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Many-Valued Coalgebraic Logic: From Boolean Algebras to Primal Varieties

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Abstract

We study many-valued coalgebraic logics with primal algebras of truth-degrees. We describe a way to lift algebraic semantics of classical coalgebraic logics, given by an endofunctor on the variety of Boolean algebras, to this many-valued setting, and we show that many important properties of the original logic are inherited by its lifting. Then, we deal with the problem of obtaining a concrete axiomatic presentation of the variety of algebras for this lifted logic, given that we know one for the original one. We solve this problem for a class of presentations which behaves well with respect to a lattice structure on the algebra of truth-degrees.

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1 Introduction

Both many-valued modal logics (see, e.g., [10, 8, 5, 12, 36]) and two-valued coalgebraic logics (see, e.g., [27, 29, 17, 18]) have received increased attention in recent years. Nonetheless, the literature on the combination of these two topics seems, as of yet, sparse (examples include [2, 1, 23]). In this paper, we use methods from universal algebra and category-theory to study algebraic semantics of many-valued coalgebraic logics.

In the classical (two-valued) case, algebraic semantics for coalgebraic logics have been described in [17] as follows. Given an endofunctor T on the category Set, an *abstract coalgebraic logic* for T consists of an endofunctor L on the variety BA of Boolean algebras together with a natural transformation δ determining the semantics (see Definition 1). One can then relate T-coalgebras and L-algebras via δ and a dual adjunction between Set and BA. In particular, we call such a coalgebraic logic *concrete* if the functor L comes equipped

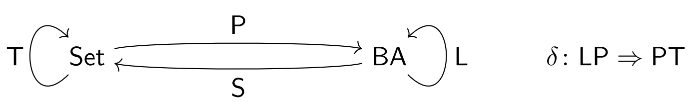

Figure 1 Classical abstract coalgebraic logic for T.

with a *presentation by operations and equations* in the sense of [4, 20, 22]. Essentially, this corresponds to an axiomatization of the variety $\mathsf{Alg}(\mathsf{L})$ of L-algebras. For example, considering classical modal logic, where $\mathsf{T} = \mathcal{P}$ is the covariant powerset functor (that is, T-coalgebras are Kripke frames), the functor L has a presentation by one unary operation $\Box$ with two equations $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$ (that is, L-algebras are modal algebras).

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It is well-known that the variety BA is generated by the two-element Boolean algebra $\mathbf{2}$, that is $\mathsf{BA} = \mathbb{HSP}(\mathbf{2})$. In this paper, we consider the many-valued case where BA is replaced by a variety $\mathcal{A} = \mathbb{HSP}(\mathbf{D})$, generated by another finite algebra $\mathbf{D}$. More specifically, we study the case where the algebra $\mathbf{D}$ is *primal*.

An algebra $\mathbf{D}$ with carrier set D is primal [11, 31, 7] if every map $f\colon D^k \to D$ is definable by a term $t_f(x_1, \ldots, x_k)$ of $\mathbf{D}$. It is well-known that the Boolean algebra $\mathbf{2}$ is primal, and primal algebras (e.g., the Post-chains, see Example 6) may be seen as many-valued generalizations of this algebra. Indeed, Hu [13] showed that if $\mathbf{D}$ is primal, then the variety $\mathcal{A}$ it generates is categorically equivalent to the variety of Boolean algebras BA (and vice versa). Utilizing such a categorical equivalence, we lift an abstract coalgebraic logic (L, δ) over BA to an abstract coalgebraic logic (L', δ') over $\mathcal{A}$ (see Figure 4). The logic thus obtained inherits many useful properties of the original one, such as (one-step) completeness and expressivity.

In particular, if L has a presentation by operations and equations, the same is true for L', so at first glance it may seem straightforward to lift concrete coalgebraic logics in a similar manner. However, as we illustrate in this paper, this task turns out to be far from trivial. While the lifting guarantees the *existence* of a presentation of L', it offers no indication of what this presentation looks like or how it can be explicitly obtained from a presentation of L. To answer these questions, we delve deeper into the algebraic structure of $\mathbf{D}$. For certain classes of functors L, we show that there is a systematic way to obtain a presentation of L' directly from a presentation of L. In particular, this method applies to classical modal logic as described above.

This work should be seen in the larger context of many-valued coalgebraic logics which have been of interest to the community for a range of potential applications, from AI and cyber-physical systems to the reasoning about software quality. Another (not necessarily coalgebraic) application of many-valued reasoning are semiring-based algorithms for solving soft constraints (see, e.g., [34] for a recent example). From the point of view of some of these applications of many-valued logics, a restriction of our approach is that the dualising algebra of truth-degrees is finite and, correspondingly, the topological duality is zero-dimensional. It remains to be seen in future work whether the techniques we develop to extend Boolean modal logics to many-valued modal logics can be generalized to a continuum of truth-degrees. Our next step, still keeping to the finite case, will be to generalize from primal to semi-primal algebras of truth values (see Question 3).

The paper is structured as follows. In Section 2, we give an overview of coalgebraic logic (Subsection 2.1) and of primal algebras (Subsection 2.2). In Section 3, we show how to lift abstract coalgebraic logics over BA to ones over $\mathcal{A}$ (see Definition 11), and we show that important properties are preserved under this lifting (see Theorem 12). In Section 4, we present some methods which, under various circumstances, allow us to obtain a presentation of the lifted logic from a presentation of the original one (see Theorems 15 and 18). We also show how these methods can be applied to classical modal logic (see Example 17) and to neighborhood semantics (see Example 19). Lastly, in Section 5, we give a short summary and collect some open questions for further research.

2 Preliminaries

In this section, we recall the most important notions used in this paper. In Subsection 2.1, we give a short summary of coalgebraic logics and their algebraic semantics [17]. We distinguish between abstract coalgebraic logics, in which the algebraic semantics correspond to an endofunctor L on a variety without further specification, and concrete coalgebraic logics, in which this functor L is given together with an explicit presentation by operations and equations [4]. We also recall two important properties of coalgebraic logics, namely one-step completeness [29, 17] and expressivity [30, 16, 35, 15].

In Subsection 2.2, we recall the definition of primality [11] and provide some examples of primal algebras which have previously been considered in logic. Note that the unary terms T_1 and T_0 defined in Example 7 reoccur in later sections of this paper. Regarding the variety generated by a primal algebra, we recall Hu's Theorem [13, 14].

2.1 Abstract and Concrete Coalgebraic Logics

Coalgebraic (modal) logic, introduced by Moss [27], offers a uniform framework for the logical study of transition systems modeled by coalgebras. In this paper, we follow the approach to coalgebraic logic developed in [17] (for an overview of the various approaches to coalgebraic logic we refer the reader to [18]). It builds on the following dual adjunction between the category Set and the variety BA of Boolean algebras, defined by two contravariant functors $\mathsf{P}\colon \mathsf{Set} \to \mathsf{BA}$ and $\mathsf{S}\colon \mathsf{BA} \to \mathsf{Set}$. Intuitively, the functor P is the contravariant powerset functor and S is the functor sending a Boolean algebra to its set of ultrafilters. Formally, we will describe them in a way which is more convenient to generalize to other algebras later on.

The functor $\mathsf{P}\colon \mathsf{Set} \to \mathsf{BA}$ assigns the Boolean algebra $\mathsf{P}(X) = \mathbf{2}^X$ to the set X, where $\mathbf{2} = (\{0,1\}, \wedge, \vee, \neg, 0, 1)$ is the two-element Boolean algebra. A map $f\colon X \to X'$ gets sent to $\mathsf{P}f\colon \mathbf{2}^{X'} \to \mathbf{2}^X$ defined by composition $\beta \mapsto \beta \circ f$.

The functor S assigns the set of homomorphisms $\mathsf{S}(\mathbf{B}) = \mathsf{BA}(\mathbf{B}, \mathbf{2})$ to a Boolean algebra $\mathbf{B} \in \mathsf{BA}$ (note that $\mathsf{BA}(\mathbf{B}, \mathbf{2})$ can be identified with the set of ultrafilters of $\mathbf{B}$) and sends a homomorphism $h\colon \mathbf{B} \to \mathbf{B}'$ to the map $\mathsf{S}h\colon \mathsf{BA}(\mathbf{B}', \mathbf{2}) \to \mathsf{BA}(\mathbf{B}, \mathbf{2})$, again defined by composition $u \mapsto u \circ h$.

It is well-known that P and S form a dual adjunction between the categories Set and BA. The corresponding natural transformations $\eta\colon 1_{\mathsf{BA}} \Rightarrow \mathsf{PS}$ and $\varepsilon\colon 1_{\mathsf{Set}} \Rightarrow \mathsf{SP}$ are given by evaluations, that is, for for all $\mathbf{B} \in \mathsf{BA}$ and $X \in \mathsf{Set}$ we have

$$\begin{aligned} \eta_{\mathbf{B}}\colon \mathbf{B} &\to \mathbf{2}^{\mathsf{BA}(\mathbf{B},\mathbf{2})} & \qquad \varepsilon_X\colon X &\to \mathsf{BA}(\mathbf{2}^X, \mathbf{2}) \\ b &\mapsto \mathsf{ev}_b & x &\mapsto \mathsf{ev}_x \end{aligned}$$

where $\mathsf{ev}_b(h) = h(b)$ and $\mathsf{ev}_x(f) = f(x)$.

Classical coalgebraic logics are built "on top" of this dual adjunction, relating coalgebras over the base category Set to algebras over the base category BA. Since we are not only interested in the classical case (that is, we aim to replace BA by other varieties later on), we use the following general definition.

▶ **Definition 1** (Abstract coalgebraic logic). *Let $\mathcal{V}$ be a variety, $\Pi\colon \mathsf{Set} \to \mathcal{V}$ and $\Sigma\colon \mathcal{V} \to \mathsf{Set}$ be two contravariant functors forming a dual adjunction and let T be an endofunctor on Set. An* abstract coalgebraic logic *for T is a pair (L, δ), consisting of an endofunctor L on $\mathcal{V}$ and a natural transformation $\delta\colon \mathsf{L}\Pi \Rightarrow \Pi\mathsf{T}$.*

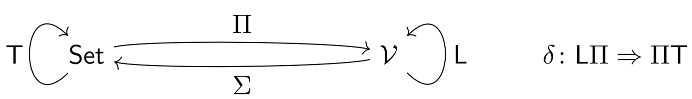

Figure 2 Abstract coalgebraic logic for T over a variety $\mathcal{V}$.

Given a T-coalgebra $\gamma\colon X \to \mathsf{T}(X)$, applying Π yields $\Pi\gamma\colon \Pi\mathsf{T}(X) \to \Pi(X)$. Composing with δ_X, we obtain an L-algebra $\Pi\gamma \circ \delta_X : \mathsf{L}\Pi(X) \to \Pi(X)$. To illustrate these notions, we recall how classical modal logic arises as a special case of a coalgebraic logic (for more details see [17]).

▶ **Example 2** (Classical modal logic). For a general introduction to classical modal logic, we refer the reader to [3]. The category of Kripke frames with bounded morphisms is isomorphic to the category $\mathsf{Coalg}(\mathcal{P})$ of coalgebras for the covariant powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$.

The variety of modal algebras, on the other hand, can be identified with the category $\mathsf{Alg}(\mathsf{L})$ of algebras for an endofunctor $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ defined as follows. If $\mathbf{B}$ is a Boolean algebra, $\mathsf{L}(\mathbf{B})$ is the free Boolean algebra generated by the set of formal expressions $\{\Box b \mid b \in B\}$, quotiented by the equations $\Box 1 = 1$ and $\Box(b_1 \wedge b_2) = \Box b_1 \wedge \Box b_2$.

The corresponding natural transformation $\delta\colon \mathsf{LP} \Rightarrow \mathsf{P}\mathcal{P}$ is defined as follows. For a set X, the component $\delta_X : \mathsf{LP}(X) \to \mathsf{P}\mathcal{P}(X)$ is the unique homomorphism which maps a generator $\Box Y$ (where $Y \subseteq X$) to $\{Z \subseteq X \mid Z \subseteq Y\}$. For a Kripke frame $\gamma\colon W \to \mathcal{P}(W)$, the algebra $\mathsf{P}\gamma \circ \delta_W$ is known as the *complex algebra* of the frame.

In this example, the category $\mathsf{Alg}(\mathsf{L})$ is a variety, since the functor L has a *presentation* by one unary operation $\Box$ and two equations $\Box 1 = 1$ and $\Box(x \wedge y) = \Box x \wedge \Box y$. For further information about *presentations of functors by operations and equations* we refer the reader to [4, 20, 22].

▶ **Definition 3** (Concrete coalgebraic logic). *A* concrete coalgebraic logic *is an abstract coalgebraic logic* (L, δ) *together with a presentation of the functor* L *by operations and equations.*

It is shown in [22, Theorem 4.7] that an endofunctor L on a variety has a presentation by operations and equations if and only if it preserves sifted colimits.

Two important properties of coalgebraic logics are (one-step) completeness [29, 17] and expressivity [30, 16, 35, 15].

▶ **Definition 4** (One-step completeness, expressivity). *A coalgebraic logic* (L, δ) *is called*

- one-step complete *if* δ *is a component-wise monomorphism, and*
- expressive *if the adjoint-transpose* $\delta^\dagger$ *of* δ *is a component-wise monomorphism.*

Classical modal logic (see Example 2) is one-step complete but not expressive. However, if we replace $\mathcal{P}$ by the finite powerset functor $\mathcal{P}_{\mathit{fin}}$ (i.e., if we only consider *image-finite* Kripke frames), the logic becomes expressive.

2.2 Primal Algebras

It is well-known that every function $f\colon \{0,1\}^k \to \{0,1\}$ (where $k \geq 1$) is term-definable in the two-element Boolean algebra $\mathbf{2}$. In 1953, Foster [11] initiated the general study of algebras with this property, introducing the following notion.

▶ **Definition 5** (Primal algebra). *A finite algebra* $\mathbf{D}$ *with carrier set* D *is called* primal *if every function* $f\colon D^k \to D$ *(where* $k \geq 1$*) is term-definable in* $\mathbf{D}$.

Next we give some examples of primal algebras which have a connection to logic, starting with a well-known example of an early many-valued logic.

▶ **Example 6** (Post chain). The $(n+1)$-element *Post chain* is the algebra

$$\mathbf{P}_n = \big(\{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, {}', 0, 1\big),$$

where $\wedge$ and $\vee$ are the usual lattice operations and the unary operation $'$ is defined by $0' = 1$ and $(\frac{i}{n})' = (\frac{i-1}{n})$ for $0 < i \leq n$. For every $n \geq 1$, the algebra $\mathbf{P}_n$ is primal [11, Theorem 35].

In our next example, we show that every finite bounded lattice can be turned into a primal algebra in a canonical way. Modal expansions of similar structures have been studied in [25].

▶ **Example 7.** Let $(L, \wedge, \vee, 0, 1)$ be a finite bounded lattice. Consider the algebra

$$\mathbf{L} = (L, \wedge, \vee, \{T_\ell\}_{\ell \in L}, \{\hat{\ell}\}_{\ell \in L}),$$

with unary operations

$$T_\ell(x) = \begin{cases} 1 & \text{if } x = \ell \\ 0 & \text{if } x \neq \ell \end{cases}$$

as well as constants $\hat{\ell}$ for every $\ell \in L$ (in particular, for the bounds 0 and 1). The algebra $\mathbf{L}$ is primal. For instance, every unary function $f\colon L \to L$ is definable by the "generalized disjunctive normal form"

$$t_f(x) = \bigvee_{\ell \in L} (T_\ell(x) \wedge \widehat{f(\ell)}).$$

We can proceed similarly with functions $f\colon L^k \to L$ of higher arity, using the terms $T_{(\ell_1,\dots,\ell_k)}(x_1, \dots, x_k) = T_{\ell_1}(x_1) \wedge \dots \wedge T_{\ell_k}(x_k)$ for every $(\ell_1, \dots, \ell_k) \in L^k$.

Other examples of primal algebras in logic which we don't describe in detail here include the four-valued bilattice studied in [32] and the "Boolean-like" algebras studied in [33].

Not surprisingly, primal algebras have a lot in common with the two-element Boolean algebra. From a category-theoretical perspective, this resemblance is subsumed by *Hu's Theorem*, which we will state now.

▶ **Theorem 8** (Hu's Theorem [13, 14]). *A variety $\mathcal{A}$ is categorically equivalent to* BA *if and only if there is a primal algebra $\mathbf{D} \in \mathcal{A}$ such that $\mathcal{A} = \mathbb{HSP}(\mathbf{D})$.*

In the following sections we will relate "classical" coalgebraic logics (L, δ), where L is an endofunctor on BA, to "primal" coalgebraic logics (L', δ') where L' is an endofunctor on the variety $\mathcal{A}$ generated by a primal algebra. Even though Theorem 8 implies that BA and $\mathcal{A}$ are categorically equivalent, we will see that this is a non-trivial task, since presentations of functors are usually not preserved under categorical equivalences.

3 Lifting Abstract Coalgebraic Logics

For the remainder of this paper, we adopt the following framework.

▶ **Assumption 9.** *Let $\mathbf{D}$ be a primal algebra, based on a bounded lattice $\mathbf{D}^\flat = (D, \wedge, \vee, 0, 1)$.*

We use $\mathcal{A} = \mathbb{HSP}(\mathbf{D})$ to denote the variety generated by $\mathbf{D}$. Note that the assumption that $\mathbf{D}$ comes equipped with a lattice structure can essentially be made without loss of generality, since every possible lattice-order on D is term-definable in a primal algebra $\mathbf{D}$.

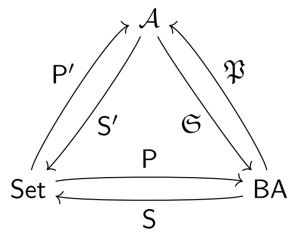

Figure 3 Functors between Set, BA and $\mathcal{A}$.

To set the scene, we now describe various functors relating our base categories Set, BA and $\mathcal{A}$. The entire constellation is summarized in Figure 3.

Due to Theorem 8, we know that $\mathcal{A}$ is categorically equivalent to BA. Since $\mathbf{D}$ is based on a bounded lattice, we have an explicit algebraic description of two functors $\mathfrak{S}\colon \mathcal{A} \to \mathsf{BA}$ and $\mathfrak{P}\colon \mathsf{BA} \to \mathcal{A}$ establishing such an equivalence [21].

The *Boolean skeleton functor* $\mathfrak{S}\colon \mathcal{A} \to \mathsf{BA}$ sends an algebra $\mathbf{A} \in \mathcal{A}$ to the Boolean algebra

$$\mathfrak{S}(\mathbf{A}) = (\mathfrak{S}(A), \wedge, \vee, T_0, 0, 1)$$

on the carrier set

$$\mathfrak{S}(A) = \{a \in A \mid T_1(a) = a\}.$$

Here, $\wedge$ and $\vee$ are the lattice operations of $\mathbf{A}$, and T_0 and T_1 are terms defining the unary operations from Example 7 (such terms exist since $\mathbf{D}$ is primal), interpreted in $\mathbf{A}$. It is shown in [26, Lemma 3.11] that $\mathfrak{S}(\mathbf{A})$ forms a Boolean algebra. To a homomorphism $g\colon \mathbf{A} \to \mathbf{A}'$ the functor $\mathfrak{S}$ assigns its restriction $\mathfrak{S}g = g|_{\mathfrak{S}(\mathbf{A})}$.

The *Boolean power functor* $\mathfrak{P}\colon \mathsf{BA} \to \mathcal{A}$ sends a Boolean algebra $\mathbf{B}$ to the Boolean power $\mathbf{D}[\mathbf{B}]$ defined as follows [11, 6]. The carrier set of $\mathbf{D}[\mathbf{B}]$ is the set of functions $\xi\colon D \to B$ which satisfy $\xi(d_1) \wedge \xi(d_2) = 0$ for all $d_1 \neq d_2$ and $\bigvee\{\xi(d) \mid d \in D\} = 1$ (for the definition of the algebra operations we refer the reader to [6]). To a Boolean homomorphism $h\colon \mathbf{B} \to \mathbf{B}'$ the functor $\mathfrak{P}$ assigns the homomorphism defined by composition $\mathfrak{P}h(\xi) = h \circ \xi$.

A proof of the fact that $\mathfrak{S}$ and $\mathfrak{P}$ form a categorical equivalence between BA and $\mathcal{A}$ may be found in [21, Corollary 4.12].

The contravariant functors $\mathsf{P}\colon \mathsf{Set} \to \mathsf{BA}$ and $\mathsf{S}\colon \mathsf{BA} \to \mathsf{Set}$ were already described in Subsection 2.1, and the contravariant functors $\mathsf{P}'\colon \mathsf{Set} \to \mathcal{A}$ and $\mathsf{S}'\colon \mathcal{A} \to \mathsf{BA}$ are defined similarly.

That is, the functor P' assigns the algebra $\mathsf{P}'(X) = \mathbf{D}^X$ to a set X and sends a map $f\colon X \to X'$ to the homomorphism $\mathsf{P}'f\colon \mathbf{D}^{X'} \to \mathbf{D}^X$ defined by composition $\alpha \mapsto \alpha \circ f$.

The functor S' assigns the set of homomorphisms $\mathsf{S}'(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{D})$ to an algebra $\mathbf{A} \in \mathcal{A}$ and sends a homomorphism $h\colon \mathbf{A} \to \mathbf{A}'$ to the map $\mathsf{S}'h\colon \mathcal{A}(\mathbf{A}', \mathbf{D}) \to \mathcal{A}(\mathbf{A}, \mathbf{D})$ defined by composition $u \mapsto u \circ h$. Like in the case where $\mathbf{D} = \mathbf{2}$, the functors P' and S' establish a dual adjunction between Set and $\mathcal{A}$. The corresponding natural transformations $\eta'\colon 1_{\mathcal{A}} \Rightarrow \mathsf{P}'\mathsf{S}'$ and $\varepsilon'\colon 1_{\mathsf{Set}} \Rightarrow \mathsf{S}'\mathsf{P}'$ are again given by evaluations (see Subsection 2.1).

We collect some useful properties of the functors appearing in Figure 3 and the natural transformations corresponding to the two dual adjunctions in the following.

▶ **Proposition 10.** *The functors* $\mathsf{P}, \mathsf{S}, \mathsf{P}', \mathsf{S}', \mathfrak{P}, \mathfrak{S}$ *and the natural transformations* $\varepsilon, \eta, \varepsilon', \eta'$ *satisfy the following properties.*

(a) $\Phi_{\mathbf{A}}\colon \mathcal{A}(\mathbf{A}, \mathbf{D}) \to \mathsf{BA}(\mathfrak{S}(\mathbf{A}), \mathbf{2})$ *given by restriction* $u \mapsto u|_{\mathfrak{S}(A)}$ *defines a natural isomorphism* $\mathsf{S}' \cong \mathsf{S}\mathfrak{S}$*. There also exists a natural isomorphism* $\mathsf{S} \cong \mathsf{S}'\mathfrak{P}$*.*

(b) $\Psi_X \colon \mathbf{2}^X \to \mathfrak{S}(\mathbf{D}^X)$, *which identifies* $\mathbf{2}^X$ *with a subset of* $\mathbf{D}^X$ *in the obvious way defines a natural isomorphism* $\mathsf{P} \cong \mathfrak{S}\mathsf{P}'$*. There also exists a natural isomorphism* $\mathsf{P}' \cong \mathfrak{P}\mathsf{P}$*.*
(c) $\varepsilon = \mathsf{S}\Psi \circ \Phi\mathsf{P}' \circ \varepsilon'$ *and* $\mathfrak{S}\eta' = \Psi\mathsf{S}' \circ \mathsf{P}\Phi \circ \eta\mathfrak{S}$*.*

Proof. In both (a) and (b), the second statement is an immediate consequence of the first one because $\mathfrak{P}$ and $\mathfrak{S}$ form a categorical equivalence. A proof of the first part of (a) can be found in [21, Proposition 4.3].

For the first part of (b), note that Ψ_X is well-defined since $\beta \in \mathbf{2}^X$ satisfies $T_1(\beta(x)) = \beta(x)$ in every component $x \in X$. Since the Boolean operations are defined component-wise, it is a homomorphism, and it is clearly injective. It is also surjective, since whenever an element $\alpha \in \mathbf{D}^X$ has a component with $\alpha(x) \notin \{0, 1\}$, we have $T_1(\alpha(x)) \neq \alpha(x)$. Naturality is straightforward by definition.

For (c), we need to show that the following diagrams commute for all $X \in \mathsf{Set}$ and $\mathbf{A} \in \mathcal{A}$.

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon_X} & \mathsf{BA}(\mathbf{2}^X, \mathbf{2}) \\ \big\downarrow{\scriptstyle \varepsilon'_X} & & \big\uparrow{\scriptstyle \mathsf{S}\Psi_X} \\ \mathcal{A}(\mathbf{D}^X, \mathbf{D}) & \xrightarrow[\Phi_{\mathbf{D}^X}]{} & \mathsf{BA}(\mathfrak{S}(\mathbf{D}^X), \mathbf{2}) \end{array} \qquad \begin{array}{ccc} \mathfrak{S}(\mathbf{A}) & \xrightarrow{\mathfrak{S}\eta'_{\mathbf{A}}} & \mathfrak{S}(\mathbf{D}^{\mathcal{A}(\mathbf{A},\mathbf{D})}) \\ \big\downarrow{\scriptstyle \eta_{\mathfrak{S}(\mathbf{A})}} & & \big\uparrow{\scriptstyle \Psi_{\mathcal{A}(\mathbf{A},\mathbf{D})}} \\ \mathbf{2}^{\mathsf{BA}(\mathfrak{S}(\mathbf{A}),\mathbf{2})} & \xrightarrow[\mathsf{P}\Phi_{\mathbf{A}}]{} & \mathbf{2}^{\mathcal{A}(\mathbf{A},\mathbf{D})} \end{array}$$

For the diagram on the left, given $x \in X$, we compute

$$\mathsf{S}\Psi_X \circ \Phi_{\mathbf{D}^X} \circ \varepsilon'_X(x) = \mathsf{S}\Psi_X \circ \Phi_{\mathbf{D}^X}(\mathsf{ev}_x) = \mathsf{S}\Psi_X(\mathsf{ev}_x|_{\mathfrak{S}(\mathbf{D}^X)}) = \mathsf{ev}_x|_{\mathfrak{S}(\mathbf{D}^X)} \circ \Psi_X,$$

which, on $\beta \in \mathbf{2}^X$, is given by $\mathsf{ev}_x|_{\mathfrak{S}(\mathbf{D}^X)} \circ \Psi_X(\beta) = \mathsf{ev}_x|_{\mathfrak{S}(\mathbf{D}^X)}(\beta) = \beta(x)$. Thus, it coincides with $\varepsilon_X(x)(\beta) = \mathsf{ev}_x(\beta) = \beta(x)$.

For the diagram on the right, given $b \in \mathfrak{S}(\mathbf{A})$, similarly we compute

$$\Psi_{\mathcal{A}(\mathbf{A},\mathbf{D})} \circ \mathsf{P}\Phi_{\mathbf{A}} \circ \eta_{\mathfrak{S}(\mathbf{A})}(b) = \Psi_{\mathcal{A}(\mathbf{A},\mathbf{D})} \circ \mathsf{P}\Phi_{\mathbf{A}}(\mathsf{ev}_b) = \Psi_{\mathcal{A}(\mathbf{A},\mathbf{D})}(\mathsf{ev}_b \circ \Phi_{\mathbf{A}}),$$

which is given on $u \in \mathcal{A}(\mathbf{A}, \mathbf{D})$ by $\Psi_{\mathcal{A}(\mathbf{A},\mathbf{D})}(\mathsf{ev}_b \circ \Phi_{\mathbf{A}})(u) = \Psi_{\mathcal{A}(\mathbf{A},\mathbf{D})}(\mathsf{ev}_b(u|_{\mathfrak{S}(\mathbf{A})})) = u(b)$. This coincides with $\mathfrak{S}\eta'_{\mathbf{A}}(b)(u) = \eta'_{\mathbf{A}}|_{\mathfrak{S}(\mathbf{A})}(b)(u) = u(b)$, finishing the proof. ◀

Suppose we are given an endofunctor T on Set and an abstract coalgebraic logic (L, δ) for T which is classical in the sense that L is an endofunctor on BA. We now lift this to an abstract coalgebraic logic (L', δ') where L' is an endofunctor on $\mathcal{A}$. The entire situation is summarized in Figure 4.

▶ **Definition 11** (Lifting of a coalgebraic logic). *Let* (L, δ) *be an abstract coalgebraic logic for* $\mathsf{T} \colon \mathsf{Set} \to \mathsf{Set}$ *with* $\mathsf{L} \colon \mathsf{BA} \to \mathsf{BA}$*. Then*

$$\mathsf{L}' = \mathfrak{P}\mathsf{L}\mathfrak{S} \text{ and } \delta' = \mathfrak{P}\delta$$

defines an abstract coalgebraic logic (L', δ') *for* T*, which we call the* lifting *of* (L, δ) *to* $\mathcal{A}$*.*

This is well-defined since, by Proposition 10(b), the natural transformation $\mathfrak{P}\delta \colon \mathfrak{P}\mathsf{L}\mathsf{P} \to \mathfrak{P}\mathsf{P}\mathsf{T}$ can be identified with one from $\mathfrak{P}\mathsf{L}\mathsf{P} \cong \mathfrak{P}\mathsf{L}\mathfrak{S}\mathsf{P}' = \mathsf{L}'\mathsf{P}'$ to $\mathfrak{P}\mathsf{P}\mathsf{T} \cong \mathsf{P}'\mathsf{T}$.

▶ **Theorem 12.** *Let* (L', δ') *be the lifting of a coalgebraic logic* (L, δ) *to* $\mathcal{A}$*.*
(a) *If* L *has a presentation by operations and equations, then* L' *has one as well.*
(b) *If* (L, δ) *is one-step complete, then so is* (L', δ')*.*
(c) *If* (L, δ) *is expressive, then so is* (L', δ')*.*

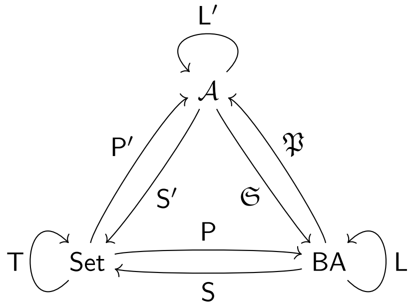

Figure 4 Classical coalgebraic logic and its lifting.

Proof.

(a) Recall that an endofunctor on a variety has a presentation if and only if it preserves sifted colimits [22, Theorem 4.7]. Of course, if L preserves sifted colimits then, by definition, so does L'.

(b) If δ is a component-wise monomorphism, then so is δ', since $\mathfrak{P}$ preserves monomorphisms.

(c) We show that $(\delta')^\dagger = \delta^\dagger\mathfrak{S}$ holds up to natural isomorphism, from which the statement follows since it implies that if $\delta^\dagger$ is a component-wise monomorphism, then so is $(\delta')^\dagger$. So we want to show that the following diagram commutes.

$$\begin{array}{ccccccc}
\mathsf{TS}' & \xrightarrow{\varepsilon'\mathsf{TS}'} & \mathsf{S}'\mathsf{P}'\mathsf{TS}' & \xrightarrow{\mathsf{S}'\delta'\mathsf{S}'} & \mathsf{S}'\mathsf{L}'\mathsf{P}'\mathsf{S}' & \xrightarrow{\mathsf{S}'\mathsf{L}'\eta'} & \mathsf{S}'\mathsf{L}' \\
\downarrow & D_1 & \downarrow & D_2 & \downarrow & D_3 & \downarrow \\
\mathsf{T}\mathsf{S}\mathfrak{S} & \xrightarrow[\varepsilon\mathsf{TS}\mathfrak{S}]{} & \mathsf{SPTS}\mathfrak{S} & \xrightarrow[\mathsf{S}\delta\mathsf{S}\mathfrak{S}]{} & \mathsf{SLPS}\mathfrak{S} & \xrightarrow[\mathsf{SL}\eta\mathfrak{S}]{} & \mathsf{SL}\mathfrak{S}
\end{array}$$

Here, by definition, the top edge of the diagram is the adjoint-transpose $(\delta')^\dagger$ and the bottom edge is $\delta^\dagger\mathfrak{S}$. All vertical arrows are natural isomorphisms obtained via Φ and Ψ from Proposition 10. The diagram D_2 commutes by definition of δ', using that $\mathsf{S}'\delta' = \mathsf{S}'\mathfrak{P}\delta$ and $\mathsf{S}'\mathfrak{P} \cong \mathsf{S}$ by Proposition 10(a). To finish the proof we show that D_1 and D_3 commute as well.

To see that D_1 commutes, we apply the first equation of Proposition 10(c) to compute

$$\mathsf{SPT}\Phi \circ \mathsf{S}\Psi\mathsf{TS}' \circ \Phi\mathsf{P}'\mathsf{TS}' \circ \varepsilon'\mathsf{TS}' = \mathsf{SPT}\Phi \circ (\mathsf{S}\Psi \circ \Phi\mathsf{P}' \circ \varepsilon')\mathsf{TS}' = \mathsf{SPT}\Phi \circ \varepsilon\mathsf{TS}',$$

which coincides with $\varepsilon\mathsf{TS}\mathfrak{S} \circ \mathsf{T}\Phi$.

Similarly, to see that D_3 commutes we apply the second equation of Proposition 10(c) to compute

$$\mathsf{SL}\eta\mathfrak{S} \circ \mathsf{SLP}\Phi \circ \mathsf{SL}\Psi\mathsf{S}' \circ \Phi\mathsf{L}'\mathsf{P}'\mathsf{S}' = \mathsf{SL}(\Psi\mathsf{S}' \circ \mathsf{P}\Phi \circ \eta\mathfrak{S}) \circ \Phi\mathsf{L}'\mathsf{P}'\mathsf{S}' = \mathsf{SL}\mathfrak{S}\eta' \circ \Phi\mathsf{L}'\mathsf{P}'\mathsf{S}',$$

which coincides with $\Phi\mathsf{L}' \circ \mathsf{S}'\mathsf{L}'\eta'$. ◀

If (L, δ) is a concrete coalgebraic logic for T with $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$, then the initial L-algebra exists and corresponds to the *Lindenbaum-Tarski* algebra of the variety $\mathsf{Alg}(\mathsf{L})$. If (L, δ) is a coalgebraic logic for T and $\gamma\colon X \to \mathsf{T}(X)$ is a coalgebra, then the unique map from the Lindenbaum-Tarski algebra into the L-algebra $\mathsf{P}\gamma \circ \delta_X$ determines semantics of formulas. In this context, it is known that one-step completeness of (L, δ) implies completeness for the resulting logic [20, Theorem 6.15]. Since the proof only uses properties of BA which are invariant under categorical equivalence, it can easily be adapted to coalgebraic logics over $\mathcal{A}$. Thus, parts (a) and (b) of Theorem 12 imply the following.

▶ **Corollary 13.** *Let* (L', δ') *be the lifting of the coalgebraic logic* (L, δ)*, where* L *has a presentation. If* (L, δ) *is complete, then so is* (L', δ')*.*

So we showed that the lifting (L', δ') of a coalgebraic logic (L, δ) inherits desirable properties from the original logic, which is satisfactory from a theoretical point of view. From a more "practical" point of view, one important question still needs to be answered, namely that of a concrete presentation of L' and its relationship to a presentation of L. Indeed, Theorem 12(a) only states that the *existence* of a presentation is preserved, without any explicit way of obtaining it from the original one. In the following section, we give some partial solutions to this problem.

4 Lifting Presentations of Functors

We aim to relate presentations of $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ to presentations of the corresponding lifted functor $\mathsf{L}' = \mathfrak{P}\mathsf{L}\mathfrak{S}\colon \mathcal{A} \to \mathcal{A}$. Not surprisingly, to do this we need to delve deeper into the algebraic structure of $\mathbf{D}$.

Since $\mathbf{D}$ is based on a bounded lattice and primal (Assumption 9), for every $d \in D$, the unary function $\tau_d\colon D \to D$ defined by

$$\tau_d(x) = \begin{cases} 1 & \text{if } d \leq x \\ 0 & \text{if } d \not\leq x \end{cases}$$

is well-defined and term-definable in $\mathbf{D}$. Note that τ_0, being of constant value 1, carries no relevant information. Thus, we only consider τ_d for $d \in D^+ := D\backslash\{0\}$ in the following. Also note that τ_1 coincides with T_1 from Example 7. Given an element $e \in D$, the map $\tau_{(\cdot)}(e)\colon D^+ \to 2$ defined by $d \mapsto \tau_d(e)$ fully determines the element e via

$$e = \bigvee\{d \mid \tau_d(e) = 1\}.$$

In the following, we characterize all maps of this form by their lattice-theoretic properties.

▶ **Lemma 14.** *Let* $\mathcal{T}\colon D^+ \to 2$ *be a map which, for all* $d_1, d_2 \in D^+$*, satisfies*

$$\mathcal{T}(d_1 \vee d_2) = \mathcal{T}(d_1) \wedge \mathcal{T}(d_2). \tag{1}$$

Then $\mathcal{T} = \tau_{(\cdot)}(e)$ *for* $e = \bigvee\{d \mid \mathcal{T}(d) = 1\}$*.*

Proof. The case $e = 0$ can only occur if $\mathcal{T}(d) = 0$ for all $d \in D^+$, which implies $\mathcal{T}(d) = 0 = \tau_d(0)$ for all $d \in D$. Now assume that $e \neq 0$. First we show that $\mathcal{T}(e) = 1$. Since e is a finite join we apply (1) to find

$$\mathcal{T}(e) = \mathcal{T}(\bigvee\{d \mid \mathcal{T}(d) = 1\}) = \bigwedge\{\mathcal{T}(d) \mid \mathcal{T}(d) = 1\} = 1.$$

Furthermore, since (1) implies that $\mathcal{T}$ is order-reversing, we have $\mathcal{T}(c) = 1$ for all $c \leq e$ as well. Now let $c \not\leq e$. Then we have $\mathcal{T}(c) = 0$, since otherwise $\mathcal{T}(c) = 1$ leads to the contradiction

$$e = \bigvee\{d \mid \mathcal{T}(d) = 1\} \geq e \vee c > e.$$

Altogether, we have shown that $\mathcal{T}(d) = 1$ if and only if $e \geq d$, so $\mathcal{T}(d) = \tau_d(e)$. ◀

Suppose that $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ has a presentation by one unary operation $\Box$ and equations which are satisfied by the terms τ_d, in the sense that all the equations obtained by replacing $\Box$ by any τ_d hold in $\mathbf{D}$. Prominent examples of such equations are $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$ from Example 2.

Under these circumstances, we can find a presentation of the corresponding lifted functor $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$ as follows. The idea is to "approach" a presentation of L' by introducing a modal operator for every $d \in D^+$, intended to correspond to $\tau_d\Box$ for the "lifted" $\Box'$. However, only if these modal operators are "consistent" in the sense of Lemma 14, we can replace them by a single operator again.

For simplicity, we only consider the case of one unary operation in the following, but there is a straightforward generalization of Theorem 15 to presentations of L by one operation which is not necessarily unary (the operations $\Box_d$ and $\Box'$ will simply have the same arity).

▶ **Theorem 15.** *Let $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ have a presentation by one unary operation $\Box$ and equations which are satisfied (in $\mathbf{D}$) by all τ_d, $d \in D^+$. Let $\mathsf{L}' = \mathfrak{P}\mathsf{L}\mathfrak{S}$.*

(a) *The functor L' can be presented by unary operations $\Box_d$ for every $d \in D^+$ and the following equations.*
- *The equations for $\Box$, where $\Box$ is replaced by $\Box_1$.*
- *$\Box_1 \tau_d(x) = \Box_d x$ for all $d \in D^+$.*
- *$T_1(\Box_d x) = \Box_d x$ for all $d \in D^+$.*

(b) *If, in the variety $\mathsf{Alg}(\mathsf{L}')$ axiomatized by the presentation of (a), the equation*

$$\Box_{d_1 \vee d_2} x = \Box_{d_1} x \wedge \Box_{d_2} x \tag{2}$$

holds, then L' can also be presented by one unary operation $\Box'$ and the following equations.
- *The equations for $\Box$, where $\Box$ is replaced by $\Box'$.*
- *$\Box' \tau_d(x) = \tau_d(\Box' x)$ for all $d \in D^+$.*

Proof.

(a) Let $\mathsf{L}^+\colon \mathcal{A} \to \mathcal{A}$ be the functor presented by the operations $\Box_d$ and equations as in the statement. We want to show that L' is naturally isomorphic to L^+. Since both these functors are finitary (because they preserve sifted colimits, in particular they preserve filtered colimits), it suffices to show that their restrictions to finite algebras are naturally isomorphic. The restrictions of P and S to the categories $\mathsf{Set}^{\mathit{fin}}$ of finite sets and $\mathsf{BA}^{\mathit{fin}}$ of finite Boolean algebras form a dual equivalence. Similarly, the restrictions of P' and S' form a dual equivalence between $\mathsf{Set}^{\mathit{fin}}$ and $\mathcal{A}^{\mathit{fin}}$. Therefore, it suffices to show that

$$\mathsf{S}'\mathsf{L}^+\mathsf{P}' \cong \mathsf{SLP},$$

since, due to Proposition 10, for the right-hand side we have further natural isomorphisms $\mathsf{SLP} \cong \mathsf{S}'\mathfrak{P}\mathsf{L}\mathfrak{S}\mathsf{P}' = \mathsf{S}'\mathsf{L}'\mathsf{P}'$. Spelling this out, we want to find a bijection between the sets of homomorphisms $\mathcal{A}(\mathsf{L}^+(\mathbf{D}^X), \mathbf{D})$ and $\mathsf{BA}(\mathsf{L}(\mathbf{2}^X), \mathbf{2})$ which is natural in $X \in \mathsf{Set}$. By definition of L^+, the set $\mathcal{A}(\mathsf{L}^+(\mathbf{D}^X), \mathbf{D})$ can be naturally identified with the collection of all maps (whose domain is simply a set of formal expressions)

$$f\colon \{\Box_d a \mid d \in D^+, a \in D^X\} \to D, \text{ where } f \text{ respects the equations of } \mathsf{L}^+.$$

Similarly, the set $\mathsf{BA}(\mathsf{L}(\mathbf{2}^X), \mathbf{2})$ can be naturally identified with the collection of all maps

$$g\colon \{\Box b \mid b \in 2^X\} \to 2, \text{ where } g \text{ respects the equations of } \mathsf{L}.$$

Given f as above, we assign to it g_f defined by

$$g_f(\Box b) = f(\Box_1 b).$$

This is well-defined, since $T_1(f(\Box_1 b)) = f(\Box_1 b)$ implies $f(\Box_1 b) \in 2$, and g_f respects the equations of L, because f does for $\Box$ replaced by $\Box_1$.
Conversely, given g as above, we assign to it f_g defined by

$$f_g(\Box_d a) = g(\Box \tau_d(a)).$$

Since the equations of L are satisfied by τ_d and respected by g, they are also respected by f_g. The remaining equations of L^+ are respected by f_g, since, for all $d \in D^+$ we can directly verify

$$f_g(\Box_1 \tau_d(a)) = g(\Box T_1(\tau_d(a))) = g(\Box \tau_d(a)) = f_g(\Box_d a),$$

where we used $T_1(\tau_d(a)) = \tau_d(a)$ since $\tau_d(a) \in 2^X$ and

$$T_1(f_g(\Box_d a)) = T_1(g(\Box \tau_d a)) = g(\Box \tau_d a) = f_g(\Box_d a),$$

where we used $T_1(g(\Box \tau_d a)) = g(\Box \tau_d a)$ since $g(\Box \tau_d a) \in 2$.
Now we show that these two assignments are mutually inverse. For this we compute

$$f_{g_f}(\Box_d a) = g_f(\Box \tau_d a) = f(\Box_1 \tau_d a) = f(\Box_d a),$$

where in the last equation we used that f respects the corresponding equation of L^+ and

$$g_{f_g}(\Box b) = f_g(\Box_1 b) = g(\Box T_1(b)) = g(\Box b),$$

where in the last equation we used $b \in 2^X$ again.
For naturality, we need to show that, given a map $m \colon X_1 \to X_2$, the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{A}(\mathsf{L}^+(\mathbf{D}^{X_1}), \mathbf{D}) & \xrightarrow{g_{(\cdot)}} & \mathsf{BA}(\mathsf{L}(\mathbf{2}^{X_1}), \mathbf{2}) \\
\Big\downarrow {\scriptstyle \mathsf{S'L^+P'}m} & & \Big\downarrow {\scriptstyle \mathsf{SLP}m} \\
\mathcal{A}(\mathsf{L}^+(\mathbf{D}^{X_2}), \mathbf{D}) & \xrightarrow[g_{(\cdot)}]{} & \mathsf{BA}(\mathsf{L}(\mathbf{2}^{X_2}), \mathbf{2})
\end{array}$$

Let $f \colon \{\Box_d a \mid d \in D^+, a \in D^{X_1}\} \to D$ be given as before. On the one hand, for $\alpha \in D^{X_2}$ and $\beta \in 2^{X_2}$ we have $\mathsf{S'L^+P'}m(f)(\Box_d \alpha) = f(\Box_d(\alpha \circ m))$ and therefore $g_{\mathsf{S'L^+P'}m(f)}(\Box \beta) = f(\Box_1(\beta \circ m))$. On the other hand, $\mathsf{SLP}m(g_f)(\Box \beta) = g_f(\Box(\beta \circ m)) = f(\Box_1(\beta \circ m))$. Thus, the diagram commutes.

(b) Let $\mathsf{L}^\star \colon \mathcal{A} \to \mathcal{A}$ be defined by one unary operation $\Box'$ and equations as in the statement and let L^+ be defined as in the proof of (a). For the same reason as before, it suffices to show

$$\mathsf{S'L^\star P'} \cong \mathsf{S'L^+P'}.$$

Again, $\mathsf{S'L^+P'}(X) = \mathcal{A}(\mathsf{L}^+(\mathbf{D}^X), \mathbf{D})$ is essentially the collection of maps

$$f \colon \{\Box_d a \mid d \in D^+, a \in D^X\} \to D, \text{ where } f \text{ respects the equations of } \mathsf{L}^+,$$

and $\mathsf{S}'\mathsf{L}^\star\mathsf{P}'(X)$ is essentially the collection of maps

$$h\colon \{\Box a \mid a \in D^X\} \to D, \text{ where } h \text{ respects the equations of } \mathsf{L}^\star.$$

Given h as above, we assign to it

$$f_h(\Box_d a) = h(\Box' \tau_d a).$$

Checking that this is well-defined is routine by now, the only non-trivial part being

$$T_1(f_h(\Box_d a) = T_1(h(\Box' \tau_d(a))) = h(\Box' T_1(\tau_d(a))) = f_h(\Box_d a),$$

which uses the fact that h respects the corresponding equation $\Box' T_1(x) = T_1(\Box' x)$ of $\mathsf{L}^\star$. Conversely, given f as above, we assign to it

$$h_f(\Box' a) = \bigvee \{c \mid f(\Box_c a) = 1\}.$$

First, given $d \in D^+$, using that $\tau_c \circ \tau_d = \tau_d$ holds for all $c \in D^+$, we note

$$h_f(\Box' \tau_d(a)) = \bigvee \{c \mid f(\Box_c \tau_d(a)) = 1\} = \bigvee \{c \mid f(\Box_1 \tau_c(\tau_d(a))) = 1\} = \bigvee \{c \mid f(\Box_d a) = 1\}.$$

Since, on the right-hand side, the formula $f(\Box_d a) = 1$ is independent of c, this join is either equal to $\bigvee \varnothing = 0$ if $f(\Box_d a) = 0$ or $\bigvee D^+ = 1$ if $f(\Box_d a) = 1$. On the other hand, by assumption we can apply Lemma 14, which yields

$$\tau_d(h_f(\Box' a)) = \tau_d(\bigvee \{c \mid f(\Box_c a) = 1\}) = f(\Box_d a)$$

as well. The two assignments thus defined are mutually inverse since

$$f_{h_f}(\Box_d a) = h_f(\Box' \tau_d(a)) = \bigvee \{c \mid f(\Box_c \tau_d(a)) = 1\} = f(\Box_d a)$$

holds again by Lemma 14 and

$$h_{f_h}(\Box' a) = \bigvee \{c \mid h(\Box' \tau_c(a)) = 1\} = \bigvee \{c \mid \tau_c(h(\Box' a)) = 1\} = h(\Box' a).$$

Analogous to (a), it is straightforward to show that the isomorphism thus defined is natural. ◀

In particular, part (b) of this theorem applies if the "original" operation $\Box$ preserves meets, as shown in the following.

▶ **Corollary 16.** *Let* L *be as in Theorem 15, such that* $\Box(x \wedge y) = \Box x \wedge \Box y$ *holds in the variety* $\mathsf{Alg}(\mathsf{L})$*. Then* $\mathsf{L}' = \mathfrak{P}\mathsf{L}\mathfrak{S}$ *can be presented by one unary operation* $\Box'$ *and the following equations.*

- *The equations for* $\Box$*, where* $\Box$ *is replaced by* $\Box'$*.*
- $\Box' \tau_d(x) = \tau_d(\Box' x)$ *for all* $d \in D^+$*.*

Proof. We verify equation (2) from Theorem 15(b) by

$$\Box_{d_1 \vee d_2} x = \Box_1 \tau_{d_1 \vee d_2}(x) = \Box_1(\tau_{d_1}(x) \wedge \tau_{d_2}(x)) = \Box_1 \tau_{d_1}(x) \wedge \Box_1 \tau_{d_2}(x) = \Box_{d_1} x \wedge \Box_{d_2} x,$$

and the statement immediately follows from there. ◀

If (L, δ) is a concrete coalgebraic logic for T, where $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ is endowed with a presentation such that the conditions of Theorem 15 are satisfied, it is now easy to describe the lifting (L', δ') as a concrete coalgebraic logic as well. The only missing piece is an explicit description of the natural transformation $\delta'\colon \mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathsf{T}$. Similar to the proof of Theorem 15, for a set X, the component $\delta'_X\colon \mathsf{L}'(\mathbf{D}^X) \to \mathbf{D}^{\mathsf{T}(X)}$ is defined on $Y \in \mathsf{T}(X)$ by

$$\delta'_X(\Box_d a)(Y) = \delta_X(\Box\tau_d(a))(Y).$$

Given that the additional condition of part (b) of Theorem 15 is also satisfied, it can be described as

$$\delta'_X(\Box a)(Y) = \bigvee\{d \mid \delta(\Box\tau_d(a)) = 1\}.$$

In the following, we show that the machinery developed works well with respect to the way classical modal logic is described as a concrete coalgebraic logic in Example 2.

▶ **Example 17** (Lifting classical modal logic)**.** Let (L, δ) be the coalgebraic logic for $\mathcal{P}$ which corresponds to classical modal logic as in Example 2, in particular $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ is presented by a unary operation $\Box$ and the equations $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

Let (L', δ') be the lifting of (L, δ) to $\mathcal{A}$. By Corollary 16, we know that L' has a presentation by a unary operation $\Box'$ and equations

$$\Box'(x \wedge y) = \Box' x \wedge \Box' y, \;\; \Box' 1 = 1 \text{ and } \tau_d(\Box' x) = \Box'\tau_d(x) \text{ for all } d \in D^+.$$

The natural transformation δ' has components $\delta'_X\colon \mathsf{L}'(\mathbf{D}^X) \to \mathbf{D}^{\mathcal{P}(X)}$, defined by

$$\delta'_X(\Box' a)(Y) = \bigvee\{d \mid \delta_X(\Box\tau_d(a))(Y) = 1\}.$$

Now, since $\delta_X(\Box\tau_d(a))(Y) = 1 \Leftrightarrow \forall y \in Y : \tau_d(a(y)) = 1 \Leftrightarrow \forall y \in Y : a(y) \geq d$ we can rewrite this as

$$\bigvee\{d \mid \delta_X(\Box\tau_d(a))(Y) = 1\} = \bigvee\{d \mid \bigwedge_{y\in Y} a(y) \geq d\} = \bigvee\{d \mid \tau_d(\bigwedge_{y\in Y} a(y)) = 1\} = \bigwedge_{y\in Y} a(y).$$

Thus, this corresponds to the usual semantics of a many-valued box over Kripke frames defined via meet (see, e.g., [5, 12]). Since we know that (L, δ) is one-step complete (and thus complete), by Theorem 12(b) (and Corollary 13) the logic (L', δ') is one-step complete (and thus complete) as well (similar results are shown in [25, 12]). Furthermore, from Theorem 12(c) we conclude that, replacing $\mathcal{P}$ by the finite-powerset functor $\mathcal{P}_{fin}$, the logic (L', δ') is expressive for image-finite frames (this can also be proved directly along the lines of [24]).

The applicability of Theorem 15 does depend on the specific choice of a presentation of L. For instance, the functor L in the example above can also be presented by one unary operator $\Diamond$ with equations $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ and $\Diamond 0 = 0$. If $\mathbf{D}$ is not linear, it is easy to check that $\tau_d(x \vee y) = \tau_d(x) \vee \tau_d(y)$ does not hold in general (simply choose incomparable elements x and y and set $d = x \vee y$). Therefore, this presentation can not be lifted by this method. However, the following order-dual version of Theorem 15 can be applied in this case.

For every $d \in D^- := D\backslash\{1\}$, the unary operation $\kappa_d\colon D \to D$ defined by

$$\kappa_d(x) = \begin{cases} 1 & \text{if } d \geq x \\ 0 & \text{if } d \not\geq x \end{cases}$$

is term-definable in $\mathbf{D}$. Not surprisingly, the following can be shown completely analogous to what we did before.

▶ **Theorem 18.** *Let* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *have a presentation by one unary operation* $\Diamond$ *and equations which are satisfied by all* κ_d, $d \in D^-$. *Let* $\mathsf{L}' = \mathfrak{P}\mathsf{L}\mathfrak{S}$.

(a) *The functor* L' *can be presented by unary operations* $\Diamond_d$ *for every* $d \in D^-$ *and the following equations.*
- *The equations for* $\Diamond$*, where* $\Diamond$ *is replaced by* $\Diamond_0$.
- $\Diamond_0 T_0(\kappa_d(x)) = \Diamond_d x$ *for all* $d \in D^-$.
- $T_1(\Diamond_d x) = \Diamond_d x$ *for all* $d \in D^-$.

(b) *If, in the variety* $\mathsf{Alg}(\mathsf{L}')$ *axiomatized by the presentation of (a), the equation*

$$\Diamond_{d_1 \wedge d_2} x = \Diamond_{d_1} x \vee \Diamond_{d_2} x \tag{3}$$

holds, then L' *can also be presented by one unary operation* $\Diamond'$ *and the following equations.*
- *The equations for* $\Diamond$*, where* $\Diamond$ *is replaced by* $\Diamond'$.
- $\Diamond' \kappa_d(x) = \kappa_d(\Diamond' x)$ *for all* $d \in D^-$.

Analogous to Corollary 16, equation (3) of Theorem 18 can be deduced if $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ holds in $\mathsf{Alg}(\mathsf{L})$. Thus, another way to concretely present the lifting (L', δ') of classical modal logic (Example 17) is by one unary operation $\Diamond'$ satisfying

$$\Diamond'(x \vee y) = \Diamond' x \vee \Diamond' y, \quad \Diamond' 1 = 1 \text{ and } \kappa_d(\Diamond' x) = \Diamond' \kappa_d(x) \text{ for all } d \in D^-.$$

The semantics of $\Diamond'$ are (as usual for many-valued diamonds over Kripke frames) defined by joins, that is, for $a \in \mathbf{D}^X$ and $Y \in \mathcal{P}(X)$ we have $\delta'_X(\Diamond' a)(Y) = \bigvee_{y \in Y} a(y)$.

We finish this section with an example to illustrate a situation where part (a) of Theorem 15 can be applied, but part (b) can not.

▶ **Example 19** (Neighborhood frames)**.** To deal with non-normal modal logics, one typically considers neighborhood semantics (for an introduction see, e.g., [28]). *Neighborhood frames* are coalgebras for the neighborhood functor $\mathcal{N}\colon \mathsf{Set} \to \mathsf{Set}$, given by $\mathcal{N} = \wp \circ \wp$, where $\wp$ is the contravariant powerset functor.

Let (L, δ) be the following concrete coalgebraic logic over $\mathcal{N}$. The functor $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ has a presentation by one unary operation $\Box$ and no (i.e., the empty set of) equations. The natural transformation δ has components $\delta_X\colon \mathsf{L}(\mathbf{2}^X) \to \mathbf{2}^{\mathcal{N}(X)}$ defined by

$$\delta_X(\Box b)(N) = N(b),$$

in other words, $\delta_X(\Box b)(N) = 1$ if and only if the subset $b \in 2^X$ is an element of the collection of neighborhoods N.

Since the presentation of L doesn't include any equations, it trivially satisfies the conditions of Theorem 15. Therefore, the lifting (L', δ') of the above logic to $\mathcal{A}$ can be described as follows. The functor $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$ has a presentation by unary operations $\Box_d$ for all $d \in D^+$ with equations

$$\Box_1 \tau_d(x) = \Box_d x \text{ and } T_1(\Box_d x) = \Box_d x \text{ for all } d \in D^+.$$

The semantics δ' can be described by

$$\delta'_X(\Box_d a)(N) = \delta_X(\Box \tau_d(a))(N) = N(\tau_d(a)),$$

which means that $\delta'_X(\Box_d a) = 1$ if and only if the subset $\{x \in X \mid a(x) \geq d\}$ is an element of the collection of neighborhoods N. Since (L, δ) is one-step complete, we again have that (L', δ') is complete.

Therefore, it can easily be shown by counter-example that $\Box_{d_1 \vee d_2} x = \Box_{d_1} x \wedge \Box_{d_2} x$ does not hold in $\mathsf{Alg}(\mathsf{L}')$, which means that the above presentation can not be simplified to the one using a single unary operation via Theorem 15(b). At this point, the question whether or not the presentation can be simplified differently remains open.

However, if we replace the functor $\mathcal{N}$ by the one which only allows collections of neighborhoods which are closed under finite intersections and supersets, we know that there is a corresponding concrete coalgebraic logic (L, δ) such that the presentation of L contains the equation $\Box(x \wedge y) = \Box x \wedge \Box y$. Thus, Corollary 16 applies in this case.

This concludes the main sections of this paper. In the last section we briefly summarize our results and discuss some potential directions for future research along similar lines.

5 Conclusion and Open Questions

We showed how to lift classical coalgebraic logics (L, δ) over BA to many-valued coalgebraic logics (L', δ') over $\mathcal{A}$, the variety generated by a primal algebra $\mathbf{D}$. On the level of abstract coalgebraic logics, it can be shown by purely category-theoretical means that the logic thus lifted inherits important properties like one-step completeness and expressivity from the original logic. On the level of concrete coalgebraic logics, we showed how one may lift a given presentation of L by operations and equations to a presentation of L', making use of algebraic properties and a lattice structure of $\mathbf{D}$. As of yet, there is no fully general method to do this. However, prominent examples like the modal logics for Kripke frames and neighborhood frames are covered by our results. In the following, we propose some open questions for future research.

As Example 19 illustrates, applying Theorem 15 does not always yield a presentation by a single unary operation. However, such a presentation could still exist in such situations.

▶ **Question 1.** *Suppose that* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *has a presentation by a single unary operation and equations. Does there always exist a presentation of* L' *by a single unary operation as well?*

If it is true, a follow-up question would be how these two presentations relate to each other in general. If it is false, a follow-up problem would be to classify the presentations of L for which it is true.

The following question arises if we start with a presentation of L with more than one, possibly infinitely-many, operations (for example, the multi-modal logic for the distribution functor described in [9]).

▶ **Question 2.** *Given that the functor* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *has a presentation by more than one operations operations and equations, can we still obtain a presentation of* L' *with methods similar to the ones developed in this paper?*

Further generalizations of results of this paper may be obtained by weakening Assumption 9 about $\mathbf{D}$ being primal. We summarize this in the following general question.

▶ **Question 3.** *Let* $\mathcal{V}$ *be some variety generated by some algebra. Is there a canonical way to lift abstract coalgebraic logics* (L, δ) *over* BA *to abstract coalgebraic logics* (L', δ') *over* $\mathcal{V}$ *and to relate presentations of* L *and* L'*?*

We plan to generalize the results of this paper to the case of $\mathbf{D}$ being *semi-primal* in future work (the first step towards that direction has been taken in [21], where we study the category-theoretical relationship between $\mathcal{V}$ and BA in this case).

Lastly, one could keep Assumption 9, but change the approach to coalgebraic logic (for an overview of the various approaches see [18]).

▶ **Question 4.** *Develop and study the theory of coalgebraic logic with a primal algebra* $\mathbf{D}$ *of truth-degrees using other approaches to coalgebraic logic.*

Many-valued nabla modalities and many-valued predicate liftings have, for example, been investigated in [2] and [1, 23]. As follow-up research, one could study the relationship between the various approaches to coalgebraic logic in the many-valued setting (similar to [19]).

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