	**(x_1,y_1)**	**(p_1,q_1)**	**(x_2,y_2)**	**(p_2,q_2)**
p	**99**	**0.99**	**99**	**0.01**
Q	100	0.99	0.01	0.01

Risk Seeking for high-probability Losses

(ii)	**(x_1,y_1)**	**(p_1,q_1)**	**(x_2,y_2)**	**(p_2,q_2)**
P	−99	0.99	−99	0.01
q	**−100**	**0.99**	**−0.01**	**0.01**

Risk Seeking for low-probability Gains

(iii)	**(x_1,y_1)**	**(p_1,q_1)**	**(x_2,y_2)**	**(p_2,q_2)**
p	1	0.01	1	0.99
q	**100**	**0.01**	**0.01**	**0.99**

Risk Aversion for low-probability Losses

(iv)	**(x_1,y_1)**	**(p_1,q_1)**	**(x_2,y_2)**	**(p_2,q_2)**
p	**−1**	**0.01**	**−1**	**0.99**
Q	−100	0.01	−0.01	0.99

* Probabilities are rounded to the nearest hundredth of the exact values necessary to make the expected values equal.

Definition 12 (Fourfold pattern): An agent who is risk-averse for gains and risk-seeking for losses at high probabilities, as defined in Definition 7, and is risk-seeking for gains and risk-averse for losses at low probabilities, as in Definition 11, exhibits the *fourfold pattern of risk preferences*.

Since a salience function in Definition 6 obeys ordering, DAS, and IPS, we have the following, which establishes that the fourfold pattern of risk attitudes is a general property of a salience function.

Corollary 1. (Fourfold Pattern of Risk Preferences): *Let $U(\mathbf{x}) = \mathbf{x}$. Then a focal thinker exhibits the fourfold pattern for any salience function μ.*

Although the fourfold pattern is explained by some well-known decision models like prospect theory (Kahneman and Tversky 1979; Tversky and Kahneman, 1992) and rank dependent utility theory (Quiggin, 1982), these models have focused on describing the phenomenon as opposed to explaining why it occurs (but see Mallpress et al., 2015). Bordalo et al. (2012) discuss the fourfold pattern as arising in their theory of salience-based risky choice. However, the interaction of ordering and DAS in generating the pattern is not clearly specified and the IPS property is not discussed at all. Instead, the pattern is explained by positing the following salience function involving a parameter $\theta > 0$ which is difficult to interpret intuitively.

$$\sigma(\mathbf{x}, \mathbf{y}) = \frac{|\mathbf{x} - \mathbf{y}|}{|\mathbf{x}| + |\mathbf{y}| + \theta},$$

In contrast, the approach outlined in the previous section provides a psychological basis for the fourfold pattern as a very general implication of a salience function and one that does not rely on any specific functional form. Instead its main properties – ordering, DAS and IPS - follow a logic that one might assume as a first principle. As shown in Proposition 1, DAS can be derived from the property that, for a fixed absolute difference, the perceptual system is more sensitive to larger ratios. Similarly, IPS can be derived from the property that, for a fixed ratio, the perceptual system is more sensitive to larger absolute differences. The ordering property is also closely related to DAS and IPS in that if one interval is contained in another, the endpoints of the larger interval have both a larger absolute difference and a larger ratio than those of the smaller interval and are therefore more salient.

7. Discussion

After providing simple and natural axioms for a general class of salience functions, we considered the implications of properties of these functions in the context of risky and intertemporal choice. Table 1 summarizes properties of observed preferences which can be formally derived from DAS, IPS, and Ordering, without any parametric assumptions regarding the salience functions. In 4.1 and 4.2 we showed that risk aversion (seeking) for gains (losses) of moderate probability as well as the common difference effect and apparent hyperbolic discounting in intertemporal choice arise because the perception of differences exhibits *diminishing absolute sensitivity* embodied in the Weber-Fechner law (e.g., the

perceived difference of 100 and 1 is larger than that between 200 and 101). Leland and Schneider (2016) show that DAS also implies ambiguity aversion in Ellsberg's (1961) paradoxes while Schneider, Leland, and Wilcox (2016) present experimental evidence that Ellsberg-type behavior can be reduced by employing lottery presentations that preclude DAS from differentially influencing the decision made in one situation versus another.

Table 1: Properties of Preferences Derived from Properties of Perception

Properties of Salience Perception	**Properties of Preferences**
Diminishing Absolute Sensitivity	Risk Aversion (Gains) / Risk Seeking (Losses)
Diminishing Absolute Sensitivity	Common Difference Effect / Hyperbolic Discounting
Diminishing Absolute Sensitivity	Ambiguity Aversion
Increasing Proportional Sensitivity	Lottery and Insurance Purchase
Increasing Proportional Sensitivity	Common Ratio Effect
Increasing Proportional Sensitivity	Magnitude Effect
Ordering, DAS, IPS	Fourfold Pattern of Risk Attitudes

We subsequently showed that the common ratio effect, the magnitude effect, and the simultaneous purchase of lottery tickets and insurance all arise because the perception of magnitude differences exhibits increasing proportional sensitivity (e.g., the perceived difference between 200 and 100 is more salient than the perceived difference between 2 and 1). Finally, we showed that diminishing absolute sensitivity, increasing proportional sensitivity, and ordering interact to produce the fourfold pattern of risk attitudes.

Appendix

Proposition 1: *Under Axioms 1-5, there exists a salience function σ that represents $\trianglerighteq_s$.*

Proof: In this proposition, we establish that Axioms 1 through 5 are sufficient for the representation. Axiom 1, well known in consumer theory, guarantees the existence (see, for instance Mas-Colell, Whinston, and Green (1995), Ch. 3, Proposition 3.C.1) of a continuous function $\sigma: X \to \mathbb{R}$ such that $(x, y) \trianglerighteq_s (x', y') \Leftrightarrow \sigma(x, y) \geq \sigma(x', y')$. Given Axiom 1, it is clear that Axiom 2 implies that σ is symmetric, and that Axiom 3 implies that σ satisfies ordering. To show that in the presence of Axiom 1, Axiom 4 implies diminishing absolute sensitivity, we can write $x' = x + \epsilon$ and $y' = y + \epsilon$ for $\epsilon > 0$. Then we have the following lemma:

> **Lemma 1:** $r(x, y) > r(x + \epsilon, y + \epsilon)$ for all $x > y > 0$, and any $\epsilon > 0$.
>
> **Proof:** Inequality $r(x, y) > r(x + \epsilon , y + \epsilon)$ holds for all $x > y > 0$ and any $\epsilon > 0$ if
>
> $$\frac{x(y + \epsilon)}{y(y + \epsilon)} > \frac{y(x + \epsilon)}{y(y + \epsilon)}.$$
>
> which requires $xy^3 + 2xy^2\epsilon + xy\epsilon^2 > xy^3 + xy^2\epsilon + y^3\epsilon + y^2\epsilon^2$. Since $x > y$, we have $2xy^2\epsilon > xy^2\epsilon + y^3\epsilon$ and $xy\epsilon^2 > y^2\epsilon^2$. Thus, $r(x, y) > r(x + \epsilon, y + \epsilon) = r(x', y')$.■

By Axiom 4, the inequality $r(x, y) > r(x', y')$ implies $(x, y) \vartriangleright_s (x', y')$ which, in the presence of Axiom 1, implies $\sigma(x, y) > \sigma(x + \epsilon, y + \epsilon)$.

To show that in the presence of Axiom 1, Axiom 5 implies increasing proportional sensitivity, we can write $x' = \alpha x$ and $y' = \alpha y$ for $\alpha > 1$. Note that for any $\alpha > 1$, we have

$\Delta(x', y') = \alpha \cdot \Delta(x, y) > \Delta(x, y)$. By Axiom 5, $\Delta(x', y') > \Delta(x, y)$ implies$(x', y') \vartriangleright_s (x, y)$ which, by Axiom 1, implies $\sigma(\alpha x, \alpha y) > \sigma(x, y)$. ■

Proposition 2: *For any salience function σ that represents $\trianglerighteq_s$:*

(i) *σ satisfies ordering if and only if $\trianglerighteq_s$ satisfies Axiom 3.*

(ii) *σ satisfies DAS if and only if $\trianglerighteq_s$ satisfies Axiom 4.*

(iii) *σ satisfies IPS if and only if $\trianglerighteq_s$ satisfies Axiom 5.*

Proof: That Axioms 3, 4 and 5 are sufficient for ordering, DAS, and IPS, respectively was confirmed in Proposition 1. It remains for us to show that Axioms 3, 4 and 5 necessarily follow from the properties of a salience function. It is clear that Axiom 3 is implied by ordering. To see that DAS implies Axiom 4, recall that by Lemma 1, $r(x, y) > r(x + \epsilon , y + \epsilon)$ for any $x, y > 0$ and any $\epsilon > 0$. Also, note that $\Delta(x, y) = \Delta(x + \epsilon , y + \epsilon)$. By DAS, $\sigma(x, y) > \sigma(x + \epsilon, y + \epsilon)$ which implies $(x, y) \vartriangleright_s (x + \epsilon, y + \epsilon)$ for any σ that represents $\trianglerighteq_s$. Thus, we have $\Delta(x, y) = \Delta(x + \epsilon , y + \epsilon)$, and $r(x, y) > r(x + \epsilon , y + \epsilon)$ which, by DAS, imply $(x, y) \vartriangleright_s (x + \epsilon, y + \epsilon)$ and Axiom 4 follows.

To see that IPS implies Axiom 5, recall that $\Delta(x', y') = \alpha \cdot \Delta(x, y) > \Delta(x, y)$ for any $x, y > 0$ and any $\alpha > 1$. Also note that $r(x, y) = r(\alpha x, \alpha y)$. By IPS, $\sigma(\alpha x, \alpha y) > \sigma(x, y)$ which implies $(\alpha x, \alpha y) \rhd_s (x, y)$ for any σ that represents $\trianglerighteq_s$. Thus, we have $r(x, y) = r(\alpha x, \alpha y)$, and $\Delta(\alpha x, \alpha y) > \Delta(x, y)$, which, by IPS, imply $(\alpha x, \alpha y) \rhd_s (x, y)$ and Axiom 5 follows. ■

Proposition 3 (Diminishing Absolute Sensitivity and Risk Preference): *Consider choices as framed in Figure 2 and let $U(\mathbf{x}) = \mathbf{x}$. Then for all $\mathbf{p_1} \in [0.5,1]$ a focal thinker is risk-averse for gains and risk-seeking for losses at moderate to high probabilities if μ satisfies ordering and DAS.*

Proof: For $U(\mathbf{x}) = \mathbf{x}$, a focal thinker chooses $\mathbf{E}(\mathbf{p})$ over $\mathbf{p}$ iff $\mu\big(\mathbf{y}, (\mathbf{x} - \mathbf{y})\mathbf{p_1} + \mathbf{y}\big) > \mu\big(\mathbf{x}, (\mathbf{x} - \mathbf{y})\mathbf{p_1} + \mathbf{y}\big)$. For $p = 0.5$, DAS implies $\mu(\mathbf{y}, \mathbf{0.5}(\mathbf{x} + \mathbf{y})) > \mu\big(\mathbf{x}, \mathbf{0.5}(\mathbf{x} + \mathbf{y})\big)$. To see that DAS implies that this inequality holds, let $\boldsymbol{\epsilon} = 0.5(x - y)$. Then $\mu(\mathbf{y} + \boldsymbol{\epsilon}, \mathbf{0.5}(\mathbf{x} + \mathbf{y}) + \boldsymbol{\epsilon}) = \mu(\mathbf{0.5}(\mathbf{x} + \mathbf{y}), \mathbf{x})$, which by symmetry, equals $\mu\big(\mathbf{x}, \mathbf{0.5}(\mathbf{x} + \mathbf{y})\big)$. For $\mathbf{p_1} \in [0.5,1]$, note that $\big[\mathbf{x}, \mathbf{x}\mathbf{p_1} + \mathbf{y}\big(\mathbf{1} - \mathbf{p_1}\big)\big] \subseteq [\mathbf{x}, \mathbf{0.5}(\mathbf{x} + \mathbf{y})]$ and thus ordering implies $\mu\big(\mathbf{x}, \mathbf{0.5}(\mathbf{x} + \mathbf{y})\big) \geq \mu\Big(\mathbf{x}, \mathbf{x}\mathbf{p_1} + \mathbf{y}\big(\mathbf{1} - \mathbf{p_1}\big)\Big)$. Thus, $\mu(\mathbf{y}, \mathbf{0.5}(\mathbf{x} + \mathbf{y})) > \mu\Big(\mathbf{x}, \mathbf{x}\mathbf{p} + \mathbf{y}\big(\mathbf{1} - \mathbf{p_1}\big)\Big)$ for all $\mathbf{p_1} \in [0.5,1]$. The result for losses follows analogously. ■

Proposition 5: *A focal thinker exhibits the general common ratio effect if and only if ϕ satisfies IPS.*

Proof: Note that $\mathbf{p} \eqsim \mathbf{q}$ under SWUP if and only if

$$\mu(\mathbf{x}, \mathbf{y})\big(U(\mathbf{y}) - U(\mathbf{x})\big)\left[\frac{\mathbf{p_1} + \mathbf{q_1}}{2}\right] = \phi(\mathbf{p_1}, \mathbf{q_1})(\mathbf{p_1} - \mathbf{q_1})\left[\frac{U(\mathbf{y}) + U(\mathbf{x})}{2}\right].$$

Also note that $\mathbf{q'} \gtrsim \mathbf{p'}$ if and only if

$$\mu(\mathbf{x}, \mathbf{y})\big(U(\mathbf{y}) - U(\mathbf{x})\big)\left[\frac{\mathbf{p_1} + \mathbf{q_1}}{2}\right] > \phi\big(\boldsymbol{\alpha}\mathbf{p_1}, \boldsymbol{\alpha}\mathbf{q_1}\big)\big(\mathbf{p_1} - \mathbf{q_1}\big)\left[\frac{U(\mathbf{y}) + U(\mathbf{x})}{2}\right].$$

By IPS, scaling $\boldsymbol{\alpha}\mathbf{p_1}$ and $\boldsymbol{\alpha}\mathbf{q_1}$ each by $\frac{1}{\boldsymbol{\alpha}}$ leads to $\phi\big(\boldsymbol{\alpha}\mathbf{p_1}, \boldsymbol{\alpha}\mathbf{q_1}\big) < \phi\big(\mathbf{p_1}, \mathbf{q_1}\big)$ for all $\boldsymbol{\alpha} \in (0,1)$. Letting $\mathbf{k} \equiv 1/\boldsymbol{\alpha}$, the common ratio effect holds if and only if $\big(\boldsymbol{\alpha}\mathbf{p_1}, \boldsymbol{\alpha}\mathbf{q_1}\big) < \phi\big(\mathbf{k}\boldsymbol{\alpha}\mathbf{p_1}, \mathbf{k}\boldsymbol{\alpha}\mathbf{q_1}\big)$ for all $\mathbf{k} > 1$. ■

Proposition 6: *A focal thinker exhibits the magnitude effect if and only if μ satisfies IPS.*

Proof: For the choice between SS and LL, $SS \eqsim_t LL$ under SWUP if and only if

$$\mu(x, y)(U(y) - U(x))\left[\frac{\delta^r + \delta^t}{2}\right] = \pi(r, t)(\delta^r - \delta^t)\left[\frac{U(y) + U(x)}{2}\right]$$

For the choice between SS' and LL', $LL' \gtrsim_t SS'$ if and only if

$$\mu(kx, ky)(U(y) - U(x))\left[\frac{\delta^r + \delta^t}{2}\right] > \pi(r, t)(\delta^r - \delta^t)\left[\frac{U(y) + U(x)}{2}\right]$$

Given $SS \eqsim_t LL$, the magnitude effect holds if and only if $\mu(kx, ky) > \mu(x, y)$ for all $k > 1$, which holds if and only if μ satisfies IPS. ■

Proposition 7 (Increasing Proportional Sensitivity and Risk Preference): *If U(x) = x, a focal thinker is risk-seeking for low-probability gains and risk-averse for low-probability losses if μ satisfies IPS.*

Proof: We show that IPS implies risk-seeking for low-probability gains. Risk aversion for low probability losses follows analogously. Given $U(x) = x$, a focal thinker strictly prefers g to f if and only if $\mu(x, kx)(x - xk)p + \mu\left(x, \frac{x}{k}\right)\left(x - \frac{x}{k}\right)(1 - p) < 0$. Since $E(f) = E(g)$, we have: $p = \frac{1-(1/k)}{k-(1/k)}$.

Thus, g is strictly preferred to f if and only if

$$\mu(x, kx)(x - xk)\left[\frac{1 - (1/k)}{k - (1/k)}\right] + \mu(x, x/k)(x - x/k)\left[\frac{k - 1}{k - (1/k)}\right] < 0,$$

which can be rewritten as $\mu(x, x/k)x\left[\frac{(k-1)(1-(1/k)}{k-(1/k)}\right] < \mu(x, kx)x\left[\frac{(k-1)(1-(1/k))}{k-(1/k)}\right]$, which implies $\mu(x, x/k) < \mu(x, kx)$. Then by symmetry and increasing proportional sensitivity, we have $\mu(x, kx) = \mu(kx, x) = \mu\left(kx, k\left(\frac{x}{k}\right)\right) > \mu\left(x, \frac{x}{k}\right)$. The result for losses follows analogously. ■

Corollary 1. (Fourfold Pattern of Risk Preferences): *Let* $U(\mathbf{x}) = \mathbf{x}$. *Then a focal thinker exhibits the fourfold pattern for any salience function* μ.

Proof: By Proposition 3, ordering and DAS imply risk aversion for high probability gains and risk seeking behavior for high-probability losses. By Proposition 7, IPS implies risk seeking for low probability gains and risk aversion for low probability losses.

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