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Optimal liquidation strategies and their implications

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Abstract

This paper studies optimal liquidation when the selling price depends on the rate of liquidation, transaction time, volume, and the asset's intrinsic value. A generic closed-form solution for maximizing the discounted liquidation proceeds is derived. To obtain financial insights, three parametric specifications that proxy for increasingly realistic market conditions are examined. In our framework, maximizing liquidation proceeds and minimizing liquidity costs are equivalent. The optimal strategies imply more rapid liquidations in less liquid markets. We also show that volatility is stochastic when market liquidity is unpredictable.

JEL classification: C61; G10; G33

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1. Introduction

This paper derives a general closed-form optimal liquidation strategy by solving a Hamilton–Jacobi–Bellman equation. To capture trading friction and the price

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impact of liquidation, we define a liquidity discount as the difference between the selling price and the asset's intrinsic value. This liquidity discount may depend on previous transactions as well as the intrinsic value of the asset itself. We explore the implications of our optimal trading strategies on the resulting liquidity costs, the effect of stochastic liquidity on volatility, and more importantly, the optimal rate of liquidation under different market conditions.

Optimal selling strategies have been considered by several authors. For example, Subramanian and Jarrow (2001) derive a condition for block trades to be optimal. Almgren and Chriss (2000) develop a mean-variance framework for liquidity costs while Bertsimas and Lo (1998) minimize expected liquidation costs. Although the objective function in this paper maximizes the expected value of the discounted liquidation proceeds, our optimal liquidation strategy also minimizes the expected liquidity costs. As a special case, we recover the optimal trading strategy of Bertsimas and Lo (1998).

The contribution of this paper is a theoretical framework for analyzing liquidation that conforms with empirical evidence. Consistent with Chan and Lakonishok's (1995) evidence regarding the division of large institutional trades into multiple trades, our optimal trading strategies involve liquidation over a period of time. Furthermore, we find it is optimal to liquidate more rapidly in less liquid markets. Empirical evidence for this behavior is documented in Keim and Madhavan (1995). Finally, as empirically documented in Jones et al. (1994), our analysis also suggests that higher levels of trading activity result in stochastic volatility when market liquidity is unpredictable.

The next section introduces the general formulation of our optimal liquidation problem along with its solution. Section 3 considers three parametric liquidity specifications and solves for their corresponding optimal liquidation strategies. Section 4 then proves the equivalence between maximizing the liquidation proceeds and minimizing the liquidation costs for the optimal strategies. Section 5 demonstrates that stochastic liquidity results in stochastic volatility. Section 6 concludes.

2. Liquidity specification and model formulation

We consider the problem of liquidating an initial holding $X_0 > 0$ over a time period $t \in [0, T]$ such that

$$X_T = 0.$$

A trader's remaining holding at time t is denoted by X_t and defined as

$$X_t = X_0 + \int_0^t u_s \, \mathrm{d}s, \tag{1}$$

where u_s is the trading strategy. Following the standard convention, u_t is *negative* for seller-initiated trades,

$$u_t < 0.$$

2.1. Liquidity specification

We denote S_t as the intrinsic value of the asset. Under the risk neutral measure, we assume S_t evolves as a geometric Brownian motion

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{2}$$

and is independent of the liquidation trading strategy. We assume that the transaction price is given by

$$P(t, S_t, X_t, u_t) = (1 + f(t, S_t, X_t)u_t)S_t, \tag{3}$$

where $f(t, S_t, X_t) > 0$ is the liquidity function that provides a quantitative characterization of the liquidity available to the liquidation trader at time t. The term $f(t, S_t, X_t)\,u_t < 0$ represents the liquidity discount in percentage terms demanded by the buyers. In other words, it is the price discount with respect to the asset's value that the liquidation trader has to offer when selling. Its dependence on S_t indicates that the liquidity available in the market is stochastic. In our framework, lower liquidity corresponds to larger values of $f(t, S_t, X_t)$. Furthermore, the greater the liquidation rate $|u_t|$, the lower is the transaction price $P(t, S_t, X_t, u_t)$ with respect to S_t.

Similar to Bertsimas and Lo (1998), our reduced-form setting does not have S_t dependent on the liquidation strategy. Eq. (2) is unaffected by transactions as trading is motivated by liquidity rather than information. The relationship $P(t, S_t, X_t, u_t) < S_t$ when selling is consistent with a common observation that prices *tend to* decrease when institutional traders liquidate their holdings. Moreover, when the liquidation is complete, $P(t, S_t, 0, 0)$ equals S_t, which implies that the price rebounds. Evidence for this behavior is documented in, for example, Holthausen et al. (1987) as well as Chan and Lakonishok (1993).

2.2. Formulation of optimal liquidation strategy

We formulate the problem of finding an optimal liquidation strategy u_t^* as a stochastic control problem. To begin, assume the existence of a riskless money market account earning a continuously compounded rate r. The proceeds from liquidating $|u_t\,\mathrm{d}t|$ units of asset in the time interval $[t, t + \mathrm{d}t)$ at the sale price $P(t, S_t, X_t, u_t)$ equals

$$-P(t, S_t, X_t, u_t)u_t\,\mathrm{d}t.$$

Therefore, the expected value of the discounted proceeds is given by

$$-\mathrm{E}\left[\int_0^T u_t(1 + f(t, S_t, X_t)u_t)S_t\,\mathrm{e}^{-rt}\,\mathrm{d}t\right]. \tag{4}$$

The objective of the trader is to maximize the expected value of the discounted proceeds from selling X_0 units by time T.

Admissible trading strategies u_t are assumed to be continuous processes such that

$$1 + f(t, S_t, X_t)u_t > 0 \quad \forall t, S_t, X_t, u_t.$$

Since the liquidity discount is small under normal market conditions, this linearized version of $\exp\{f(t, S_t, X_t)u_t\}$ is useful for attaining analytical tractability. The optimal trading strategies that are analytically obtained can then be used for ensuring that transaction prices are positive by requiring T to be larger than some function of X_0 (see Section 3.4).

Let $\mathscr{U}_t$ be the set of all admissible trading strategies with time $t \geqslant 0$. The stochastic control problem is formulated as

$$\sup_{u_t \in \mathscr{U}_t} -\mathrm{E}\left[\int_0^T u_t(1 + f(t, S_t, X_t)u_t)S_t\, \mathrm{e}^{-rt}\, \mathrm{d}t\right]$$

$$\text{subject to} \quad \int_0^T u_t\, \mathrm{d}t = -X_0,$$

$$\mathrm{d}X_t = u_t\, \mathrm{d}t,$$

$$\frac{\mathrm{d}S_t}{S_t} = r\, \mathrm{d}t + \sigma\, \mathrm{d}W_t. \tag{5}$$

Since we are maximizing the discounted *expected* proceeds from the sale, conditional on a specified function $f(t, S_t, X_t)$, the resulting optimal trading strategy will not be random.

The value function for a given trading strategy u_t, $J(t, S_t, X_t, u_t) : R_+ \times R_+ \times R \times \mathscr{U} \mapsto R$, is defined as

$$J(t, S_t, X_t, u_t) = -\mathrm{E}\left[\int_t^T u_w(1 + f(w, S_w, X_w)u_w)S_w\, \mathrm{e}^{-rw}\, \mathrm{d}w \,\middle|\, \int_t^T u_w\, \mathrm{d}w = -X_t\right],$$

while the optimal value function $V(t, S_t, X_t) : R_+ \times R_+ \times R \mapsto R$ is defined as

$$V(t, S_t, X_t) = \sup_{u_t \in \mathscr{U}_t} J(t, S_t, X_t, u_t).$$

The above constrained stochastic control problem is generally difficult to solve. However, it is well known that the unconstrained optimal value function $V(t, S_t, X_t)$ satisfies the following HJB equation:

$$\frac{\partial V}{\partial t} + \sup_{u \in \mathscr{U}_t}[-u_t S_t(1 + f(t, S_t, X_t)u_t)\, \mathrm{e}^{-rt} + \mathscr{A}^u(V(t, S_t, X_t))] = 0, \tag{6}$$

where $\mathscr{A}^u$ is the differential operator:[1]

$$\mathscr{A}^u(V(t, S_t, X_t)) = u_t \frac{\partial V(t, S_t, X_t)}{\partial X_t} + rS_t \frac{\partial V(t, S_t, X_t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(t, S_t, X_t)}{\partial S_t^2}.$$

We begin by solving the unconstrained problem and then derive the optimal solution from all possible solutions to the HJB equation such that the constraint is

[1]See, for example, Duffie (2001).

satisfied. The validity of this approach requires the optimal solution to be given by the HJB equation implied by the unconstrained problem. At any point in time, if the asset holding reduces to zero, the value function $V(t, S_t, 0) = 0$. This economic consideration constitutes a boundary condition associated with the stochastic control problem.

Theorem 2.1. *For a given liquidity function $f(t, S_t, X_t)$, the optimal trading strategy u_t^* is*

$$u_t^* = \frac{1}{2f(t, S_t, X_t)} \left(\frac{e^{rt}}{S_t} \frac{\partial V}{\partial X_t} - 1 \right), \tag{7}$$

where $V \equiv V(t, S_t, X_t)$ solves the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{e^{rt}}{4S_t f(t, S_t, X_t)} \left(\frac{\partial V}{\partial X_t} - S_t e^{-rt} \right)^2 + rS_t \frac{\partial V}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} = 0, \tag{8}$$

with the boundary condition

$$V(t, S_t, 0) = 0 \quad \forall t, S_t. \tag{9}$$

Proof. Since $f(t, S_t, X_t) > 0$, the inner maximization problem in Eq. (6) is a quadratic concave programming problem with respect to u_t. Applying the first-order condition to the inner maximization problem in Eq. (6) yields the optimal trading strategy u_t^* given in Eq. (7). Substituting the optimal solution u_t^* into Eq. (6) implies that $V(t, S_t, X_t)$ satisfies Eq. (8) with the boundary condition in Eq. (9). □

2.3. Solution for optimal trading strategy

By Theorem 2.1, the optimal trading strategy that solves the nonlinear partial differential equation in Eq. (8) has three variables, namely t, S_t, and X_t. In general, Eq. (8) is intractable for an arbitrary liquidity function $f(t, S_t, X_t)$. However, if $f(t, S_t, X_t)$ manifests some economically reasonable properties, then closed-form solutions are obtainable.

We consider liquidity functions of the following form:

$$f(t, S_t, X_t) = h(t) g(X_t) \left(\frac{S_t}{S_0} \right)^{\beta}, \tag{10}$$

where $h(t)$ and $g(X_t)$ are strictly positive functions whose first-order derivatives exist within their respective domains. Having S_t in the liquidity function is motivated by the fact that market liquidity is typically stochastic, as documented by Goldstein and Kavajecz (2000); Chordia et al. (2001) as well as many others. Therefore, it is important to ascertain how stochastic liquidity affects optimal trading strategies. The parameter β characterizes the degree of randomness in market liquidity.

For reasons that will become apparent later, we also assume the following technical conditions:

$$\left|\int_0^{X_t} \sqrt{g(x)}\,\mathrm{d}x\right| < \infty,$$

$$\left|\int_0^{T} \frac{\mathrm{e}^{kt}}{h(t)}\,\mathrm{d}t\right| < \infty,$$

where k is defined as

$$k \equiv -\beta(r + \tfrac{1}{2}(\beta+1)\sigma^2).$$

Given the liquidity function in Eq. (10) and the above technical conditions, the following theorem provides the starting point of our analysis.

Theorem 2.2. *If* $f(t, S_t, X_t)$ *is specified as in Eq.* (10), *then the optimal trading strategy* $u_t^* \equiv u^*(t, X_t)$ *equals*

$$u_t^* = -\frac{\mathrm{e}^{kt}\int_0^{X_t}\sqrt{g(x)}\,\mathrm{d}x}{h(t)\sqrt{g(X_t)}\int_t^T(\mathrm{e}^{ks}/h(s))\,\mathrm{d}s}. \tag{11}$$

The optimal expected value of the liquidation proceeds is

$$V(t, S_t, X_t) = S_t X_t\,\mathrm{e}^{-rt} - \frac{S_0\,\mathrm{e}^{(k-r)t}\left(\int_0^{X_t}\sqrt{g(x)}\,\mathrm{d}x\right)^2}{\int_t^T(\mathrm{e}^{ks}/h(s))\,\mathrm{d}s}\left(\frac{S_t}{S_0}\right)^{\beta+1}. \tag{12}$$

Proof. Conjecture a solution to Eq. (8) of the form

$$V(t, S_t, X_t) = S_t X_t\,\mathrm{e}^{-rt} + \left(\frac{S_t}{S_0}\right)^{\beta+1} M(t)N(X_t),$$

where $M(\cdot)$ and $N(\cdot)$ are differentiable functions. By the verification theorem, we need only to verify that $V(t, S_t, X_t)$ in Eq. (12) is a solution to Eq. (8). Substituting the relevant partial derivatives of $V(t, S_t, X_t)$ into Eq. (8) yields

$$N(X_t)\left[\frac{\mathrm{d}M(t)}{\mathrm{d}t} + (r-k)M(t)\right] + \frac{\mathrm{e}^{rt}}{4S_0 h(t)g(X_t)}M^2(t)\left(\frac{\mathrm{d}N(X_t)}{\mathrm{d}X_t}\right)^2 = 0, \tag{13}$$

which implies that there exists a constant $D > 0$ such that

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} + (r-k)M(t) + \frac{D\,\mathrm{e}^{rt}}{S_0 h(t)}M^2(t) = 0, \tag{14}$$

$$\left(\frac{\mathrm{d}N(X_t)}{\mathrm{d}X_t}\right)^2 - 4DN(X_t)g(X_t) = 0. \tag{15}$$

Solving the first ordinary differential in Eq. (14) yields

$$M(t) = \frac{S_0 M(0)\,\mathrm{e}^{(k-r)t}}{S_0 + DM(0)\int_0^t(\mathrm{e}^{ks}/h(s))\,\mathrm{d}s}, \tag{16}$$

with D and $M(0)$ to be determined by the boundary condition which is given as Eq. (9). The value function $V(t, S_t, 0) = 0$ for all t and S_t implies that $N(0) = 0$. Therefore, the solution of Eq. (15) is

$$N(X_t) = D\left(\int_0^{X_t} \sqrt{g(x)}\,\mathrm{d}x\right)^2. \tag{17}$$

Hence, the optimal solution in Eq. (7) becomes

$$u_t^* = \frac{DM(0)\,\mathrm{e}^{kt}\int_0^{X_t}\sqrt{g(x)}\,\mathrm{d}x}{h(t)\sqrt{g(X_t)}\left(S_0 + DM(0)\int_0^t(\mathrm{e}^{ks}/h(s))\,\mathrm{d}s\right)}. \tag{18}$$

To determine the constants, D and $M(0)$, we use the fact that $\mathrm{d}X_t = u_t^*\,\mathrm{d}t$ and rewrite Eq. (18) as

$$\frac{\sqrt{g(X_t)}}{\int_0^{X_t}\sqrt{g(x)}\,\mathrm{d}x}\frac{\mathrm{d}X_t}{\mathrm{d}t} = \frac{DM(0)\,\mathrm{e}^{kt}}{h(t)(S_0 + DM(0)\int_0^t(\mathrm{e}^{ks}/h(s))\,\mathrm{d}s)}.$$

Integrating both sides leads to

$$\int_0^{X_t}\sqrt{g(x)}\,\mathrm{d}x = C\left(S_0 + DM(0)\int_0^t \frac{\mathrm{e}^{ks}}{h(s)}\,\mathrm{d}s\right),$$

where C is a non-zero constant. Since $X_T = 0$, it follows that

$$S_0 + DM(0)\int_0^T \frac{\mathrm{e}^{ks}}{h(s)}\,\mathrm{d}s = 0,$$

which implies

$$DM(0) = -\frac{S_0}{\int_0^T(\mathrm{e}^{ks}/h(s))\,\mathrm{d}s}.$$

Consequently,

$$DM(0)\int_0^t \frac{\mathrm{e}^{ks}}{h(s)}\,\mathrm{d}s = -S_0 - DM(0)\int_t^T \frac{\mathrm{e}^{ks}}{h(s)}\,\mathrm{d}s.$$

Substituting the two functions, $M(t)$ and $N(X_t)$, into the relevant equations yields Eqs. (11) and (12), respectively. □

Proposition 2.1. *The expected value of the discounted proceeds from executing the optimal trading strategy equals*

$$V(0, S_0, X_0) = S_0X_0 - \frac{S_0}{\int_0^T(\mathrm{e}^{ks}/h(s))\,\mathrm{d}s}\left(\int_0^{X_0}\sqrt{g(x)}\,\mathrm{d}x\right)^2. \tag{19}$$

Proof. Eq. (19) is simply the value function, Eq. (12), at time $t = 0$. □

The first term S_0X_0 represents the hypothetical proceeds from selling the entire amount at a given initial price S_0 as a block trade. The second term may be interpreted as the liquidity cost which erodes the liquidation proceeds. In Section 4, we prove that the second term is indeed the liquidity-induced transaction cost.

3. Liquidation in different market conditions

To obtain financial insight into the optimal trading strategies, this section discusses three specific market conditions with increasing complexity but greater corresponding realism. These formulations of market condition are special cases of Eq. (10):

$$\text{Market 1.}\quad f(t, S_t, X_t) = \gamma,$$

$$\text{Market 2.}\quad f(t, S_t, X_t) = \frac{\gamma \sinh^2(\alpha(X_t - X_0))}{t},$$

$$\text{Market 3.}\quad f(t, S_t, X_t) = \frac{\gamma \sinh^2(\alpha(X_t - X_0))}{t}\left(\frac{S_t}{S_0}\right)^{\beta}.$$

The first functional form describes a market in which liquidity is independent of time, past trades, and S_t. Market liquidity is determined only by the parameter $\gamma > 0$, with larger values indicating greater liquidity discounts. Markets 2 and 3 describe trading environments[2] that are more realistic than Market 1. The parameter α is a positive constant that characterizes the market depth. With α exogenously determined, the amount of asset X_0 to be liquidated cannot be very much larger than $1/\alpha$. This is because market depth is limited and excessive selling causes financial distress when the discount $\sinh^2(\alpha(X_t - X_0))u_t/t$ becomes so large in magnitude that the liquidating price $P(t, S_t, X_t, u_t)$ is close to zero.

3.1. *Optimal liquidation with constant price impact*

We begin with the following proposition in the simplest case where the liquidity discount is independent of previous trades and the intrinsic value of the asset.

Proposition 3.1. *The optimal trading strategy when liquidity is independent of past trades and the intrinsic value is*

$$u_t^* = \frac{-X_0}{T},$$

while the expected value of the discounted proceeds equals

$$V(0, S_0, X_0) = S_0 X_0 - \frac{\gamma S_0 X_0^2}{T}.$$

Proof. From Theorem 2.2, the optimal trading strategy is

$$u_t^* = \frac{X_t}{t - T}.$$

[2]It is noteworthy that $\lim_{t\to 0} \sinh^2(\alpha(X_t - X_0))/t \to 0$ as the trading strategy u_t is finite and continuous for all t.

Because $\mathrm{d}X_t = u_t\,\mathrm{d}t$, the optimal trading strategy is the one that sells a constant amount per unit time with the following implication:

$$X_t = a(t - T) \quad \text{for a constant } a.$$

At time $t = 0$, we have $a = -X_0/T$, which implies

$$u_t^* = a = \frac{-X_0}{T}.$$

This solution also satisfies the liquidation constraint. Thus, it is the optimal trading strategy. From Proposition 2.1, the optimal trading strategy generates the proceeds

$$V(0, S_0, X_0) = S_0 X_0 - \frac{\gamma S_0 X_0^2}{T}. \qquad \square$$

Observe that the optimal strategy u_t^* is independent of γ while the expected value of the discounted proceeds is smaller when γ is larger. This parameter characterizes the market's liquidity per unit sold, irrespective of any transaction sequence.

Corollary 3.1. *The holding X_t as a result of executing the optimal trading strategy in Proposition* 3.1 *is*

$$X_t = X_0\left(1 - \frac{t}{T}\right). \tag{20}$$

Proof. This result is obtained from integrating $\mathrm{d}X_t/\mathrm{d}t = -X_0/T$. $\square$

Proposition 3.1 suggests a linear trading strategy that sells the same amount per unit time is optimal. Consequently, the holding X_t is a decreasing linear function of time t. This is intuitive in a market without price impacts from past trades. This result parallels the linear solution of Bertsimas and Lo (1998) found by dynamic programming in discrete time.

3.2. Optimal liquidation with price impact from cumulative volume

We now consider a more realistic specification for liquidity when the cumulative volume traded induces a price impact. The optimal trading strategy for Market 2 is given by the following proposition.

Proposition 3.2. *The optimal trading strategy when liquidity is determined by past trades is*

$$u_t^* = \frac{2(\cosh(\alpha X_0) - 1)t}{\alpha T^2 \sinh(\alpha(X_t - X_0))} \tag{21}$$

with the initial rate of trading

$$u_0^* = -\frac{\sqrt{2(\cosh(\alpha X_0) - 1)}}{\alpha T}.$$

Proof. See Appendix A.

As previously discussed, larger α values are associated with less liquid markets. In these circumstances, our optimal liquidation strategy indicates that traders should sell more units at the beginning of the trading horizon. In other words, the initial rate of liquidation $|u_0^*|$ increases as α increases. This result is consistent with the empirical evidence in Keim and Madhavan (1995) that less liquid markets are associated with more concentrated trading.

The factor $1 + f(t, S_t, X_t)u_t^*$ in the transaction price is a decreasing function of time:

$$1 + \frac{2\gamma(\cosh(\alpha X_0) - 1)}{\alpha T^2} \sinh(\alpha(X_t - X_0)).$$

Since $\sinh(\alpha(X_t - X_0)) < 0$ and $|X_t - X_0|$ becomes larger with time, the difference between the asset's value S_t and the transaction price $P(t, S_t, X_t, u_t^*)$ becomes larger. In other words, liquidity discount in dollars, which is $f(t, S_t, X_t)u_t^* S_t$, becomes larger with time. Therefore, as Proposition 3.2 suggests, it is better to liquidate more rapidly in the beginning.

Corollary 3.2. *The expected value of the discounted proceeds from using the optimal trading strategy in Proposition* 3.2 *equals*

$$V(0, S_0, X_0) = S_0 X_0 - \frac{2\gamma S_0}{\alpha^2 T^2}(\cosh(\alpha X_0) - 1)^2.$$

Proof. Applying Proposition 2.1 yields the expected value of the discounted proceeds. □

Obviously, larger γ values lower the proceeds as in Proposition 3.1. In addition, longer time horizon T results in higher proceeds.

Corollary 3.3. *The holding X_t as a result of executing the optimal trading strategy in Proposition* 3.2 *is*

$$X_t = X_0 - \frac{1}{\alpha} \operatorname{acosh}\left(1 + (\cosh(\alpha X_0) - 1)\left(\frac{t}{T}\right)^2\right). \tag{22}$$

Proof. With $Y_t \equiv X_t - X_0$ and $Y_t = \int_0^t u_s \, ds$, the optimal trading strategy in Eq. (21) can be expressed as

$$\sinh(\alpha Y_t) \, dY_t = \frac{2(\cosh(\alpha X_0) - 1)}{\alpha T^2} t \, dt.$$

Integrating both sides yields

$$\frac{\cosh(\alpha Y_t) - 1}{\alpha} = \frac{(\cosh(\alpha X_0) - 1)t^2}{\alpha T^2},$$

which results in

$$\cosh(\alpha Y_t) = 1 + (\cosh(\alpha X_0) - 1)\left(\frac{t}{T}\right)^2.$$

Since $Y_t \leqslant 0$ and $\alpha > 0$,

$$Y_t = -\frac{1}{\alpha} \operatorname{acosh}\left(1 + (\cosh(\alpha X_0) - 1)\left(\frac{t}{T}\right)^2\right),$$

which is Eq. (22). □

A smaller α implies a more liquid market. Therefore, Eq. (22) ought to reduce to a linear function of t as α approaches zero. Indeed, Eq. (22) reduces to Eq. (20) in the limit $\alpha \to 0$. To support this claim, we first note that the argument of the inverse hyperbolic cosine $\operatorname{acosh}(\cdot)$ in Eq. (22) equals

$$1 + \frac{1}{2}\left(\frac{\alpha X_0 t}{T}\right)^2 + o(\alpha^2),$$

where $o(c)$ denotes a collection of terms whose orders are higher than c. Since the inverse hyperbolic cosine $\operatorname{acosh}(x) = \log(x + \sqrt{x^2 - 1})$, the square root term in the argument of $\operatorname{acosh}(\cdot)$ equals $\alpha X_0 t/T + o(\alpha)$. Eq. (22) then becomes $X_t = X_0 - (1/\alpha)\log(1 + \alpha X_0 t/T + o(\alpha))$. Finally, since $\log(1 + x) = x + o(x)$, we obtain

$$X_t = X_0\left(1 - \frac{t}{T}\right) + o(\alpha).$$

Therefore, $\alpha \to 0$ from above gives rise to a linear reduction of asset holding with respect to time t as in Corollary 3.1.

3.3. *Optimal liquidation with stochastic price impact*

We now extend the previous functional form for the liquidity discount to incorporate the asset's intrinsic value S_t. Liquidity is now stochastic. In Market 3, the optimal trading strategy depends on the interest rate r and volatility σ of the asset through $k \equiv -\beta(r + (1 + \beta)\sigma^2/2)$ as seen in the following proposition.

Proposition 3.3. *The optimal trading strategy when liquidity depends on past trades and the asset value S_t is*

$$u_t^* = \frac{k^2(\cosh(\alpha X_0) - 1)t\, e^{kt}}{\alpha(1 - (1 - kT)\, e^{kT})\, \sinh(\alpha(X_t - X_0))} \tag{23}$$

with the initial trading strategy

$$u_0^* = -\frac{1}{\alpha}\sqrt{\frac{k^2(\cosh(\alpha X_0) - 1)}{1 - (1 - kT)e^{kT}}}.$$

Proof. See Appendix B.

The implications of stochastic liquidity in terms of the liquidation proceeds are obtained from the following corollary.

Corollary 3.4. *The expected value of the discounted proceeds from using the optimal trading strategy in Proposition* 3.3 *equals*

$$V(0, S_0, X_0) = S_0 X_0 - \gamma S_0 \frac{k^2(\cosh(\alpha X_0) - 1)^2}{\alpha^2(1 - (1 - kT)\,\mathrm{e}^{kT})}. \tag{24}$$

Proof. Applying Proposition 2.1 yields the expected value of the discounted proceeds. □

Corollary 3.5. *The holding* X_t *as a result of executing the optimal trading strategy in Proposition* 3.3 *is*

$$X_t = X_0 - \frac{1}{\alpha}\,\mathrm{acosh}\left(1 + (\cosh(\alpha X_0) - 1)\frac{1 - (1 - kt)\mathrm{e}^{kt}}{1 - (1 - kT)\mathrm{e}^{kT}}\right).$$

Proof. We write $Y_t \equiv X_t - X_0$. Since $Y_t = \int_0^t u_s\,\mathrm{d}s$ and from Proposition 3.3, we have

$$\mathrm{d}Y_t = \frac{k^2(\cosh(\alpha X_0) - 1)}{\alpha(1 - (1 - kT)\mathrm{e}^{kT})}\,\frac{t\,\mathrm{e}^{kt}}{\sinh(\alpha Y_t)}\,\mathrm{d}t.$$

After integration, we obtain

$$\frac{\cosh(\alpha Y_t) - 1}{\alpha} = \frac{k^2(\cosh(\alpha X_0) - 1)}{\alpha(1 - (1 - kT)\mathrm{e}^{kT})}\left(\frac{t\,\mathrm{e}^{kt}}{k} - \frac{\mathrm{e}^{kt}}{k^2} + \frac{1}{k^2}\right).$$

Re-arranging the above equation completes the proof. □

Depending on the sign of k, stochastic liquidity may favor or disfavor discretionary liquidation. This is because $k^2/(1 - (1 - kT)\mathrm{e}^{kT}) > 0$ is a strictly decreasing function of k. The second term of Eq. (24) will be smaller as k increases. It follows that the larger[3] k is, the larger will be the expected proceeds and vice versa.

When k approaches zero, the term $k^2\mathrm{e}^{kt}/(1 - (1 - kT)\,\mathrm{e}^{kT})$ in u_t^* in Eq. (23) becomes

$$\frac{2\mathrm{e}^{kt}}{T^2(1 + kT + \mathrm{o}(k))} \quad \longrightarrow \quad \frac{2}{T^2} \quad \text{as } k \to 0.$$

Also, Corollary 3.5 reduces to Corollary 3.3 with $(1 - (1 - kt)\mathrm{e}^{kt})/(1 - (1 - kT)\mathrm{e}^{kT})$ becoming $(t/T)^2$. Therefore, when k or $\beta \to 0$, Market 3 reduces to Market 2.

In summary, the results concerning the three markets may be understood in the following way:

$$\text{Market 3} \quad \overset{\beta \to 0}{\longrightarrow} \quad \text{Market 2} \quad \overset{\alpha \to 0}{\longrightarrow} \quad \text{Market 1}.$$

[3]Positive k corresponds to $\beta \in (-(1 + 2r/\sigma^2), 0)$.

3.4. *Liquidation horizon and amount of asset to be liquidated*

The admissibility of the optimal trading strategy u_t^* can be expressed as a relationship between the liquidation horizon T and the amount X_0 to be liquidated. Under Market 1, the admissibility condition leads to

$$T > \gamma X_0.$$

In reality, T has to be much larger than γX_0 to ensure that the selling price is not much lower than the asset's value. Similarly, under Market 2, to satisfy the condition $1 + f(t, S_t, X_t)u_t^* > 0$ at all time t, and $\sinh(\alpha(X_t - X_0))$ being largest when $X_T = 0$, we obtain

$$T > \sqrt{\frac{2\gamma(\cosh(\alpha X_0) - 1)\sinh(\alpha X_0)}{\alpha}}.$$

Hence, given parameters α and γ, T and X_0 are related in a nonlinear fashion. Under Market 3, the time horizon has to be even longer.

4. Liquidity costs

Bertsimas and Lo's (1998) approach to finding the optimal trading strategy minimizes transaction costs, whereas our approach maximizes the proceeds from the liquidation. It is natural to ascertain whether these two approaches are equivalent. To this end, we consider the difference between $P(t, S_t, X_t, u_t)$ and the intrinsic value S_t as a measure of liquidity-induced transaction costs. For a trading strategy u_t, the following quantity constitutes a reasonable economic definition[4] of the expected value of the discounted liquidity costs at any point in time t,

$$\begin{aligned} L(t, S_t, X_t, u_t) &\equiv \mathrm{E}\left[\int_t^T u_z(P(z, S_z, X_z, u_z) - S_z)\mathrm{e}^{-rz}\,\mathrm{d}z\right] \\ &= \mathrm{E}\left[\int_t^T u_z^2 S_z f(z, S_z, X_z)\,\mathrm{e}^{-rz}\,\mathrm{d}z\right]. \end{aligned} \tag{25}$$

Theorem 4.1. *Assume the liquidating strategy u_t is deterministic. Maximizing the expected value of the discounted liquidation proceeds and minimizing the expected value of the discounted liquidity costs are equivalent.*

Proof. It is straightforward to verify that the expected liquidation proceeds in Eq. (4) can be rewritten as a sum of two terms as follows:

$$-\mathrm{E}\left[\int_t^T u_z S_z\,\mathrm{e}^{-rz}\,\mathrm{d}z\right] - L(t, S_t, X_t, u_t).$$

[4]This definition of liquidity cost appears in a paper by Çetin et al. (2003).

With $S_t = S_0\,\mathrm{e}^{(r-(1/2)\sigma^2)t+\sigma W_t}$, $\mathrm{E}[\mathrm{e}^{\sigma W_t}] = \mathrm{E}[\mathrm{e}^{\sigma w_t}] = \mathrm{e}^{\sigma^2 t/2}$ and $\int_0^T u_t\,\mathrm{d}t = -X_0$, applying Fubini's theorem to the first term results in

$$S_0 X_0 - L(0, S_0, X_0, u_0).$$

Since $S_0 X_0$ is a constant, maximizing the expected proceeds $S_0 X_0 - L(0, S_0, X_0, u_0)$ that yields deterministic u_t^* as in Section 3 is equivalent to minimizing $L(0, S_0, X_0, u_0)$. □

The next proposition verifies that the liquidity costs defined by Eq. (25) are identical to the proceeds of a block trade at time $t = 0$ minus the expected value of the discounted proceeds generated by the optimal trading strategies.

Proposition 4.1. *The second term in each of the three discounted expected proceeds, namely,*

$$\begin{aligned} &\textit{Market } 1. \quad \frac{\gamma S_0 X_0^2}{T}, \\ &\textit{Market } 2. \quad \frac{2\gamma S_0}{\alpha^2 T^2}(\cosh(\alpha X_0) - 1)^2, \\ &\textit{Market } 3. \quad \frac{\gamma S_0 k^2 (\cosh(\alpha X_0) - 1)^2}{\alpha^2 (1 - (1 - kT)\mathrm{e}^{kT})}, \end{aligned}$$

is equivalent to the liquidity-induced transaction cost defined in Eq. (25) *under the three respective market conditions.*

Proof. Based on the optimal trading strategy in Proposition 3.1, the expected liquidation cost in Eq. (25) is evaluated as

$$\mathrm{E}\left[\gamma \int_0^T u_t^2\,\mathrm{e}^{-rt} S_t\,\mathrm{d}t\right] = \frac{\gamma S_0 X_0^2}{T},$$

which is equivalent to the expected cost of a block trade at time $t = 0$. The corresponding proofs for the second and third market conditions are given in Appendix C. □

5. Stochastic liquidity and stochastic volatility

It is a common knowledge that information arrival is random. As information affects market liquidity, the random nature of its arrival will cause liquidity to fluctuate. Our stochastic liquidity is formulated in the same spirit as Pastor and Stambaugh (2003) who treat liquidity as a stochastic state variable. As much as S_t is an unobservable state variable, market liquidity $f(t, S_t, X_t)$ in our formulation is a stochastic function of S_t. We show that stochastic liquidity will result in an effective volatility different from σ in Eq. (2).

For notational convenience, we denote the transaction price in Eq. (3) as

$$P \equiv P(t, S_t, X_t, u_t) = G S_t,$$

where

$$G \equiv G(t, S_t, X_t, u_t) \equiv 1 + f(t, S_t, X_t)u_t.$$

The change in transaction price is

$$\mathrm{d}P = S_t\,\mathrm{d}G + G\,\mathrm{d}S_t + \mathrm{d}[G, S_t].$$

The squared return is

$$\left(\frac{\mathrm{d}P}{P}\right)^2 = \left(\frac{\mathrm{d}S_t}{S_t}\right)^2 + \left(\frac{\mathrm{d}G}{G}\right)^2 + \left(\frac{\mathrm{d}[G, S_t]}{P}\right)^2 + 2\left(\frac{\mathrm{d}S_t}{S_t}\frac{\mathrm{d}G}{G} + \frac{\mathrm{d}S_t}{S_t}\frac{\mathrm{d}[G, S_t]}{P} + \frac{\mathrm{d}G}{G}\frac{\mathrm{d}[G, S_t]}{P}\right).$$

Being terms of higher order than $\mathrm{d}t$, $(\mathrm{d}[G, S_t]/P)^2$ as well as the two cross terms of $\mathrm{d}[G, S_t]/P$ with $\mathrm{d}S_t/S_t$ and $\mathrm{d}G/G$ do not contribute to the volatility. Hence, we write

$$\mathrm{E}\left[\left(\frac{\mathrm{d}P}{P}\right)^2\right] = \mathrm{E}\left[\left(\frac{\mathrm{d}S_t}{S_t}\right)^2 + \left(\frac{\mathrm{d}G}{G}\right)^2 + 2\,\frac{\mathrm{d}S_t}{S_t}\,\frac{\mathrm{d}G}{G}\right]. \tag{26}$$

Since u_t in general has the form of $\mathrm{d}u_t = a_t\,\mathrm{d}t + b_t\,\mathrm{d}X_t$ with a_t and b_t depending only on time t, applying Itô's formula yields

$$\mathrm{d}G = \sigma S_t \frac{\partial G}{\partial S_t}\,\mathrm{d}W_t + \left(\frac{\partial G}{\partial t} + rS_t\frac{\partial G}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 G}{\partial S_t^2} + \frac{\partial G}{\partial X_t}u_t + \frac{\partial G}{\partial u_t}\left(\frac{\partial u_t}{\partial t} + u_t\frac{\partial u_t}{\partial X_t}\right)\right)\mathrm{d}t + \mathrm{o}(\mathrm{d}t).$$

Only the first term in the above equation is relevant, with

$$\mathrm{E}\left[\left(\frac{\mathrm{d}G}{G}\right)^2\right] = \sigma^2\left(\frac{S_t}{G}\frac{\partial G}{\partial S_t}\right)^2\mathrm{d}t$$

and

$$\mathrm{E}\left[\frac{\mathrm{d}S_t}{S_t}\frac{\mathrm{d}G}{G}\right] = \sigma^2\left(\frac{S_t}{G}\frac{\partial G}{\partial S_t}\right)\mathrm{d}t.$$

Since the function G is deterministic in the first two market conditions, $\partial G/\partial S_t = 0$ and the volatility of the return $\mathrm{d}P/P$ is the same as the volatility σ of $\mathrm{d}S_t/S_t$. In contrast, when liquidity is stochastic, the volatility will be different from σ. We write

$$\mathrm{Var}\left[\frac{\mathrm{d}P}{P}\right] = \mathrm{E}\left[\left(\frac{\mathrm{d}P}{P}\right)^2\right] \equiv \sigma_*^2(t, S_t, X_t, u_t)\,\mathrm{d}t.$$

The following theorem characterizes the effective volatility $\sigma_*(t, S_t, X_t, u_t)$.

Theorem 5.1. *The effective volatility* σ_* *is*

$$\sigma_* \equiv \sigma_*(t, S_t, X_t, u_t) = \left(1 + \beta\, \frac{f(t, S_t, X_t)u_t}{1 + f(t, S_t, X_t)\, u_t}\right)\sigma. \tag{27}$$

Proof. Since $G \equiv 1 + h(t)g(X_t)(S_t/S_0)^\beta u_t$,

$$\begin{aligned}\frac{S_t}{G}\frac{\partial G}{\partial S_t} &= \frac{\beta}{G}\, h(t)g(X_t)u_t\left(\frac{S_t}{S_0}\right)^\beta \\ &= \beta\frac{G-1}{G}.\end{aligned}$$

Consequently, the last two terms on the right side of Eq. (26) equal

$$\left(\left(\beta\frac{G-1}{G}\right)^2 + 2\beta\frac{G-1}{G}\right)\sigma^2\, \mathrm{d}t.$$

Since $\mathrm{E}[(\mathrm{d}S_t/S_t)^2] = \sigma^2\, \mathrm{d}t$ and

$$\frac{G-1}{G} = \frac{f(t, S_t, X_t)u_t}{1 + f(t, S_t, X_t)u_t},$$

we obtain Eq. (27). □

Although Theorem 5.1 is not restricted to optimal trading strategies, it provides an insight into the combined effect of stochastic liquidity and discretionary liquidity trading on volatility. Research findings of, for example, Jones et al. (1994) suggest that volatility is higher when trading activity is more intense than usual. As liquidation trading leads to higher volume, the stylized volume–volatility relationship suggests that $\sigma_* > \sigma$. In turn, we gather from Theorem 5.1 that the volatility can only be higher if β is negative as u_t is negative.

6. Conclusion

We provide a mathematical analysis of optimal liquidation when the market demands a liquidity discount from the liquidation trader. Using stochastic control, a generic closed-form solution for the optimal trading strategy that maximizes the expected value of the discounted sales proceeds is derived. In our framework, optimizing the proceeds and minimizing transaction costs is equivalent. Three special cases for the liquidity discount are considered in increasing order of realism to conform with empirical evidence.

The optimal trading strategies yield several interesting economic results. First, they conform to existing empirical evidence as greater liquidity engenders more gradual liquidation. In contrast, less liquid markets have larger price impacts, and our analysis reveals that it is optimal to liquidate more rapidly. Second, if the liquidity function also depends on the intrinsic value of the asset, the liquidity discount is not necessarily higher. Third, stochastic liquidity and trading activity result in an effective volatility that is stochastic.

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Appendix A

A.1. Proof of Proposition 3.2

The proof follows the same procedure as Proposition 3.1. Theorem 2.2 implies that the optimal trading strategy equals

$$u_t^* \equiv u(t, X_t) = \frac{t\int_0^{X_t} \sinh(\alpha(X - X_0))\,\mathrm{d}X}{\sinh(\alpha(X_t - X_0))((t^2 - T^2)/2)}.$$

The essence of the proof is to produce a more explicit expression for the numerator. Since $\mathrm{d}X_t = u_t\,\mathrm{d}t$, we have

$$\frac{\alpha \sinh(\alpha(X_t - X_0))}{\cosh(\alpha(X_t - X_0)) - \cosh(\alpha X_0)}\,\mathrm{d}X_t = \frac{2t}{t^2 - T^2}\,\mathrm{d}t,$$

which leads to

$$|\cosh(\alpha(X_t - X_0)) - \cosh(\alpha X_0)| = a|t^2 - T^2| \quad \text{for } a > 0. \tag{28}$$

The initial condition at $t = 0$ for X_t implies that

$$a = \frac{1}{T^2}(\cosh(\alpha X_0) - 1).$$

Substituting a into Eq. (28) yields

$$\cosh(\alpha(X_t - X_0)) = \begin{cases} \cosh(\alpha X_0) + (\cosh(\alpha X_0) - 1)\dfrac{t^2 - T^2}{T^2} & \text{for } 0 \leqslant X_t \leqslant 2X_0, \\ \cosh(\alpha X_0) - (\cosh(\alpha X_0) - 1)\dfrac{t^2 - T^2}{T^2} & \text{for } X_t < 0 \text{ or } X_t \geqslant 2X_0. \end{cases} \tag{29}$$

The case of $X_t < 0$ should be rejected since the trader's holding is positive. Hence,

$$u_t^* = \frac{2(\cosh(\alpha X_0) - 1)t}{\alpha T^2 \sinh(\alpha(X_t - X_0))}.$$

From (29), one can rewrite X_t in terms of t and substitute the expression for u_t^* into the liquidity constraint to verify the solution. Therefore, u_t^* is the optimal trading strategy. Taking the limit as $t \to 0$ and applying L'Hôpital's rule, we obtain

$$u_0^* \equiv u^*(0, X_0) = \frac{2(\cosh(\alpha X_0) - 1)}{\alpha^2 T^2 u^*(0, X_0)}.$$

Since $\cosh(x) \geqslant 1$ for all x, the initial optimal rate of selling is

$$u_0^* = \frac{-\sqrt{2(\cosh(\alpha X_0) - 1)}}{\alpha T}. \quad \square$$

Appendix B

B.1. *Proof of Proposition 3.3*

Applying Theorem 2.2 to the third specification of market liquidity, the optimal trading strategy is

$$u_t^* = \frac{(\cosh(\alpha(X_t - X_0) - \cosh(\alpha X_0)))t\,e^{kt}}{\alpha \sinh(\alpha(X_t - X_0)) \int_t^T s\,e^{ks}\,ds}.$$

Hence, $dX_t = u_t\,dt$ implies

$$\frac{\alpha \sinh(\alpha(X_t - X_0))}{\cosh(\alpha(X_t - X_0)) - \cosh(\alpha X_0)}\,dX_t = \frac{t\,e^{kt}}{\int_t^T s\,e^{ks}\,ds}\,dt.$$

Reorganizing the above equation yields the following relation:

$$|\cosh(\alpha(X_t - X_0)) - \cosh(\alpha X_0)| = a\left|\frac{T}{k}e^{kT} - \frac{t}{k}e^{kt} - \frac{1}{k^2}e^{kT} + \frac{1}{k^2}e^{kt}\right|, \tag{30}$$

for some constant $a > 0$. The initial condition implies that

$$a = \frac{k^2(\cosh(\alpha X_0) - 1)}{1 - (1 - kT)e^{kT}}.$$

Hence, substituting a into Eq. (30) and performing the same analysis as in Appendix A implies that

$$u_t^* = \frac{k^2(\cosh(\alpha X_0) - 1)t\,e^{kt}}{\alpha(1 - (1 - kT)e^{kT}) \sinh(\alpha(X_t - X_0))}.$$

Similarly, one can verify that the liquidity constraint holds and that u_t^* is the optimal solution. Once again, applying L'Hôpital's rule as $t \to 0$ yields

$$u_0^* = -\frac{1}{\alpha}\sqrt{\frac{k^2(\cosh(\alpha X_0) - 1)}{1 - (1 - kT)e^{kT}}}. \quad \square$$

Appendix C

C.1. Proof of Proposition 4.1

Market 2: The expected value of the discounted liquidity costs equals

$$\gamma \mathrm{E}\left[\int_0^T S_t \frac{\sinh^2(\alpha(X_t - X_0))}{t} u_t^{*2} \mathrm{e}^{-rt}\, \mathrm{d}t\right],$$

where the optimal trading strategy under Market 2 is

$$u_t^* = \frac{2(\cosh(\alpha X_0) - 1)t}{\alpha T^2 \sinh(\alpha(X_t - X_0))}.$$

Since the hyperbolic sine function cancels, the expected liquidation cost becomes

$$\frac{4\gamma}{\alpha^2 T^4}(\cosh(\alpha X_0) - 1)^2\, \mathrm{E}\left[\int_0^T t S_t\, \mathrm{e}^{-rt}\, \mathrm{d}t\right]$$

which reduces to

$$\frac{4\gamma}{\alpha^2 T^4}(\cosh(\alpha X_0) - 1)^2 S_0 \int_0^T t\, \mathrm{d}t.$$

Therefore, the second term in Corollary 3.2 becomes the expected liquidity cost

$$\frac{2\gamma S_0}{\alpha^2 T^2}(\cosh(\alpha X_0) - 1)^2.$$

Market 3: For the optimal trading strategy u_t^* derived in Proposition 3.3, the expected value of the discounted liquidity costs for Market 3 is

$$\gamma \mathrm{E}\left[\int_0^T S_t \frac{\sinh^2(\alpha(X_t - X_0))}{t} \left(\frac{S_t}{S_0}\right)^{\beta} u_t^{*2}\, \mathrm{e}^{-rt}\, \mathrm{d}t\right],$$

which equals

$$\gamma \frac{k^4(1 - \cosh(\alpha X_0))^2}{\alpha^2(1 - (1 - kT)\,\mathrm{e}^{kT})^2} S_0 \mathrm{E}\left[\int_0^T t\, \mathrm{e}^{-rt}\, \mathrm{e}^{2kt}\, \mathrm{e}^{(1+\beta)(r - \sigma^2/2)t}\, \mathrm{e}^{(1+\beta)\sigma W_t}\right].$$

Since

$$\mathrm{E}[\mathrm{e}^{(1+\beta)\sigma W_t}] = \exp\{\tfrac{1}{2}((1 + \beta)\sigma)^2 t\},$$

the integral becomes

$$\int_0^T t \exp\left\{\left(-r + 2k + (1 + \beta)\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{2}((1 + \beta)\sigma)^2\right) t\right\} \mathrm{d}t,$$

which reduces to

$$\int_0^T t\, \mathrm{e}^{kt}\, \mathrm{d}t = \frac{1}{k^2}(1 - (1 - kT)\mathrm{e}^{kT}).$$

Therefore, the expected value of the discounted liquidity costs for Market 3 is

$$\gamma \frac{k^2(1-\cosh(\alpha X_0))^2}{\alpha^2(1-(1-kT)\,\mathrm{e}^{kT})} S_0$$

as in the second term of Corollary 3.3. □

References

Almgren, R.F., Chriss, N., 2000. Optimal execution of portfolio transactions. Journal of Risk 3, 5–39.

Bertsimas, D., Lo, A., 1998. Optimal control of execution costs. Journal of Financial Markets 1, 1–50.

Çetin, U., Jarrow, R.A., Protter, P., 2003. Liquidity risk and arbitrage pricing theory. Finance and Stochastics 8, 311–341.

Chan, L.K.C., Lakonishok, J., 1993. Institutional trades and intraday stock price behavior. Journal of Financial Economics 33, 173–199.

Chan, L.K.C., Lakonishok, J., 1995. The behavior of stock prices around institutional trades. Journal of Finance 50, 1147–1174.

Chordia, T., Roll, R., Subrahmanyam, A., 2001. Market liquidity and trading activity. Journal of Finance 56, 501–530.

Duffie, D., 2001. Dynamic Asset Pricing Theory. Princeton University Press, NJ.

Goldstein, M.A., Kavajecz, K.A., 2000. Eighths, sixteenths, and market depth: changes in tick size and liquidity provision on the NYSE. Journal of Financial Economics 56, 125–149.

Holthausen, R.W., Leftwich, R.W., Mayers, D., 1987. The effect of large block transactions on security prices: a cross-sectional analysis. Journal of Financial Economics 19, 237–267.

Jones, C., Gautam, K., Lipson, M., 1994. Transactions, volume, and volatility. Review of Financial Studies 7, 631–651.

Keim, D.B., Madhavan, A., 1995. Anatomy of the trading process: empirical evidence on the behavior of institutional traders. Journal of Financial Economics 37, 371–398.

Pastor, L., Stambaugh, R.F., 2003. Liquidity risk and expected stock returns. Journal of Political Economy 111, 642–685.

Subramanian, A., Jarrow, R.A., 2001. The liquidity discount. Mathematical Finance 11, 447–474.