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# Uniqueness of Equilibrium in Directed Search Models\*

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## Abstract

We study a decentralized trading model as in Peters (1984), where a finite number of heterogeneous capacity-constrained sellers compete for a finite number of homogeneous buyers, by posting prices. This “directed search” model is known to admit symmetric equilibria; yet, uniqueness has proved elusive. This study makes two contributions: a substantive contribution is to establish uniqueness of symmetric equilibrium; a methodological contribution is to develop a tool based on directional derivatives to characterize equilibrium.

Keywords: Nash equilibrium, matching, price distribution, search

JEL: C70, D390, D490, E390

## 1 Introduction

This study concerns equilibrium in finite markets where heterogeneous sellers compete for homogeneous buyers by posting prices. The central characteristic

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of the economy is that market participants face a trade-off between the price posted and the probability of trading; see [1, 6, 7, 8].

More precisely, the market is composed of a finite population of capacity-constrained sellers and mobility-constrained buyers. Sellers are endowed with (or can produce) one homogeneous good; consumption of this good—usually assumed indivisible—gives utility to buyers who can, in turn, transfer utility to sellers. The market is modeled as a sequential game of complete information where sellers take the lead by simultaneously and independently posting (and committing to) a price, i.e., a utility level for any buyer who wishes to trade. Buyers see all prices, and simultaneously and independently visit one seller, i.e., they can direct their search. Once meetings occur, sellers trade at the posted price satisfying capacity constraints through random rationing.

Models of this type have been adopted to tackle a variety of issues in labor and IO, such as wage and price dispersion, market efficiency, and competing mechanisms; see [3, 5, 9] for some recent examples. Many equilibria exist in this setting, some of which are symmetric and some of which are not; see the examples in [1]. The focus in this literature has been (strongly) symmetric equilibrium, existence of which is established in [4, 7]. A significant open question is whether symmetric equilibrium is unique. Establishing uniqueness has so far proved elusive because of the analytical intractability associated to working with finite numbers of buyers and heterogeneous sellers; see [8]. In fact, most studies focus on limit economies where sellers are homogeneous and the number of players gets large. Our study fills this important theoretical gap in the literature on markets where search can be directed.

The analysis we conduct provides two contributions. A substantive contribution is to present a theorem establishing uniqueness of symmetric equilibrium in directed search economies with finite numbers of players and heterogeneous sellers such as those in [1, 6, 7, 8]. The result can be extended to economies with infinite, heterogeneous populations. The analysis provides a methodological contribution, also. In the type of markets we study global concavity is not a property of sellers' payoff functions; therefore, standard methods of analysis

cannot be applied to determine uniqueness. We develop a tool to handle this type of situation; the technique is based on use of directional derivatives along judiciously selected equilibrium “price paths.”

The paper proceeds as follows. Section 2 presents the model and lays out some notation. Section 3 offers some preliminaries involving properties of demand. Section 4 contains the main results and Section 5 concludes.

## 2 The Model

Consider an economy with finitely many sellers and buyers. Buyers are homogeneous, but sellers could be heterogenous. Let  $\mathcal{J} = \{1, \dots, \bar{J}\}$ ,  $2 \leq \bar{J} < \infty$  be the set of sellers, and  $\mathcal{I} = \{1, \dots, I\}$ ,  $2 \leq I < \infty$  be the set of buyers. Each seller has one indivisible good which cannot be consumed by sellers, but are desirable to buyers.

Buyers and sellers play a sequential game of complete information, over three stages. In the first stage, sellers simultaneously and independently post and commit to a price for the indivisible good they sell. Let  $v_j$  be the (indirect) utility for a buyer who purchases the good offered by seller  $j$  at the price posted by that seller. Hence,  $\mathbf{v} = (v_1, \dots, v_{\bar{J}}) \in \mathbb{R}_+^{\bar{J}}$  is the strategy profile of all sellers and  $\mathbf{v}_{-j}$  denotes the strategy profile of every other seller, when we fix seller  $j$ . Denote the set of all *feasible* promised utilities as

$$\mathcal{V} := \prod_{j \in \mathcal{J}} [v_j, \bar{v}_j] \subset \mathbb{R}_+^{\bar{J}}, \quad \text{where } 0 \leq v_j < \bar{v}_j \text{ for all } j \in \mathcal{J}.$$

In the second stage, buyers observe the posted prices, or equivalently, the promised utilities  $\mathbf{v}$  and then simultaneously and independently choose to visit a single seller. Let  $\pi_j(\mathbf{v})$  be the probability that any buyer chooses to visit seller  $j \in \mathcal{J}$  after observing  $\mathbf{v} \in \mathcal{V}$  when buyers act symmetrically. We denote  $\boldsymbol{\pi}(\mathbf{v}) = (\pi_1(\mathbf{v}), \dots, \pi_{\bar{J}}(\mathbf{v})) \in \Delta^{\bar{J}-1}$  as the symmetric strategy profile of buyers.

In the third stage, matches are realized and a trade may take place in each match.

Consider a match between one seller and  $i$  buyers. Due to capacity constraint, at most one buyer will trade with the seller.

For any  $j \in \mathcal{J}$ , we denote  $\mathcal{H}(\pi_j)$ , the conditional probability that a buyer trades conditional on visiting seller  $j$ , when *every other buyer* visits that same seller with probability  $\pi_j$ . Hence

$$\mathcal{H}(\pi_j)v_j$$

is the buyer's payoff conditional on visiting seller  $j$ .<sup>1</sup>

Let  $\phi_j(v_j)$  be the payoff to seller  $j$  conditional on trading with a buyer when seller  $j$  promises utility  $v_j \in [\underline{v}_j, \bar{v}_j]$  to buyer. We normalize  $\phi_j(\bar{v}_j) = 0$ . Let  $\mathcal{M}(\pi_j)$  denote the *unconditional* probability that seller  $j$  trades, given that buyers visit the seller with probability  $\pi_j$ . In this case

$$\Pi_j(\mathbf{v}) := \mathcal{M}(\pi_j)\phi_j(v_j)$$

is the payoff to seller  $j$  conditional on promising  $v_j$  to any buyer.

The probability functions  $\mathcal{H}$  and  $\mathcal{M}$  satisfy the following properties, (i)  $\mathcal{M}$  is  $C^2$ ,  $\mathcal{M}' > 0$  and  $\mathcal{M}'' < 0$ ; (ii)  $\mathcal{H}$  is  $C^2$ ,  $\mathcal{H}' < 0$  and  $\mathcal{H}'' > 0$ ; (iii)  $\mathcal{H}(\pi)^{-1}$  is convex; (iv)  $\mathcal{H}(\pi)v$  is quasiconcave, these properties are discussed in [2]. Furthermore, assume that  $\phi_j$  is  $C^2$ ,  $\phi_j' < 0$  and  $\phi_j'' \leq 0$  for all  $j \in \mathcal{J}$ .

Consider the subgame where buyers choose sellers based on the promised utility vector  $\mathbf{v} \in \mathcal{V}$ . We call this subgame the “buyer's game”, as in [7].

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<sup>1</sup>The literature generally assumes random rationing, so

$$\mathcal{H}(\pi) := \sum_{i=0}^{I-1} \frac{(I-1)!}{i!(I-1-i)!} \pi^i (1-\pi)^{I-1-i} \frac{1}{i+1}.$$

If, in addition, if there is some external “shock” that may prevent trade with the chosen buyer, then we have

$$\mathcal{H}(\pi) := \sum_{i=0}^{I-1} \frac{(I-1)!}{i!(I-1-i)!} \pi^i (1-\pi)^{I-1-i} \rho(i+1),$$

where  $\rho$  is an assignment rule, see [2].

**Definition 1.** Given  $\mathbf{v} \in \mathcal{V}$ , a symmetric equilibrium in the buyer's game is a vector  $\boldsymbol{\pi}(\mathbf{v})$  such that:  $\sum_{j \in \mathcal{J}} \pi_j(\mathbf{v}) = 1$ ; if  $\pi_j(\mathbf{v}) > 0$  for  $j \in \mathcal{J}$ , then  $\mathcal{H}(\pi_j(\mathbf{v}))v_j = \max_{k \in \mathcal{J}} \mathcal{H}(\pi_k(\mathbf{v}))v_k$ .

It is known that the buyer's equilibrium exists and is unique [7]; see [2] for an alternative proof. From now on, with a small abuse in notation, we let  $\boldsymbol{\pi}(\mathbf{v}) := (\pi_1(\mathbf{v}), \dots, \pi_{\bar{J}}(\mathbf{v}))$  denote the unique buyer's equilibrium, for a given  $\mathbf{v} \in \mathcal{V}$ . Let denote the set of sellers in the market when buyers adopt the equilibrium strategy profile  $\boldsymbol{\pi}(\mathbf{v})$  as

$$\Gamma(\mathbf{v}) := \{i \in \mathcal{J} | \pi_i(\mathbf{v}) > 0\}.$$

### 3 Local properties of market demand $\boldsymbol{\pi}(\mathbf{v})$

In this section, we study local properties of buyers' equilibrium strategy profile  $\boldsymbol{\pi}(\mathbf{v})$  when the set of sellers in the market (= market structure) is unaffected by small variations in the promised utility vector  $\mathbf{v}$ . To do so, consider any open ball  $B(\mathbf{v}^*) \subset \mathbb{R}_+^{\bar{J}}$  centered at some vector of promised utilities  $\mathbf{v}^* \in \mathbb{R}_+^{\bar{J}}$  such that

$$\text{for any } \mathbf{v} \in B(\mathbf{v}^*), \quad \Gamma(\mathbf{v}) = \Gamma(\mathbf{v}^*),$$

i.e., the market structure is unaffected in the open ball  $B(\mathbf{v}^*)$ .<sup>2</sup> Therefore, without loss of generality, we may denote  $\Gamma(\mathbf{v}) = \{1, \dots, J\}$ , where  $J \leq \bar{J}$ , i.e. sellers  $1, \dots, J$  are in the market and sellers  $J+1, \dots, \bar{J}$  are out of the market.

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<sup>2</sup>Note that such a neighborhood  $B$  need not exist for *all* feasible profiles of promised utilities  $\mathbf{v} \in \mathcal{V}$ . For example, it may happen that for some profile  $\mathbf{v}$ , a small variation in the promised utility of one seller would change the market structure. In fact, the definition above considers only profiles  $\mathbf{v}^*$  where this cannot happen and it can be demonstrated that almost any feasible profile  $\mathbf{v} \in \mathcal{V}$  satisfies this property. To see this, note that we can find a subset of  $\mathcal{V}$ , let us call it  $\tilde{\mathcal{V}}$ , which satisfies two desirable properties: (i)  $\tilde{\mathcal{V}}$  is a dense subset of  $\mathcal{V}$ , i.e., the closure of  $\tilde{\mathcal{V}}$  is  $\mathcal{V}$ ; and (ii) for any  $v^* \in \tilde{\mathcal{V}}$ , we can find an open ball  $B(\mathbf{v}^*) \subset \mathcal{V}$  such that  $\boldsymbol{\pi}(\mathbf{v})$  is smooth on  $B(\mathbf{v}^*)$ . These two properties imply that the set of profiles  $\tilde{\mathcal{V}}$  covers almost every feasible profile of promised utilities and at *any* point in this set of profiles  $\tilde{\mathcal{V}}$ , we can find a nonempty neighborhood where the market structure is invariant.

This means that for any  $\mathbf{v} \in B(\mathbf{v}^*)$ , we have  $\pi_j(\mathbf{v}) > 0$  for any  $j = 1, \dots, J$ , and  $\pi_k(\mathbf{v}) = 0$  for any  $k = J + 1, \dots, \bar{J}$ .

Start by noticing that by Definition 1 the equilibrium vector  $\boldsymbol{\pi}(\mathbf{v})$  is the unique solution to the system of  $J$  equations,

$$\begin{cases} \mathcal{H}(\pi_j(\mathbf{v}))v_j - \mathcal{H}(\pi_J(\mathbf{v}))v_J = 0, & j = 1, \dots, J - 1, \\ \sum_{1 \leq j \leq J} \pi_j(\mathbf{v}) - 1 = 0. \end{cases} \quad (1)$$

The first  $J - 1$  equations are simply the required indifference of buyers across sellers, while the last equation is the requirement that  $\pi_j$  is in a mixed strategy. Define the partial derivatives  $h_j(\mathbf{v}) := \mathcal{H}'(\pi_j)v_j$  for every  $j = 1, \dots, J$ , where  $h_j < 0$  since  $\mathcal{H}' < 0$ . From now on we omit the arguments  $\mathbf{v}$  from  $\pi_j$  and  $h_j$ , if there is no possible confusion.

Following the analysis in [4], implicit differentiation on the system in (1) gives

$$\begin{pmatrix} h_1 & 0 & 0 & \dots & 0 & -h_J \\ 0 & h_2 & 0 & \dots & 0 & -h_J \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & h_{J-1} & -h_J \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix} \begin{pmatrix} d\pi_1 \\ d\pi_2 \\ \vdots \\ d\pi_{J-1} \\ d\pi_J \end{pmatrix} = \begin{pmatrix} \mathcal{H}(\pi_J) \\ \mathcal{H}(\pi_J) \\ \vdots \\ \mathcal{H}(\pi_J) \\ 0 \end{pmatrix} dv_J.$$

By Cramer's rule, we can find how the choice  $\pi_j$  of any buyer changes when some element of the vector  $\mathbf{v}$  changes. For any  $j = 1, \dots, J$  and  $i \neq j$ , we have

$$\begin{aligned} \frac{\partial \pi_j}{\partial v_j} &= -\frac{\sum_{s \neq j} \prod_{k \neq s, j} h_k}{\sum_{1 \leq s \leq J} \prod_{k \neq s} h_k} \mathcal{H}(\pi_j) = -\frac{\mathcal{H}(\pi_j)}{h_j + \frac{1}{\sum_{k \neq j} \frac{1}{h_k}}} > 0, \\ \frac{\partial \pi_i}{\partial v_j} &= \frac{\prod_{k \neq i, j} h_k}{\sum_{1 \leq s \leq J} \prod_{k \neq s} h_k} \mathcal{H}(\pi_j) < 0. \end{aligned} \quad (2)$$

Note that since we are considering a “local” change in  $\mathbf{v}$ , i.e.  $\mathbf{v} \in B(\mathbf{v}^*)$  and  $\sum_{i=1}^J \pi_i = 1$ , we have  $\sum_{i=1}^J \frac{\partial \pi_i}{\partial v_j} = 0$ . This means that a change in  $v_j$  simply redis-



tributes demand across sellers in the market, but does not change the number of sellers in the market. Therefore, by the laws of probability, we have

$$\sum_{j=1}^J \pi_j(\mathbf{v}) = \sum_{j=1}^J \pi_j(\mathbf{v}^*) = 1.$$

By equalities in (2), we can directly obtain two important identities that shows how demand at seller  $j$  changes when  $v_j$  changes as we “control” for price changes elsewhere in the market.

**Proposition 1** (Demand is locally homogeneous of degree zero in prices). *For any  $\mathbf{v} \in B(\mathbf{v}^*) \subset \mathbb{R}_+^J$ ,*

$$v_j \frac{\partial \pi_i}{\partial v_j} = v_i \frac{\partial \pi_j}{\partial v_i}, \quad (3)$$

$$v_1 \frac{\partial \pi_1}{\partial v_1} + \dots + v_J \frac{\partial \pi_J}{\partial v_J} = 0. \quad (4)$$

*Proof.* By (2),  $\frac{\partial \pi_i}{\partial v_j} = \kappa_{ij} \mathcal{H}(\pi_j)$  for any  $j = 1, \dots, J$ ,  $i \neq j$ , where  $\kappa_{ij} = \kappa_{ji} =$

$\frac{\prod_{k \neq i, j} h_k}{\sum_{1 \leq s \leq J} \prod_{k \neq s} h_k}$ . Thus we have  $\frac{\partial \pi_i}{\partial v_j} = \frac{\mathcal{H}(\pi_j)}{\mathcal{H}(\pi_i)} \frac{\partial \pi_j}{\partial v_i}$ . Since  $\mathcal{H}(\pi_j) v_j = \mathcal{H}(\pi_i) v_i$  by

the definition of equilibrium in Definition 1, we have (3). Since  $\sum_{i=1}^J \frac{\partial \pi_i}{\partial v_j} = 0$ , by substituting (3) and multiplying by  $v_j$ , we get (4).  $\square$

Expression (3) shows the relative magnitude of demand changes induced by (own) price changes, all else equal for any pair of sellers  $i$  and  $j$ . If seller  $j$  varies  $v_j$  and every other seller posts  $\mathbf{v}_{-j}$ , buyers will respond by adjusting the probabilities with which they visit seller  $j$ . This response depends on the posted price  $v_j$ ,  $\pi_i$  and  $\pi_j$ . Expression (3) shows that in buyer’s equilibrium, the relative response of buyers  $\frac{\partial \pi_j \backslash \partial v_i}{\partial \pi_i \backslash \partial v_j}$  is equal to the relative “price”  $\frac{v_j}{v_i}$ .

To understand the meaning of (4), note that the distribution of demand in the market is fully identified by  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_J)$ . Expression (4) considers the impact on market demand of a market-wide raises their prices  $\mathbf{v}$  by the same proportion. (4) says that demand is homogeneous of degree zero in “prices”,

since it is homogeneous of degree zero in the promised utilities. To do so, we took a directional derivative. The total derivative of  $\pi_j$  at  $\mathbf{v} \in B(\mathbf{v}^*)$  is  $\sum_{i=1}^J \frac{\partial \pi_j}{\partial v_i}$ . Since we wish to consider how  $\pi_j$  changes when every element of  $\mathbf{v}$  changes by the same proportion, then we consider the directional derivative  $\sum_{i=1}^J v_i \frac{\partial \pi_j}{\partial v_i}$ . This tells us how  $\pi_j$  changes at  $\mathbf{v}$  as the price ratios are kept constant, i.e. as we move “in the direction” of vector  $\mathbf{v}$ . The figure illustrates (4) for the case of  $J = 2$ .

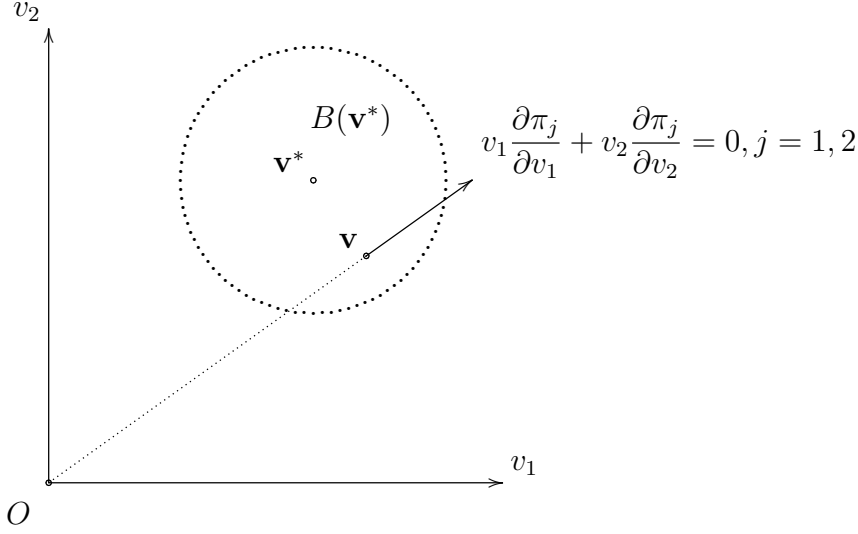


Figure 1: A directional derivative of  $\pi_j$  at  $\mathbf{v}$  in the direction of  $\mathbf{v}$

We present an additional, useful, characterization of demand.

**Lemma 1.** For any  $\mathbf{v} \in B(\mathbf{v}^*) \subset \mathbb{R}_+^J$  and  $i \neq j \leq J$ , we have

$$\frac{\partial \pi_i}{\partial v_i} = c_{ij}(\boldsymbol{\pi}) A_{ij}(\mathbf{v}),$$

where:

- $c_{ij}(\boldsymbol{\pi}) = a(\pi_j) B_{ij}(\boldsymbol{\pi}) + \mathcal{H}(\pi_i)$ .
- $A_{ij}(\mathbf{v}) = A_{ji}(\mathbf{v}) > 0$ ,  $B_{ij}(\boldsymbol{\pi}) = B_{ij}(\boldsymbol{\pi}) \geq 0$ ,  $a(\pi_j) > 0$ , for any  $\pi_j \in (0, 1)$ .
- If  $v_k < v_j$  ( $k \neq i$ ), then  $B_{ij}(\boldsymbol{\pi}) < B_{ik}(\boldsymbol{\pi})$ .
- If  $v_i < v_k$  ( $k \neq j$ ), then  $B_{ij}(\boldsymbol{\pi}) > B_{kj}(\boldsymbol{\pi})$ , and  $c_{ij}(\boldsymbol{\pi}) > c_{kj}(\boldsymbol{\pi})$ .
- If  $v_i \leq v_j$ , then  $a(\pi_j) < a(\pi_j)$  and  $c_{ij}(\boldsymbol{\pi}) > c_{ji}(\boldsymbol{\pi})$ .

**Proof of Lemma 1.** See the the Appendix. □

**Proposition 2.** For any vector  $\mathbf{v} \in B(\mathbf{v}^*) \subset \mathbb{R}_+^{\bar{J}}$  with  $v_i > v_j$ ,

$$\frac{\partial \pi_i}{\partial v_i} < \frac{\partial \pi_j}{\partial v_j}.$$

*Proof.* By Lemma 1,

$$\frac{\partial \pi_i}{\partial v_i} = c_{ij}(\boldsymbol{\pi}) A_{ij}(\mathbf{v}), \quad \frac{\partial \pi_j}{\partial v_j} = c_{ji}(\boldsymbol{\pi}) A_{ji}(\mathbf{v}),$$

where  $c_{ij}(\boldsymbol{\pi}) < c_{ji}(\boldsymbol{\pi})$  and  $A_{ij}(\mathbf{v}) = A_{ji}(\mathbf{v})$ . Hence

$$\frac{\partial \pi_i}{\partial v_i} < \frac{\partial \pi_j}{\partial v_j}.$$

□

The meaning of Proposition 2 is that demand is more responsive to price changes at more expensive sellers. Suppose two sellers  $i, j$  promise different utilities  $v_i > v_j$ . Consider a given increase in promised utility (same price cut). Such change raises demand more at the more expensive seller. The intuition for this result lies in the use of random rationing to assign the product. Sellers with higher prices (= lower  $v$ ) expect less customers (= lower  $\pi$ ). Hence, for a given price cut, the high-priced seller will be able to attract more buyers (buyers are more likely to “earn” that price cut at sellers with less customers).

Finally we characterize market demand, when only a subset of sellers changes their prices in equal proportions, while the remaining sellers keep their prices fixed.

**Proposition 3.** For any vector  $\mathbf{v} \in B(\mathbf{v}^*) \subset \mathbb{R}_+^{\bar{J}}$ ,

$$v_j \frac{\partial \pi_j(\mathbf{v})}{\partial v_j} + \sum_{i \neq j} \lambda_i v_i \frac{\partial \pi_j(\mathbf{v})}{\partial v_i} \geq 0, \tag{5}$$

$$v_j \frac{\partial}{\partial v_j} \left( \frac{\partial \pi_j(\mathbf{v})}{\partial v_j} \right) + \sum_{i \neq j} \lambda_i v_i \frac{\partial}{\partial v_i} \left( \frac{\partial \pi_j(\mathbf{v})}{\partial v_j} \right) < 0, \tag{6}$$

for every  $j = 1, \dots, J$  and any  $\lambda_i \in \{0, 1\}$ ,  $i = 1, \dots, J$ , where the inequality in (5) is strict if  $\lambda_i = 0$  for some  $i \neq j$ .

**Proof of Proposition 3.** See the Appendix. □

**Corollary 4.** For any vector  $\mathbf{v} \in B(\mathbf{v}^*) \subset \mathbb{R}_+^{\bar{J}}$ ,

$$\frac{\partial^2 \pi_j}{\partial v_j^2} < 0.$$

*Proof.* With  $\lambda_i = 0$  for all  $i \neq j$  at (6), we have  $v_j \frac{\partial^2 \pi_j}{\partial v_j^2} < 0$ . □

That is demand  $\pi_j(v_j, \mathbf{v}_{-j})$  is concave in  $v_j$  conditional on  $\pi_j > 0$ . This is fundamental for proving existence of equilibrium in the entire game.

Finally we compare the behavior of equilibrium demand at seller  $i$  relative to demand of seller  $j$ , when seller  $i$  faces a demand reduction and seller  $j$  faces the opposite situation of a demand increase, i.e. we have

$$0 < \pi_j^* < \pi_j < \pi_i < \pi_i^*.$$

To do so, consider two different promised utilities  $\tilde{\mathbf{v}}$  and  $\tilde{\mathbf{v}}^*$  with associated open sets  $B(\tilde{\mathbf{v}}), B(\tilde{\mathbf{v}}^*) \subset \mathbb{R}_+^{\bar{J}}$ . Such vectors may support different sets of sellers in the market. That is

- For any  $\mathbf{v} \in B(\tilde{\mathbf{v}})$ , we have  $\Gamma := \Gamma(\tilde{\mathbf{v}}) = \Gamma(\mathbf{v})$ ,
- For any  $\mathbf{v}^* \in B(\tilde{\mathbf{v}}^*)$ , we have  $\Gamma^* := \Gamma(\tilde{\mathbf{v}}^*) = \Gamma(\mathbf{v}^*)$ .

When no confusion arises, we denote  $f \equiv f(\mathbf{v})$  and  $f^* \equiv f(\mathbf{v}^*)$  for any function of  $\mathbf{v}$  and  $\mathbf{v}^*$ . So for example,  $\boldsymbol{\pi}$  and  $\boldsymbol{\pi}^*$  are equilibrium probabilities under  $\mathbf{v}$  and  $\mathbf{v}^*$ , respectively.

**Proposition 5.** Let  $\mathbf{v} \neq \mathbf{v}^*$  be any two vectors with  $\mathbf{v} \in B(\tilde{\mathbf{v}})$ ,  $\mathbf{v}^* \in B(\tilde{\mathbf{v}}^*)$ , and consider sellers  $i, j \in \Gamma \cap \Gamma^*$ . If  $\pi_j^* < \pi_j < \pi_i < \pi_i^*$ , then

$$\frac{c_{ij}(\boldsymbol{\pi})}{c_{ij}(\boldsymbol{\pi}^*)} \frac{c_{ji}(\boldsymbol{\pi}^*)}{c_{ji}(\boldsymbol{\pi})} > 1.$$

*Proof of Proposition 5.* See the Appendix. □

## 4 The main result

In the first stage of the game, each seller  $j$  chooses promised utility  $v_j \in [\underline{v}_j, \bar{v}_j] \subset \mathbb{R}$  to maximize the payoff

$$\Pi_j(v_j, \mathbf{v}_{-j}) = \mathcal{M}(\pi_j(v_j, \mathbf{v}_{-j}))\phi_j(v_j),$$

taking as given  $\mathbf{v}_{-j}$  and the buyers' optimal response  $\boldsymbol{\pi}(v_j, \mathbf{v}_{-j})$  defined in Definition 1. We focus on outcomes where sellers adopt pure strategies.

**Definition 2.** A directed search equilibrium is a vector  $\mathbf{v} \in \mathcal{V}$  such that  $\Pi_j(v_j, \mathbf{v}_{-j}) \geq \Pi_j(v'_j, \mathbf{v}_{-j})$  for all  $v'_j \in [\underline{v}_j, \bar{v}_j]$  and all  $j \in \mathcal{J}$ , where  $\boldsymbol{\pi}(\mathbf{v})$  satisfies Definition 1.

Existence of directed search equilibrium in pure strategies is established in [4], who also explain why mixed strategy equilibria cannot generally be ruled out.<sup>3</sup> The analysis in [4] establishes uniqueness of equilibrium for the case of homogeneous sellers under the following assumption:

**Assumption 1.** If  $\bar{\mathbf{v}} := (\bar{v}_1, \dots, \bar{v}_J)$ , then  $\boldsymbol{\pi}(\bar{\mathbf{v}}) \gg 0$ .

We retain this assumption, and call  $\bar{\mathbf{v}}$  the (promised utilities associated with) “predatory” price vector. If Assumption 1 holds, then seller  $j$  can always attract buyers by posting promised utility close to  $\bar{v}_j$ , even if everyone else “dumps” production on the market by posting their highest possible promised utility  $\bar{\mathbf{v}}_{-j}$ .

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<sup>3</sup>The problem is that the sellers' payoff function  $\Pi_j(v_j, \mathbf{v}_{-j})$ , which is quasi-concave in  $v_j$  when other sellers use pure strategies  $\mathbf{v}_{-j}$ , need not be quasi-concave when other sellers use mixed strategies [4, p.11-12]

Some remarks are in order. First, the assumption *does not* imply that a seller cannot feasibly transfer all of the profit to buyers. Each seller  $j$  is free to raise promised utility to attract buyers, up to the point where the seller breaks even, i.e., the value  $\bar{v}_j$ . Assumption 1 implies that even if a seller is just breaking even, then he can still attract some buyers. In particular, this means that the assumption does not impose any restrictions on the sellers' behavior; for instance, if some seller posts a very low promised utility out of equilibrium, other sellers can still force him out of the market by raising their promised utilities.

Second, Assumption 1 is implicit in [1] and related models, because it holds if sellers are homogeneous. Therefore, the assumption does not rule out applications of our analysis to the symmetric model in [1].

Third, Assumption 1 also holds if there is no big difference in predatory prices  $\bar{v}_j$ . For example, it holds if

$$\mathcal{H}\left(\frac{1}{J-1}\right)\bar{v}_M < \mathcal{H}(0)\bar{v}_m,$$

where  $\bar{v}_M := \max\{\bar{v}_1, \dots, \bar{v}_J\}$  and  $\bar{v}_m := \min\{\bar{v}_1, \dots, \bar{v}_J\}$ .<sup>4</sup>

Fourth, the equilibrium may not be unique when Assumption 1 is not satisfied. The reason is that there is a kink in the seller's payoff function as prices get sufficiently close to predatory prices. To see this consider that [4] show how, when Assumption 1 does not hold (Assumption 3 in their paper), equilibria exist in which identical buyers act symmetrically, but identical sellers do not. This immediately gives rise to equilibrium multiplicity. To see this, consider two buyers and three sellers  $j = 1, 2, 3$  where sellers 1, 2 have utility  $\phi(v) = 1 - v$  and seller 3 has utility  $\hat{\phi}(v) = \hat{U} - v$  where  $\hat{U} < 1$ . The analysis in [4] shows that there is an equilibrium in which the promised utility vector is

$$(v_1, v_2, v_3) = \mathbf{v} := \left(\frac{1}{2} + \varepsilon, \frac{1}{2} + \kappa, \hat{U}\right)$$

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<sup>4</sup>The left hand side shows the case when every other seller “gangs-up” against the weak seller who can only offer  $v_m$ , by offering  $\mathbf{v}_M$ . Given that no other buyer visits the weak seller, it is still optimal for a buyer to deviate and visit that seller.

where  $0 < \varepsilon < \kappa$  small, and where  $\hat{U}$  corresponds to the buyers' payoff in equilibrium, when only sellers 1 and 2 are in the market. In this equilibrium the first order conditions of sellers 1 and 2 are not satisfied with equality; there is a kink in their payoff at the point  $\mathbf{v}$ . Sellers 1 and 2 could lower their offers, if they wanted to, but they choose not to do so because lowering their promised utilities would get seller 3 into the market, reducing expected demand, which delivers a lower payoff. Hence, there is a jump in the first order conditions at  $\mathbf{v}$ .

It is immediate that another equilibrium also exists, characterized by

$$(v_1^*, v_2^*, v_3^*) = \mathbf{v}^* := \left( \frac{1}{2} + \kappa, \frac{1}{2} + \varepsilon, \hat{U} \right).$$

But we can find other equilibria, by considering small perturbations of the offers of sellers 1 and 2, which keep the buyers' expected payoff fixed at  $\hat{U}$ . This can be done, by continuity of the first order conditions.

**Proposition 6.** *Let Assumption 1 hold. If  $\mathbf{v} = (v_1, \dots, v_J)$  is a directed search equilibrium, then  $v_j \in (0, \bar{v}_j)$  for all  $j \in \mathcal{J}$ , and  $\Gamma(\mathbf{v}) = \mathcal{J}$ .*

*Proof.* First, we show that  $v_j > 0$  for all  $j \in \mathcal{J}$ , in equilibrium.

- If  $\underline{v}_j > 0$  for all  $j \in \mathcal{J}$ , then this is immediate. Hence consider the case where  $\underline{v}_j = 0$  for some  $j$ . If some  $v_k > 0$  and  $v_j = 0$ , then seller  $j$  can improve his payoff above  $\Pi_j = 0$ , which is the payoff when  $v_j = 0$ .
- Now consider the case in which  $v_i = 0$  for all  $i \in \mathcal{J}$ . If seller  $j$  deviates with  $v_j = \epsilon > 0$ , then seller  $j$  captures the whole market ( $\pi_j = 1$ ). By continuity of the payoff function  $\phi_j$ , seller  $j$  can always do better by deviating marginally from  $v_j = 0$ :

$$\underbrace{\mathcal{M}\left(\frac{1}{J}\right)\phi_j(0)}_{\text{equilibrium payoff}} < \underbrace{\lim_{\epsilon \rightarrow 0} \mathcal{M}(1)\phi_j(\epsilon)}_{\text{payoff from small deviation}} = \mathcal{M}(1)\phi_j(0).$$

- Next, we show that  $v_j < \bar{v}_j$  for all  $j \in \mathcal{J}$  in equilibrium. Let  $v_j = \bar{v}_j$  be an equilibrium for some  $j \in \mathcal{J}$ . We have  $\Pi_j(\bar{v}_j, \mathbf{v}_{-j}) = \mathcal{M}(\pi_j(\bar{v}_j, \mathbf{v}_{-j}))\phi_j(\bar{v}_j) =$

0, since  $\phi_j(\bar{v}_j) = 0$  by normalization. By continuity of  $\pi_j$ , there exists a small  $\epsilon > 0$  such that  $\pi_j(\bar{v}_j - \epsilon, \bar{\mathbf{v}}_{-j}) > 0$ , and

$$\Pi_j(\bar{v}_j - \epsilon, \mathbf{v}_{-j}) = \mathcal{M}(\pi_j(\bar{v}_j - \epsilon, \mathbf{v}_{-j}))\phi(\bar{v}_j - \epsilon) > \Pi_j(\bar{v}_j, \mathbf{v}_{-j}) = 0.$$

Therefore  $v_j = \bar{v}_j$  is not a best response to  $\mathbf{v}_{-j}$ .

- Finally we prove by means of contradiction that  $\Gamma(\mathbf{v}) = \mathcal{J}$ . Suppose seller  $j \notin \Gamma(\mathbf{v})$ , i.e.  $\Gamma(\mathbf{v}) \subsetneq \mathcal{J}$ . Since  $\pi_j(\mathbf{v}) = 0$ ,  $\Pi_j(\mathbf{v}) = 0$ . But then, by Assumption 1, we have  $\pi_j(\bar{\mathbf{v}}) > 0$ , so that  $\Pi_j(\bar{v}_j - \epsilon, \mathbf{v}_{-j}) > 0$  for small  $\epsilon > 0$ .

□

**Theorem 7.** *Let Assumption 1 hold. If  $\mathbf{v}$  is a directed search equilibrium, then  $\mathbf{v}$  is also the unique equilibrium.*

The remainder of this section is devoted to prove Theorem 7 through a sequence of results.

Under the assumption made in Section 2, a directed search equilibrium  $\mathbf{v}$  exists. The proof is in [4]. Now let Assumption 1 hold. Consider  $\pi_j \in (0, 1)$  for all  $j \in \mathcal{J}$ . We have that

- $\pi_j(v_j, \mathbf{v}_{-j})$  is strictly increasing and strictly concave in  $v_j$ .
- $\mathcal{M}(\pi_j)$  is strictly increasing and strictly concave in  $\pi_j$ .
- $\phi_j(v_j)$  is strictly decreasing and concave in  $v_j \in [v_j, \bar{v}_j]$ .

Hence seller  $j$ 's payoff function  $\Pi_j(v_j, \mathbf{v}_{-j}) = \mathcal{M}(\pi_j(v_j, \mathbf{v}_{-j}))\phi_j(v_j)$  is strictly concave in  $v_j$ .

To establish uniqueness of equilibrium, we use a proof by means of contradiction. Suppose there are two distinct equilibria  $\mathbf{v}, \mathbf{v}^* \in \mathcal{V}$ . By Proposition 6,  $\mathbf{v}, \mathbf{v}^* \gg 0$  and  $\boldsymbol{\pi}(\mathbf{v}), \boldsymbol{\pi}(\mathbf{v}^*) \gg 0$ . We will study the first order conditions from seller  $j$ 's maximization problem, considering two cases:

- (i)  $\mathbf{v}^* > \mathbf{v}$ , i.e.  $v_j^* \geq v_j$  for all  $j \in \mathcal{J}$  and  $v_i^* > v_i$  for at least one  $i \in \mathcal{J}$ ,



(ii)  $v_i^* > v_i$  for at least one  $i \in \mathcal{J}$  and  $v_j^* < v_j$  for at least one  $j \in \mathcal{J}$ .

As a matter of notation, we omit  $\mathbf{v}$  and  $\mathbf{v}^*$  as arguments, if there is no possible confusion, e.g.  $\pi \equiv \pi(\mathbf{v})$  and  $\pi^* \equiv \pi(\mathbf{v}^*)$ . We also use the notation  $\frac{\partial \pi_j}{\partial v_j}(\mathbf{v})$  directional limits: If  $v_j = \underline{v}_j$ , then consider the directional limit of  $\frac{\partial \pi_j}{\partial v_j}$  at  $\mathbf{v} = (\underline{v}_j, \mathbf{v}_{-j})$ ,

$$\lim_{w_j \rightarrow \underline{v}_j^+} \frac{\partial \pi_j}{\partial v_j}(w_j, \mathbf{v}_{-j}).$$

If  $v_j > \underline{v}_j$ , then  $\frac{\partial \pi_j}{\partial v_j}(\mathbf{v})$  exists and  $\frac{\partial \pi_j}{\partial v_j}(\mathbf{v}) = \lim_{w_j \rightarrow v_j^+} \frac{\partial \pi_j}{\partial v_j}(w_j, \mathbf{v}_{-j})$ , since  $v_j < \bar{v}_j$ .

#### 4.1 Case (i): $v_j^* \geq v_j$ , $v_i^* > v_i$ for all $j$ and some $i$

Given  $\mathbf{v}$  and  $\mathbf{v}^*$ , consider a seller, say seller  $L$ , who has the greatest percentage price change when moving from  $\mathbf{v}^*$  to  $\mathbf{v}$ , i.e.  $\frac{v_L}{v_L^*} = \min_{j \in \mathcal{J}} \left\{ \frac{v_j}{v_j^*} \right\}$ . We will study the optimality of seller  $L$ 's choices. Intuitively if there is a market-wide decrease in promised utilities (going from  $\mathbf{v}^*$  to  $\mathbf{v}$ ), it is natural to question the optimality of the choice of seller  $L$ , who proportionally lowers promised utility to such an extreme degree, more than anyone else.

To study the optimality of seller  $L$ 's choices, we need to characterize  $\pi_L$  and  $\pi_L^*$ . The problem is that we cannot fully characterize  $\pi_L$  relative to  $\pi_L^*$  due to the general equilibrium effects of a market-wide promised utility change from  $\mathbf{v}^*$  to  $\mathbf{v}$ . Therefore, we adopt a recursive procedure, by judiciously constructing a path between  $\pi_L^*$  and  $\pi_L$  which allows us to characterize  $\pi_L^*$  and  $\pi_L$ . To do so, we exploit the results in Proposition 3 about the properties of  $\pi_j$ , when we control for the change in promised utilities. Hence we construct a path between any  $\pi_j^*$  and  $\pi_j$  by *proportionally* decreasing  $v_j^*$  for every seller  $j \in \mathcal{J}$  in a sequence of steps until the promised utility of seller  $j$  reaches  $v_j$ . This gives rise to finitely many market-wide promised utilities  $\mathbf{v} \leq \mathbf{v}^k \leq \mathbf{v}^*$ ,  $k = 0, 1, \dots$  such that

- $\mathbf{v}^0 = (v_1^0, \dots, v_j^0) := \mathbf{v}^* = (v_1^*, \dots, v_j^*)$ ,

- $\mathbf{v}^{k+1} := (b_1^k v_1^k, b_2^k v_2^k, \dots, b_j^k v_j^k)$  with

$$b_j^k = \begin{cases} 1 & , \text{ if } v_j^k = v_j \\ b^k := \max\{x \mid x = \frac{v_j}{v_j^k}, v_j < v_j^k\} & , \text{ otherwise.} \end{cases}$$

Intuitively, we decrease market-wide promised utilities from  $\mathbf{v}^*$  to  $\mathbf{v}$  sequentially, partitioning sellers into two groups:

- some sellers have not yet reached  $v_j$ ; hence they all decrease promised utility by the same proportion  $b^k$  in the  $k + 1^{\text{th}}$  step (i.e. raise their price in equal proportions);
- the remaining sellers have already reduced their promised utility to  $v_j$ ; hence do not lower it further. In the last step, the only sellers left out of this group are those such that  $\frac{v_j}{v_j^*} = \frac{v_L}{v_L^*}$ ; i.e. those sellers who have in the extreme promised utility change.

For a geometric interpretation, let  $\mathbf{w} \in \mathbb{R}_+^J$  be any point on the line segment connecting two consecutive points  $\mathbf{v}^k > \mathbf{v}^{k+1}$ , i.e.

$$\mathbf{w} = \mathbf{v}^{k+1} + \alpha(\mathbf{v}^k - \mathbf{v}^{k+1}), \quad \alpha \in [0, 1].$$

Figure 2 illustrates the procedure for the simplest case of two sellers.

The path formed by connecting all consecutive points  $\mathbf{v}^k$  allows us to build a path between  $\pi_L^*$  and  $\pi_L$ ; once this is done, we can characterize  $\pi_L$  and  $\pi_L^*$  using the results in Proposition 3.<sup>5</sup>

For any  $\mathbf{w}$  on the path between  $\mathbf{v}^k$  and  $\mathbf{v}^{k+1}$ , the directional derivatives of  $\pi_L(\mathbf{w})$

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<sup>5</sup>It is possible that even if all sellers are active at both points  $\mathbf{v}$  and  $\mathbf{v}^*$ , some sellers may not be active on some segments of the path between  $\mathbf{v}$  and  $\mathbf{v}^*$ . One can prove that this does cause jumps in the derivative of demand functions for sellers that remain in the market, but does not alter our analysis. For the proof, and an example, see the Supplementary Information.

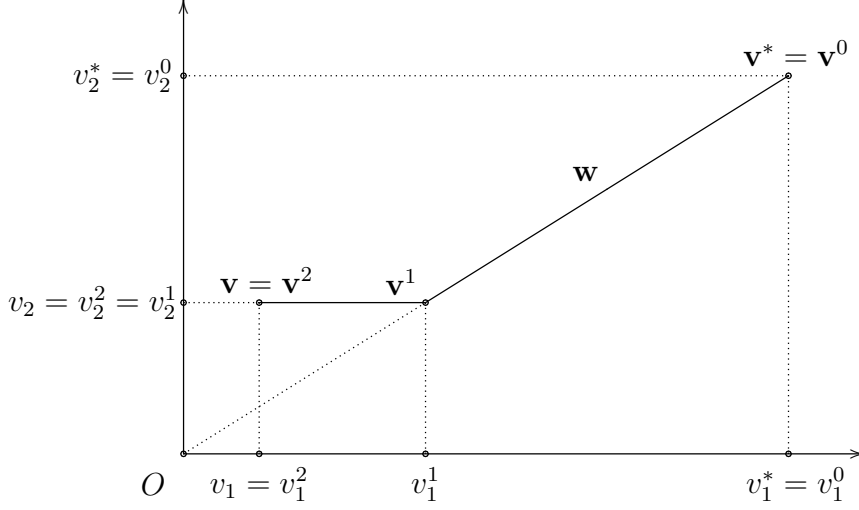


Figure 2: The price path between  $\mathbf{v}^*$  and  $\mathbf{v}$

and  $\frac{\partial \pi_L}{\partial v_L}(\mathbf{w})$  in the direction of vector  $(\mathbf{v}^k - \mathbf{v}^{k+1})$  are

$$(\mathbf{v}^k - \mathbf{v}^{k+1}) \cdot \left( \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_j} \right) (\pi_L(\mathbf{w})) = C(\mathbf{w}) \left( w_L \frac{\partial}{\partial v_L} + \sum_{j \neq L} \lambda_j^k w_j \frac{\partial}{\partial v_j} \right) (\pi_L(\mathbf{w})),$$

$$(\mathbf{v}^k - \mathbf{v}^{k+1}) \cdot \left( \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_j} \right) \left( \frac{\partial \pi_L}{\partial v_L}(\mathbf{w}) \right) = C(\mathbf{w}) \left( w_L \frac{\partial}{\partial v_L} + \sum_{j \neq L} \lambda_j^k w_j \frac{\partial}{\partial v_j} \right) \left( \frac{\partial \pi_L}{\partial v_L}(\mathbf{w}) \right),$$

for some  $C(\mathbf{w}) > 0$  and  $\lambda_j^k \in \{0, 1\}$ . Note that  $\lambda_j^k = 0$  is possible because for some  $j \neq L$ , we may have  $v_j^k = v_j^{k+1} = v_j$ . For example, consider  $v_2^k$  and  $v_2^{k+1}$  for  $k = 1$  in Figure 2.

By Proposition 3, we have that for seller  $L$

$$\pi_L(\mathbf{v}^{k+1}) \leq \pi_L(\mathbf{v}^k), \text{ and } \frac{\partial \pi_L(\mathbf{v}^{k+1})}{\partial v_L} > \frac{\partial \pi_L(\mathbf{v}^k)}{\partial v_L} \text{ for all } k = 0, 1, \dots$$

Consequently, we have

$$\pi_L \leq \pi_L^*, \text{ and } \frac{\partial \pi_L}{\partial v_L} > \frac{\partial \pi_L^*}{\partial v_L}.$$

Hence by the properties discussed in Section 2, we have

$$\begin{aligned}
& \mathcal{M}'(\pi_L) \geq \mathcal{M}'(\pi_L^*) > 0, \text{ and } 0 < \mathcal{M}(\pi_L) \leq \mathcal{M}(\pi_L^*), \\
& \frac{\partial \pi_L}{\partial v_L} > \frac{\partial \pi_L^*}{\partial v_L} > 0, \\
& \phi_L(v_L) > \phi_L(v_L^*) > 0, \text{ and } \phi'_L(v_L^*) \leq \phi'_L(v_L) \leq 0, \\
& \mathcal{M}(\pi_L^*)\phi'_L(v_L^*) \leq \mathcal{M}(\pi_L)\phi'_L(v_L) \leq 0.
\end{aligned} \tag{7}$$

We are now ready to demonstrate that if  $\mathbf{v}$  and  $\mathbf{v}^*$  are both equilibrium, then there exists a contradiction. By the first order conditions, since  $\mathbf{v}$  and  $\mathbf{v}^*$  are both equilibria, we must have

$$\begin{aligned}
\Pi'_L(v_L, \mathbf{v}_{-L}) &:= \mathcal{M}'(\pi_L) \frac{\partial \pi_L}{\partial v_L} \phi_j(v_L) + \mathcal{M}(\pi_L) \phi'_L(v_L) \leq 0, \\
\Pi'_L(v_L^*, \mathbf{v}_{-L}^*) &:= \mathcal{M}'(\pi_L^*) \frac{\partial \pi_L^*}{\partial v_L} \phi_L(v_L^*) + \mathcal{M}(\pi_L^*) \phi'_L(v_L^*) = 0,
\end{aligned}$$

where the first inequality follows from observing that seller  $L$  may be constrained at  $\mathbf{v}$ , i.e. he posts  $v_L = \underline{v}_L$  when every other seller plays  $\mathbf{v}_{-L}$ . Clearly  $v_L^* > v_L$ , so the first order condition for seller  $L$  must vanish because  $v^*$  is an equilibrium and this seller is unconstrained given  $\mathbf{v}_{-j}^*$ . But these first order conditions imply the following contradiction,

$$0 \geq \mathcal{M}'(\pi_L) \frac{\partial \pi_L}{\partial v_L} \phi_L(v_L) + \mathcal{M}(\pi_L) \phi'_L(v_L) > \mathcal{M}'(\pi_L^*) \frac{\partial \pi_L^*}{\partial v_L} \phi_L(v_L^*) + \mathcal{M}(\pi_L^*) \phi'_L(v_L^*) = 0$$

by the observations in (7).

## 4.2 Case (ii): $v_i^* > v_i$ , $v_j^* < v_j$ for some pair $(i, j)$

There are four separate cases to consider:

- (1)  $v_i < v_j$  and  $v_i^* > v_j^*$ ,
- (2)  $v_i > v_j$  and  $v_i^* > v_j^*$ ,
- (3)  $v_i < v_j$  and  $v_i^* < v_j^*$ ,
- (4)  $v_i > v_j$  and  $v_i^* < v_j^*$ .

The following first order conditions must hold in each of the above cases. For seller  $i$ , we must have:

$$\begin{aligned}\Pi'_i(v_i, \mathbf{v}_{-i}) &= \mathcal{M}'(\pi_i) \frac{\partial \pi_i}{\partial v_i} \phi_i(v_i) + \mathcal{M}(\pi_i) \phi'_i(v_i) \leq 0, \\ \Pi'_i(v_i^*, \mathbf{v}_{-i}^*) &= \mathcal{M}'(\pi_i^*) \frac{\partial \pi_i^*}{\partial v_i} \phi_i(v_i^*) + \mathcal{M}(\pi_i^*) \phi'_i(v_i^*) = 0.\end{aligned}$$

For seller  $j$ , we must have the opposite, i.e.

$$\begin{aligned}\Pi'_j(v_j, \mathbf{v}_{-j}) &= \mathcal{M}'(\pi_j) \frac{\partial \pi_j}{\partial v_j} \phi_j(v_j) + \mathcal{M}(\pi_j) \phi'_j(v_j) = 0, \\ \Pi'_j(v_j^*, \mathbf{v}_{-j}^*) &= \mathcal{M}'(\pi_j^*) \frac{\partial \pi_j^*}{\partial v_j} \phi_j(v_j^*) + \mathcal{M}(\pi_j^*) \phi'_j(v_j^*) \leq 0.\end{aligned}$$

For each of the four cases, we may work under assumption that

$$\pi_i < \pi_i^* \text{ and } \pi_j > \pi_j^*. \quad (8)$$

This assumption is without loss of generality (see the Appendix). To see this, note that if only two sellers are the only sellers who change promised utilities going from  $\mathbf{v}^*$  to  $\mathbf{v}$ , then any other demand configuration is inconsistent with Definition 1 (by buyers' equilibrium). A seller who lower promised utility cannot get greater demand, and vice versa. If more than two sellers change promised utilities going from  $\mathbf{v}^*$  to  $\mathbf{v}$ , then other configurations of demand can be reduced to the one proposed above by properly choosing sellers  $i$  and  $j$ , that is, with three or more sellers changing the promised utility as considered in this section, we can always find a pair of sellers whose demand moves in opposite directions. We want to focus on this specific pair  $(i, j)$  because, intuitively, if it is optimal for seller  $j$  to offer higher promised utility at  $\mathbf{v}$  than at  $\mathbf{v}^*$  in order to get *higher* demand, then it is meaningful to question whether seller  $i$  (who acts exactly in the opposite manner, and experience a *decrease* in demand) is acting optimally. Note that  $\Pi'_i(v_i, \mathbf{v}_{-i}) \leq \Pi'_i(v_i^*, \mathbf{v}_{-i}^*)$  can only be satisfied when

$$\frac{\partial \pi_i}{\partial v_i} < \frac{\partial \pi_i^*}{\partial v_i}, \quad (9)$$

since  $\mathcal{M}'(\pi_i) > \mathcal{M}'(\pi_i^*)$ ,  $\phi_i(v_i) > \phi_i(v_i^*)$ , and  $\mathcal{M}(\pi_i)\phi_i'(v_i) \geq \mathcal{M}(\pi_i^*)\phi_i'(v_i^*)$  from the properties of payoff functions, discussed in Section 2. Similarly  $\Pi_j'(v_j, \mathbf{v}_{-j}) \geq \Pi_j'(v_j^*, \mathbf{v}_{-j}^*)$  can only be satisfied when

$$\frac{\partial \pi_j}{\partial v_j} > \frac{\partial \pi_j^*}{\partial v_j}. \quad (10)$$

Now suppose  $\mathbf{v}$  and  $\mathbf{v}^*$  are both equilibria. We show the existence of contradiction for all possible four cases.

**Case (1):**  $v_i < v_j$  and  $v_i^* > v_j^*$

By Proposition 2, we have:

$$\frac{\partial \pi_j}{\partial v_j} < \frac{\partial \pi_i}{\partial v_i} \quad \text{and} \quad \frac{\partial \pi_i^*}{\partial v_i} < \frac{\partial \pi_j^*}{\partial v_j}.$$

Together with (9), (10), we have

$$\frac{\partial \pi_i}{\partial v_i} < \frac{\partial \pi_i^*}{\partial v_i} < \frac{\partial \pi_j^*}{\partial v_j} < \frac{\partial \pi_j}{\partial v_j} < \frac{\partial \pi_i}{\partial v_i},$$

which is a contradiction.

**Case (2):**  $v_i > v_j$  and  $v_i^* > v_j^*$

By Proposition 2,

$$\frac{\partial \pi_i}{\partial v_i} < \frac{\partial \pi_j}{\partial v_j} \quad \text{and} \quad \frac{\partial \pi_i^*}{\partial v_i} < \frac{\partial \pi_j^*}{\partial v_j}.$$

Together with (9), (10), we have

$$\frac{\partial \pi_i}{\partial v_i} < \frac{\partial \pi_i^*}{\partial v_i} < \frac{\partial \pi_j^*}{\partial v_j} < \frac{\partial \pi_j}{\partial v_j}.$$

By Lemma 1, we have

$$c_{ij}(\boldsymbol{\pi})A_{ij}(\mathbf{v}) < c_{ij}(\boldsymbol{\pi}^*)A_{ij}(\mathbf{v}^*) < c_{ji}(\boldsymbol{\pi}^*)A_{ji}(\mathbf{v}^*) < c_{j,i}(\boldsymbol{\pi})A_{ji}(\mathbf{v}).$$

Since  $A_{ij}(\mathbf{v}) = A_{ji}(\mathbf{v})$  and  $A_{ij}(\mathbf{v}^*) = A_{ji}(\mathbf{v}^*)$ ,

$$\frac{c_{ij}(\boldsymbol{\pi})}{c_{ij}(\boldsymbol{\pi}^*)} \frac{c_{ji}(\boldsymbol{\pi}^*)}{c_{ji}(\boldsymbol{\pi})} < 1.$$

Because  $v_j < v_i$ , we have  $\pi_j < \pi_i$ . Given (8), we have

$$\pi_j^* < \pi_j < \pi_i < \pi_i^*.$$

Therefore, by Proposition 5, we have

$$\frac{c_{ij}(\boldsymbol{\pi})}{c_{ij}(\boldsymbol{\pi}^*)} \frac{c_{ji}(\boldsymbol{\pi}^*)}{c_{ji}(\boldsymbol{\pi})} > 1,$$

which gives us the desired contradiction.

**Case (3):**  $v_i < v_j$  and  $v_i^* < v_j^*$

The proof is similar to the proof for Case (2) above.

**Case (4):**  $v_i > v_j$  and  $v_i^* < v_j^*$

Given that we are considering the case  $v_i < v_i^*$ ,  $v_j^* < v_j$ , we have

$$v_i < v_i^* < v_j^* < v_j < v_i,$$

which is a contradiction.

This concludes the proof of Theorem 7.

## 5 Conclusion

Our analysis has filled an important gap in the theory of directed search. We have showed that symmetric equilibrium—in heterogeneous and finite markets—is unique. Studying equilibrium in these is challenging because of the externality associated to individual pricing decisions; a change in price by a given seller alters the queue of buyers at competing sellers. Heterogeneity greatly complicates the

analysis because this feedback effect has a dissimilar marginal impact across the population of sellers. The uniqueness result we have reported can be easily extended to heterogeneous “large” economies—where the pricing externality is not operative.

An additional, methodological contribution, of this study is the development of a technique that is helpful to study equilibrium when payoff functions are not globally concave. The technique is based on the use of directional derivatives along equilibrium “price paths” that have some desirable properties. It our hope that such a technique can be of use in other problems where global concavity is an issue.

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# Appendix

## Lemma 1

*Proof of Lemma 1.* Without loss of generality, we may assume that  $i = 1$  and  $j = J$ .

Case 1 ( $J = 2$ ):

$$\frac{\partial \pi_1}{\partial v_1} = \mathcal{H}(\pi_1) \left( -\frac{1}{h_1 + h_J} \right), \text{ and } \frac{\partial \pi_J}{\partial v_J} = \mathcal{H}(\pi_J) \left( -\frac{1}{h_1 + h_J} \right).$$

With  $A_{1J}(\mathbf{v}) = \left( -\frac{1}{h_1 + h_J} \right)$ ,  $a(\pi_i) = \mathcal{H}(\pi_i)$ , and  $B_{1J}(\boldsymbol{\pi}) = 0$ , the lemma is satisfied.

Case 2 ( $J > 2$ ):

$$\begin{aligned} \frac{\partial \pi_1}{\partial v_1} &= -\mathcal{H}'(\pi_J) \mathcal{H}(\pi_1) v_J \frac{\sum_{2 \leq s \leq J-1} \prod_{k \neq s} h_k}{\sum_{1 \leq s \leq J} \prod_{k \neq s} h_k} - \mathcal{H}(\pi_1) \frac{h_2 \cdots h_{J-1}}{\sum_{1 \leq s \leq J} \prod_{k \neq s} h_k} \\ &= \left\{ -\mathcal{H}'(\pi_J) \mathcal{H}(\pi_1) v_J \left( -\frac{1}{h_2} - \cdots - \frac{1}{h_{J-1}} \right) + \mathcal{H}(\pi_1) \right\} \left( -\frac{h_2 \cdots h_{J-1}}{\sum_{1 \leq s \leq J} \prod_{k \neq s} h_k} \right) \\ &= \tilde{c}_{1J}(\mathbf{v}) A_{1J}(\mathbf{v}), \end{aligned}$$

where

$$\begin{aligned} A_{1J}(\mathbf{v}) &:= \left( -\frac{h_2 \cdots h_{J-1}}{\sum_{1 \leq s \leq J} \prod_{k \neq s} h_k(\mathbf{v})} \right) \\ \tilde{c}_{1J}(\mathbf{v}) &:= -\mathcal{H}'(\pi_J) \mathcal{H}(\pi_1) v_J \left( -\frac{1}{h_2(\mathbf{v})} - \cdots - \frac{1}{h_{J-1}(\mathbf{v})} \right) + \mathcal{H}(\pi_1) \\ &= -\frac{\mathcal{H}'(\pi_J)}{(\mathcal{H}(\pi_J))^2} \mathcal{H}(\pi_1) \mathcal{H}(\pi_J) \left( -\frac{\mathcal{H}(\pi_J) v_J}{\mathcal{H}'(\pi_2) v_2} - \cdots - \frac{\mathcal{H}(\pi_J) v_J}{\mathcal{H}'(\pi_{J-1}) v_{J-1}} \right) + \mathcal{H}(\pi_1). \end{aligned}$$

Denote

$$a(\pi_i) := -\frac{\mathcal{H}'(\pi_i)}{(\mathcal{H}(\pi_i))^2} > 0$$

$$B_{1J}(\boldsymbol{\pi}) := \mathcal{H}(\pi_1)\mathcal{H}(\pi_J) \left( -\frac{\mathcal{H}(\pi_2)}{\mathcal{H}'(\pi_2)} - \dots - \frac{\mathcal{H}(\pi_{J-1})}{\mathcal{H}'(\pi_{J-1})} \right) > 0,$$

Define

$$c_{1J}(\boldsymbol{\pi}) := a(\pi_J)B_{1J}(\boldsymbol{\pi}) + \mathcal{H}(\pi_1).$$

We have

$$\begin{aligned} \frac{\partial \pi_1}{\partial v_1} &= (a(\pi_J)B_{1J}(\boldsymbol{\pi}) + \mathcal{H}(\pi_1))A_{1J}(\mathbf{v}) \\ &= c_{1J}(\boldsymbol{\pi})A_{1J}(\mathbf{v}). \end{aligned}$$

Note that

$$A_{1J}(\mathbf{v}) = A_{J1}(\mathbf{v}), \text{ and } B_{1J}(\boldsymbol{\pi}) = B_{J1}(\boldsymbol{\pi}).$$

Moreover

$$B_{1J}(\boldsymbol{\pi}) = \mathcal{H}(\pi_1)\mathcal{H}(\pi_J) \left( -\frac{\mathcal{H}(\pi_2)}{\mathcal{H}'(\pi_2)} - \frac{\mathcal{H}(\pi_3)}{\mathcal{H}'(\pi_3)} - \dots - \frac{\mathcal{H}(\pi_{J-1})}{\mathcal{H}'(\pi_{J-1})} \right),$$

$$B_{2J}(\boldsymbol{\pi}) = \mathcal{H}(\pi_2)\mathcal{H}(\pi_J) \left( -\frac{\mathcal{H}(\pi_1)}{\mathcal{H}'(\pi_1)} - \frac{\mathcal{H}(\pi_3)}{\mathcal{H}'(\pi_3)} - \dots - \frac{\mathcal{H}(\pi_{J-1})}{\mathcal{H}'(\pi_{J-1})} \right).$$

If  $v_1 < v_2$ , then  $\pi_1 < \pi_2$ ,  $\mathcal{H}(\pi_1) > \mathcal{H}(\pi_2)$ , and

$$-\frac{1}{\mathcal{H}'(\pi_2)} > -\frac{1}{\mathcal{H}'(\pi_1)}.$$

Hence  $B_{1J}(\boldsymbol{\pi}) > B_{2J}(\boldsymbol{\pi})$ . Therefore, for any  $v_i$ ,  $2 \leq i < J$  with  $v_1 < v_i$ , we have

$$B_{1J}(\boldsymbol{\pi}) > B_{iJ}(\boldsymbol{\pi}).$$

Similarly  $B_{J1}(\boldsymbol{\pi}) < B_{i1}(\boldsymbol{\pi})$  for  $v_J > v_i$ , hence we can derive

$$B_{1J}(\boldsymbol{\pi}) = B_{J1}(\boldsymbol{\pi}) < B_{i1}(\boldsymbol{\pi}) = B_{1i}(\boldsymbol{\pi}).$$

By convexity of  $\mathcal{H}^{-1}(\pi_i)$  (see [2]),

$$\frac{\partial a(\pi_i)}{\partial \pi_i} = \frac{\partial}{\partial \pi_i} \left( -\frac{\mathcal{H}'(\pi_i)}{(\mathcal{H}(\pi_i))^2} \right) = -\frac{\mathcal{H}''(\pi_i)\mathcal{H}(\pi_i) - 2(\mathcal{H}'(\pi_i))^2}{(\mathcal{H}(\pi_i))^3} > 0.$$

Therefore, if  $v_1 \leq v_J$ , then  $\pi_1 < \pi_J$ ,  $a(\pi_1) < a(\pi_J)$ ,  $\mathcal{H}(\pi_1) > \mathcal{H}(\pi_J)$ , and  $c_{1J}(\boldsymbol{\pi}) > c_{J1}(\boldsymbol{\pi})$ .  $\square$

### Proposition 3

*Proof of Proposition 3.* Without loss of generality, we may show it for  $j = 1$ .

Note that

$$v_1 \frac{\partial \pi_1(\mathbf{v})}{\partial v_1} + \dots + v_J \frac{\partial \pi_1(\mathbf{v})}{\partial v_J} = 0, \text{ for any } \mathbf{v} \in B(\mathbf{v}^*) \subset \mathbb{R}_+^J. \quad (11)$$

Hence we have (5) with  $\lambda_i = 1$ , for all  $i = 2, \dots, J$ . By differentiating the equation (11) with respect to  $v_1$ , omitting the argument  $\mathbf{v}$ , we have

$$\frac{\partial}{\partial v_1} \left( v_1 \frac{\partial \pi_1}{\partial v_1} + \dots + v_J \frac{\partial \pi_1}{\partial v_J} \right) = 0,$$

i.e. by changing the order of differentiation, we have

$$v_1 \frac{\partial}{\partial v_1} \left( \frac{\partial \pi_1}{\partial v_1} \right) + \dots + v_J \frac{\partial}{\partial v_J} \left( \frac{\partial \pi_1}{\partial v_1} \right) = -\frac{\partial \pi_1}{\partial v_1} < 0, \quad (12)$$

Since from the previous results we have  $\frac{\partial \pi_1}{\partial v_1} > 0$ . Therefore (6) is satisfied with  $\lambda_i = 1$ , for all  $i = 2, \dots, J$ . Now we consider cases with some  $\lambda_i = 0$  for some  $i \neq 1$ . If  $\lambda_i = 0$ , for all  $i = 2, \dots, J$ , then (5) and (6) are direct. Therefore we only need to show these inequalities with some  $\lambda_l = 0$  and  $\lambda_m = 1$  for some

$l, m = 2, \dots, J$ . Without loss of generality, we may assume that

$$\begin{cases} \lambda_i = 0 & , \text{ for } i = 2, \dots, k-1, \\ \lambda_i = 1 & , \text{ for } i = k, \dots, J. \end{cases}$$

From (11)

$$v_1 \frac{\partial \pi_1}{\partial v_1} + \dots + v_{k-1} \frac{\partial \pi_1}{\partial v_{k-1}} = -v_k \frac{\partial \pi_1}{\partial v_k} - \dots - v_J \frac{\partial \pi_1}{\partial v_J} > 0,$$

which is true since  $\frac{\partial \pi_1}{\partial v_j} < 0$  for all  $j > 1$ . Hence we only need to show

$$v_1 \frac{\partial}{\partial v_1} \left( \frac{\partial \pi_1}{\partial v_1} \right) + \dots + v_{k-1} \frac{\partial}{\partial v_{k-1}} \left( \frac{\partial \pi_1}{\partial v_1} \right) < 0.$$

From (12), we have

$$\begin{aligned} v_1 \frac{\partial}{\partial v_1} \left( \frac{\partial \pi_1}{\partial v_1} \right) + \dots + v_{k-1} \frac{\partial}{\partial v_{k-1}} \left( \frac{\partial \pi_1}{\partial v_1} \right) \\ = -\frac{\partial \pi_1}{\partial v_1} - v_J \frac{\partial}{\partial v_J} \left( \frac{\partial \pi_1}{\partial v_1} \right) - \dots - v_k \frac{\partial}{\partial v_k} \left( \frac{\partial \pi_1}{\partial v_1} \right). \end{aligned}$$

Therefore it is enough to show that

$$\frac{\partial \pi_1}{\partial v_1} + v_J \frac{\partial}{\partial v_J} \left( \frac{\partial \pi_1}{\partial v_1} \right) + \dots + v_k \frac{\partial}{\partial v_k} \left( \frac{\partial \pi_1}{\partial v_1} \right) > 0.$$

Using the expression for  $\frac{\partial \pi_1}{\partial v_1}$  we get  $v_i \frac{\partial^2 \pi_1}{\partial v_i \partial v_1} =$

$$\frac{-\mathcal{H}'(\pi_1) v_i \frac{\partial \pi_1}{\partial v_i} \left( h_1 + \frac{1}{\frac{1}{h_2} + \dots + \frac{1}{h_J}} \right)}{\left( h_1 + \frac{1}{\frac{1}{h_2} + \dots + \frac{1}{h_J}} \right)^2} + \frac{\mathcal{H}(\pi_1) v_i \left( \frac{\partial h_1}{\partial v_i} + \frac{\frac{\partial h_2}{\partial v_i} \frac{1}{h_2^2} + \dots + \frac{\partial h_J}{\partial v_i} \frac{1}{h_J^2}}{\left( \frac{1}{h_2} + \dots + \frac{1}{h_J} \right)^2} \right)}{\left( h_1 + \frac{1}{\frac{1}{h_2} + \dots + \frac{1}{h_J}} \right)^2}.$$

So we have

$$\begin{aligned}
& \left( \frac{\partial \pi_1}{\partial v_1} + v_J \frac{\partial^2 \pi_1}{\partial v_J \partial v_1} + \dots + v_k \frac{\partial^2 \pi_1}{\partial v_k \partial v_1} \right) \left( h_1 + \frac{1}{\frac{1}{h_2} + \dots + \frac{1}{h_J}} \right)^2 \\
&= \frac{\partial \pi_1}{\partial v_1} \left( h_1 + \frac{1}{\frac{1}{h_2} + \dots + \frac{1}{h_J}} \right)^2 \\
&+ \sum_{i=k, \dots, J} \left\{ -\mathcal{H}'(\pi_1) v_i \frac{\partial \pi_1}{\partial v_i} \left( h_1 + \frac{1}{\frac{1}{h_2} + \dots + \frac{1}{h_J}} \right) \right. \\
&+ \left. \mathcal{H}(\pi_1) v_i \left( \frac{\partial h_1}{\partial v_i} + \frac{\frac{\partial h_2}{\partial v_i} \frac{1}{h_2^2} + \dots + \frac{\partial h_J}{\partial v_i} \frac{1}{h_J^2}}{\left( \frac{1}{h_2} + \dots + \frac{1}{h_J} \right)^2} \right) \right\}, \\
&= \frac{\partial \pi_1}{\partial v_1} \left\{ h_1^2 + 2 \frac{h_1}{\frac{1}{h_2} + \dots + \frac{1}{h_J}} + \left( \frac{1}{\frac{1}{h_2} + \dots + \frac{1}{h_J}} \right)^2 \right\} \\
&- \sum_{i=k, \dots, J} \mathcal{H}'(\pi_1) v_i \frac{\partial \pi_1}{\partial v_i} \left( h_1 + \frac{1}{\frac{1}{h_2} + \dots + \frac{1}{h_J}} \right) \\
&+ \sum_{i=k, \dots, J} \mathcal{H}(\pi_1) v_i \left( \frac{\partial h_1}{\partial v_i} + \frac{\mathcal{H}''(\pi_i) \frac{\partial \pi_i}{\partial v_i} v_i \frac{1}{h_i^2}}{\left( \frac{1}{h_2} + \dots + \frac{1}{h_J} \right)^2} + \frac{\mathcal{H}'(\pi_i) \frac{1}{h_i^2}}{\left( \frac{1}{h_2} + \dots + \frac{1}{h_J} \right)^2} \right) \\
&+ \sum_{i=k, \dots, J} \mathcal{H}(\pi_1) v_i \left( \frac{\sum_{\substack{j=2, \dots, J \\ j \neq i}} \frac{\partial h_j}{\partial v_i} \frac{1}{h_j^2}}{\left( \frac{1}{h_2} + \dots + \frac{1}{h_J} \right)^2} \right).
\end{aligned}$$

It can be shown that the sum of the first terms in the first three lines is positive; The second term in the third line is positive; The sum of the last terms in the first three lines is positive; And, the sum of all remaining terms is positive. (The relevant algebraic manipulations are available from the authors upon request).

Therefore we have the desired inequality

$$\frac{\partial \pi_1}{\partial v_1} + v_J \frac{\partial}{\partial v_J} \left( \frac{\partial \pi_1}{\partial v_1} \right) + \dots + v_k \frac{\partial}{\partial v_k} \left( \frac{\partial \pi_1}{\partial v_1} \right) > 0.$$

□

## Proposition 5

*Proof of Proposition 5.* We are interested in studying the case in which at least two sellers, say  $i$  and  $j$ , are both in  $\Gamma$  and in  $\Gamma^*$ . Without loss of generality, we assume  $i = 1$ ,  $j = J$  and

- $\Gamma = \{1, \dots, K\} \cup \{J\}$ ,  $1 \leq K < J$ ,
- $\Gamma^* = \{1\} \cup \{L, \dots, J\}$ ,  $1 < L \leq J$ .

Define sets of other sellers in the market,

$$\Gamma_{1,J} := \Gamma \setminus \{1, J\}, \text{ and } \Gamma_{1,J}^* := \Gamma^* \setminus \{1, J\}.$$

Case 1: Suppose  $\Gamma_{1,J}, \Gamma_{1,J}^* \neq \emptyset$ . By Lemma 1

$$c_{1J}(\boldsymbol{\pi}) = \mathcal{H}(\pi_1) \mathcal{H}(\pi_J) \frac{\mathcal{H}'(\pi_J)}{(\mathcal{H}(\pi_J))^2} \left( \frac{\mathcal{H}(\pi_J)}{\mathcal{H}'(\pi_J)} + \sum_{j \in \Gamma_{1,J}} \frac{\mathcal{H}(\pi_j)}{\mathcal{H}'(\pi_j)} \right),$$

$$c_{J1}(\boldsymbol{\pi}) = \mathcal{H}(\pi_1) \mathcal{H}(\pi_J) \frac{\mathcal{H}'(\pi_1)}{(\mathcal{H}(\pi_1))^2} \left( \frac{\mathcal{H}(\pi_1)}{\mathcal{H}'(\pi_1)} + \sum_{j \in \Gamma_{1,J}} \frac{\mathcal{H}(\pi_j)}{\mathcal{H}'(\pi_j)} \right).$$

Hence

$$\begin{aligned}
& \frac{1}{\mathcal{H}(\pi_1)\mathcal{H}(\pi_J)\mathcal{H}(\pi_1^*)\mathcal{H}(\pi_J^*)} c_{1J}(\boldsymbol{\pi}) c_{J1}(\boldsymbol{\pi}^*) = \\
& \quad \frac{\mathcal{H}'(\pi_J)}{(\mathcal{H}(\pi_J))^2} \left( \frac{\mathcal{H}(\pi_J)}{\mathcal{H}'(\pi_J)} + \sum_{j \in \Gamma_{1,J}} \frac{\mathcal{H}(\pi_j)}{\mathcal{H}'(\pi_j)} \right) \frac{\mathcal{H}'(\pi_1^*)}{(\mathcal{H}(\pi_1^*))^2} \left( \frac{\mathcal{H}(\pi_1^*)}{\mathcal{H}'(\pi_1^*)} + \sum_{i \in \Gamma_{1,J}^*} \frac{\mathcal{H}(\pi_i^*)}{\mathcal{H}'(\pi_i^*)} \right), \\
& \frac{1}{\mathcal{H}(\pi_1)\mathcal{H}(\pi_J)\mathcal{H}(\pi_1^*)\mathcal{H}(\pi_J^*)} c_{1J}(\boldsymbol{\pi}^*) c_{J1}(\boldsymbol{\pi}) = \\
& \quad \frac{\mathcal{H}'(\pi_J^*)}{(\mathcal{H}(\pi_J^*))^2} \left( \frac{\mathcal{H}(\pi_J^*)}{\mathcal{H}'(\pi_J^*)} + \sum_{i \in \Gamma_{1,J}^*} \frac{\mathcal{H}(\pi_i^*)}{\mathcal{H}'(\pi_i^*)} \right) \frac{\mathcal{H}'(\pi_1)}{(\mathcal{H}(\pi_1))^2} \left( \frac{\mathcal{H}(\pi_1)}{\mathcal{H}'(\pi_1)} + \sum_{j \in \Gamma_{1,J}} \frac{\mathcal{H}(\pi_j)}{\mathcal{H}'(\pi_j)} \right).
\end{aligned}$$

One can show that

$$\begin{aligned}
& \left( -\frac{\mathcal{H}'(\pi_J)}{(\mathcal{H}(\pi_J))^2} \right) \left( -\frac{\mathcal{H}(\pi_J)}{\mathcal{H}'(\pi_J)} - \sum_{j \in \Gamma_{1,J}} \frac{\mathcal{H}(\pi_j)}{\mathcal{H}'(\pi_j)} \right) \\
& \quad \times \left( -\frac{\mathcal{H}'(\pi_1^*)}{(\mathcal{H}(\pi_1^*))^2} \right) \left( -\frac{\mathcal{H}(\pi_1^*)}{\mathcal{H}'(\pi_1^*)} - \sum_{i \in \Gamma_{1,J}^*} \frac{\mathcal{H}(\pi_i^*)}{\mathcal{H}'(\pi_i^*)} \right) \\
& \qquad \qquad \qquad > \\
& \left( -\frac{\mathcal{H}'(\pi_1)}{(\mathcal{H}(\pi_1))^2} \right) \left( -\frac{\mathcal{H}(\pi_1)}{\mathcal{H}'(\pi_1)} - \sum_{j \in \Gamma_{1,J}} \frac{\mathcal{H}(\pi_j)}{\mathcal{H}'(\pi_j)} \right) \\
& \quad \times \left( -\frac{\mathcal{H}'(\pi_J^*)}{(\mathcal{H}(\pi_J^*))^2} \right) \left( -\frac{\mathcal{H}(\pi_J^*)}{\mathcal{H}'(\pi_J^*)} - \sum_{i \in \Gamma_{1,J}^*} \frac{\mathcal{H}(\pi_i^*)}{\mathcal{H}'(\pi_i^*)} \right),
\end{aligned} \tag{13}$$

which completes the proof of this part. (Details of the calculations are available upon request).

Case 2:  $\Gamma_{1,J} = \emptyset$  or  $\Gamma_{1,J}^* = \emptyset$ . In this case, we have, respectively,

$$-\sum_{j \in \Gamma_{1,J}} \frac{\mathcal{H}(\pi_j)}{\mathcal{H}'(\pi_j)} = 0 \text{ or } -\sum_{i \in \Gamma_{1,J}^*} \frac{\mathcal{H}(\pi_i^*)}{\mathcal{H}'(\pi_i^*)} = 0.$$

and the proof is the same as for case 1. □



## Deriving the inequalities in (8)

Consider two candidate equilibria such that  $v_i^* > v_i$ , and  $v_j^* < v_j$  for some pair  $(i, j)$ . We wish to prove that we can always choose the pair  $(i, j)$  so that the inequalities  $\pi_i < \pi_i^*$  and  $\pi_j > \pi_j^*$  are true.

The proof is by contradiction. Suppose that every seller who offers lower promised utility at  $\mathbf{v}$  relative to  $\mathbf{v}^*$ , expects at least as much demand at  $\mathbf{v}$  as he would expect at  $\mathbf{v}^*$ . Pick any such seller, and call this seller  $a$ , i.e., we have  $v_a < v_a^*$  and  $\pi_a(\mathbf{v}) \geq \pi_a(\mathbf{v}^*)$ . Then, there must be at least one seller, call her seller  $b$ , who does not offer lower promised utility at  $\mathbf{v}$  and does not expect higher demand at  $\mathbf{v}$  relative to  $\mathbf{v}^*$ . That is, we must have  $v_b \geq v_b^*$  and  $\pi_b(\mathbf{v}) \leq \pi_b(\mathbf{v}^*)$ .

Since the conditional probability  $\mathcal{H}$  that a buyer trades is decreasing in  $\pi$ , we have that

$$\frac{\mathcal{H}(\pi_a(\mathbf{v}))}{\mathcal{H}(\pi_b(\mathbf{v}))} = \frac{v_b}{v_a} \quad \Rightarrow \quad \frac{\mathcal{H}(\pi_a(\mathbf{v}^*))}{\mathcal{H}(\pi_b(\mathbf{v}^*))} > \frac{v_b^*}{v_a^*}$$

which contradicts the requirement that in symmetric equilibrium buyers' payoffs satisfy

$$\frac{\mathcal{H}(\pi_a(\mathbf{v}^*))v_a^*}{\mathcal{H}(\pi_b(\mathbf{v}^*))v_b^*} = 1.$$