The Effect of Taxes on the Pricing of Defaultable Debt

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The effect of taxes on the pricing of defaultable debt

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Empirical studies have documented the dependence of corporate credit spreads on default risk, equity premiums, and taxes. However, taxes have previously not been incorporated into reduced-form credit risk models. Therefore, we first extend the existing literature by considering a default intensity that depends on taxes as well as the default-free short rate and a market index. Consequently, we establish a theoretical basis to explain previous empirical findings regarding the significant impact of taxation on defaultable bond prices. Unlike previous models, tax implications for defaultable debt cannot be constructed from a sum of tax effects on zero coupon bonds. Our empirical tests then illustrate the importance of taxation. In particular, the impact of taxation increases as a function of the debt's maturity and coupon rate.

1 Introduction

In the last decade, credit risk modeling has become an important branch of financial research. These models are usually calibrated using defaultable corporate debt whose credit spreads may be decomposed into premiums for default, illiquidity, and taxes. This paper makes two contributions to the credit risk literature. We first derive a theoretical model for pricing defaultable debt that incorporates taxes. Second, we find empirical evidence that tax effects, in addition to default, heavily influence the pricing of defaultable debt. Interestingly, our empirical results indicate that our parsimonious model's performance is comparable to Janosi, Jarrow and Yildirim (2002) who parameterize liquidity with as many as four additional coefficients.

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The reduced-form credit risk approach was first proposed by Jarrow and Turnbull (1995) while Lando (1998) extends the framework to incorporate a stochastic intensity process modeled as a simple linear function of the default-free short rate. This enhancement introduces correlation between default and the default-free short interest rate. Duffie and Singleton (1999) develop a reduced-form model under the assumption that a fraction of the defaultable bond's market value is recovered. This is usually described as the fractional recovery of market value (FRMV) assumption and provides a more convenient comparison between prices of default-free and defaultable debt. Existing theories of default-free term structure modeling, such as the Heath, Jarrow and Morton (1992) model are then readily applied to defaultable term structures. Jarrow and Turnbull (2000) base their subsequent model on the FRMV recovery assumption and generalize the default intensity to depend on the default-free short rate and the cumulative return of a market index.

In a recent study, Elton, Gruber, Agrawal and Mann (2001) present stylized empirical results regarding the dependence of corporate bond spreads on default risks, equity premiums, and taxes. Although they demonstrate the significant impact of taxes on credit spreads, hence defaultable bond prices, existing reduced-form credit risk models have not accounted for their impact. Consequently, this paper develops a pricing model for defaultable coupon bonds that incorporates taxation. Furthermore, features such as a correlation between the default process and the default-free short rate, as well as the influence of stock market returns on the default process are retained.

Upon default, investors receive a capital loss tax rebate on the bond's principal. This tax rebate causes the principal to differ and proportional recovery rates for coupons. Thus, pricing a defaultable coupon bond as the sum of defaultable zero coupon bonds is not appropriate. In almost all existing credit risk models, defaultable coupon bonds are treated as a sum of defaultable zero coupon bonds with different maturities. This approach has the advantage of usually being analytically tractable. However, some critical considerations, apart from taxation, are not addressed in this framework. First, the linear aggregation of risky zero-coupon bond prices to form risky coupon bond price, as in equation (5) of Janosi, Jarrow and Yildrim (2002), implicitly assumes that default and coupon rates are unrelated. Indeed, default rates only influence the pricing of zero coupon defaultable debt, but these "building blocks" are common across all coupon bearing defaultable bonds, irrespective of their coupon payments. Moreover, Duffie and Singleton (1999) prove that if dependence exists between coupon rates and default premiums, then defaultable bond prices are generally non-linear in the coupon cashflows. Second, it is also implicitly assumed that the discount rate between two future dates may be found using the standard procedure for computing forward rates from zero coupon bonds. However, this bootstrapping procedure may severely violate default conditions practiced in the real world, as pointed out by Jarrow and Turnbull (2000). In our model, there is no need for bootstrapping since our methodology is applied directly to coupon bond data.

This paper is structured as follows. In Section 2 we present a theoretical model for pricing defaultable coupon bonds with tax effects while Section 3 provides empirical tests of our model's performance. We compare our tax-based model with the liquidity models of Janosi, Jarrow and Yildirim (2002) to determine the marginal contribution of incorporating liquidity versus taxation. This comparison is useful as their study also allows for correlation between the default process, the short rate, and a stock market index. Section 4 contains our conclusions. Some details of the derivations are relegated to the appendices.

2 Pricing defaultable debt with tax effects

In this section, we propose and derive a theoretical pricing model for defaultable debt which explicitly incorporates taxation, notably a different implied recovery rate for interim coupons and the principal, resulting from the tax loss and generated by default. Motivated by past research, our model continues to incorporate the default-free short rate and a market index into the default intensity. Without loss of generality, we consider defaultable debt with semi-annual coupons US\$$C$ and principal values of US\$100. The total number of coupon payments is N with the last coupon payment time coinciding with principal repayment.

A summary of our model's notation is given below for easy reference.

- T_j: coupon payment dates with $T_j - T_{j-1} = 0.5, \forall j = 1, 2, 3, \dots, N$
- R: recovery rate of principal and accrued interest
- t_s: state tax rate
- t_g: federal tax rate
- τ: default time
- $\lambda(t)$: default process intensity
- $r(t)$: default-free short rate
- $M(t)$: stock market index (eg, S&P500)
- ρ: correlation coefficient between $r(t)$ and $M(t)$
- $B(t, T_j)$: present value of a tax-free default-free zero coupon bond with maturity at T_j
- $V(t, T_j)$: present value of a taxable defaultable corporate bond with maturity at T_j

In addition, the assumptions of our model are listed below.

(A1) Before default, coupon payments are subject to both state and federal taxes. The state taxes are net of federal taxes implying that the marginal impact of state taxes is $t_s(1 - t_g)$. However, the principal repayment is not taxable.

(A2) At default, bond-holders receive a fraction R of the principal value and accrued interest on the coupon.

(A3) At default, the lost principal constitutes a capital loss and state taxes are recoverable. This results in a tax rebate of $100(1 - R)\, t_s(1 - t_g)$.

(A4) The default intensity is a linear function of the default-free short rate and cumulative changes in the S&P500 stock market index.

(A5) There exists an equivalent risk-neutral martingale measure **Q**.

Assumptions (A1), (A2), and (A3) distinguish our model from previous research. As shown in Elton, Gruber, Agrawal and Mann (2001), a major difference between default-free and defaultable debt lies in the fact that corporate debt is subject to federal and state taxes. Although all coupon payments are taxable, if the firm defaults, the amount of lost principal, $100(1 - R)$ represents a capital loss with taxes being recoverable. Our first contribution is to develop a theoretical model incorporating tax and credit risk that satisfies the usual no-arbitrage term structure conditions.

We make use of three lemmas from Lando (1998) which are contained in Appendix A. The symbol E_t^Q denotes an expectation at time t under the risk-neutral martingale measure **Q**.

We study the pricing defaultable debt with semiannual coupon payments US\$$C$, principal value of US\$100, and N remaining coupon payments. Conditional on no default prior to or at time T_{j-1}, if default occurs over the period $(T_{j-1}, T_j]$, the payment to the bond-holder at the default time τ is

$$f_j(\tau) = R\left[C(\tau - T_{j-1}) + 100\right] + 100(1 - R)\, t_s (1 - t_g) \tag{1}$$

for $T_{j-1} < \tau \le T_j$. The payment $f_j(\tau)$ consists of two parts, the first term being the fractional recovery of accrued interest and principal value, the second component representing the tax rebate generated by the capital loss. The value of this claim at time T_{j-1} is

$$v_j(T_{j-1}) = E_{T_{j-1}}^Q\left[\exp\left(-\int_{T_{j-1}}^{\tau} r(u)\,\mathrm{d}u\right) f_j(\tau)\right]$$

which is evaluated using Lemma 3 in Appendix A as

$$v_j(T_{j-1}) = 1_{\{\tau > T_{j-1}\}}\, E_{T_{j-1}}^Q\left[\int_{T_{j-1}}^{T_j} f_j(s)\lambda(s)\exp\left(-\int_{T_{j-1}}^{s}\left[r(u) + \lambda(u)\right]\mathrm{d}u\right)\mathrm{d}s\right]$$

Consequently, the value of the claim $v_j(T_{j-1})$ at time t is

$$v_j(t) = E_t^Q\left[\exp\left(-\int_t^{T_{j-1}} r(x)\,\mathrm{d}x\right) v_j(T_{j-1}) 1_{\{\tau > T_{j-1}\}}\right]$$

Using Lemma 1 in Appendix A, this expression becomes

$$
\begin{aligned}
v_j(t) &= 1_{\{\tau>t\}} E_t^Q\left[\exp\left(-\int_t^{T_{j-1}}[r(x)+\lambda(x)]\mathrm{d}x\right) v_j(T_{j-1})\right] \\
&= 1_{\{\tau>t\}} E_t^Q\left[\exp\left(-\int_t^{T_{j-1}}[r(x)+\lambda(x)]\mathrm{d}x\right)\times\right. \\
&\qquad\left.\int_{T_{j-1}}^{T_j} f_j(s)\lambda(s)\exp\left(-\int_{T_{j-1}}^{s}[r(u)+\lambda(u)]\mathrm{d}u\right)\mathrm{d}s\right] \\
&= 1_{\{\tau>t\}} E_t^Q\left[\int_{T_{j-1}}^{T_j} f_j(s)\lambda(s)\exp\left(-\int_t^{s}[r(u)+\lambda(u)]\mathrm{d}u\right)\mathrm{d}s\right]
\end{aligned} \tag{2}
$$

The claims $v_j(t)$ represent payments at various default times after adjustments involving the recovery rate and taxes. They form one component of defaultable debt in the event that default occurs before maturity. Adding together all coupon payments, the price of a defaultable coupon bond is

$$V(t,T_N) = V_1(t,T_N) + V_2(t,T_N)$$

where

$$
\begin{aligned}
V_1(t,T_N) &= \sum_{j=1}^{N} v_j(t) \\
&= 1_{\{\tau>t\}} \sum_{j=1}^{N} E_t^Q\left[\int_{T_{j-1}}^{T_j} f_j(s)\lambda(s)\exp\left(-\int_t^{s}[r(u)+\lambda(u)]\mathrm{d}u\right)\mathrm{d}s\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&V_2(t,T_N) \\
&= E_t^Q\left\{\left[\sum_{j=1}^{N}\left(C(1-t_s)(1-t_g)\mathrm{e}^{-\int_t^{T_j} r(u)\mathrm{d}u}\right) + 100\mathrm{e}^{-\int_t^{T_N} r(u)\mathrm{d}u}\right]1_{\{\tau>T_N\}}\right\} \\
&= 1_{\{\tau>t\}} \sum_{j=1}^{N} E_t^Q\left[C(1-t_s)(1-t_g)\mathrm{e}^{-\int_t^{T_j}[r(u)+\lambda(u)]\mathrm{d}u}\right] \\
&+ 1_{\{\tau>t\}} E_t^Q\left[100\mathrm{e}^{-\int_t^{T_N}[r(u)+\lambda(u)]\mathrm{d}u}\right]
\end{aligned}
$$

The term $V_1(t, T_N)$ may be interpreted as the value of a bond that defaults for certain at a random time. In addition, $V_2(t, T_N)$ may be interpreted as the value of a taxable defaultable bond weighted by the probability of no default.

Denote the price of tax-free defaultable zero coupon bond as

$$\tilde{B}(t,T_j) = E_t^Q\left[\mathrm{e}^{-\int_t^{T_j}[r(u)+\lambda(u)]\mathrm{d}u}\right]$$

We first express both $V_1(t, T_N)$ and $V_2(t, T_N)$ in terms of $\tilde{B}(t,T_j)$. It is straightforward to obtain

$$\begin{aligned} V_2(t,T_N) &= 1_{\{\tau>t\}}\left\{\sum_{j=1}^{N}\left[C(1-t_s)(1-t_g)\tilde{B}(t,T_j)\right] + 100\tilde{B}(t,T_N)\right\} \\ &= 1_{\{\tau>t\}}\left\{\sum_{j=1}^{N-1}\left[C(1-t_s)(1-t_g)\tilde{B}(t,T_j)\right]\right\} \\ &\quad + 1_{\{\tau>t\}}\left[100 + C(1-t_s)(1-t_g)\right]\tilde{B}(t,T_N) \end{aligned}$$

For $V_1(t, T_N)$, however, the derivation is more involved.

$$\begin{aligned} V_1(t,T_N) = 1_{\{\tau>t\}}\sum_{j=1}^{N}\tilde{B}(t,T_{j-1})\int_{T_{j-1}}^{T_j}\Big[f_j(s)\exp\Big(E_t^Q[Y_2] + \tfrac{1}{2}\mathrm{var}_t^Q[Y_2] + {} \\ \mathrm{cov}_t^Q[Y_1,Y_2]\Big)\Big(E_t^Q[X] + \mathrm{cov}_t^Q[X,Y]\Big)\Big]\mathrm{d}s \end{aligned}$$

where

$$\begin{aligned} X &= \lambda(s) \\ Y &= -\int_t^s\left[r(u)+\lambda(u)\right]\mathrm{d}u \\ Y_1 &= -\int_t^{T_{j-1}}\left[r(u)+\lambda(u)\right]\mathrm{d}u \\ Y_2 &= -\int_{T_{j-1}}^s\left[r(u)+\lambda(u)\right]\mathrm{d}u \end{aligned}$$

The details of the above derivation are shown in Appendix B. We re-express

$$V_1(t,T_N) = 1_{\{\tau>t\}}\left[\sum_{j=1}^{N-1}\tilde{B}(t,T_j)\,\pi_j + \pi_0\right]$$

where

$$\pi_j = \int_{T_j}^{T_{j+1}} \left\{ f_{j+1}(s)\exp\left(E_t^Q[Y_2] + \tfrac{1}{2}\mathrm{var}_t^Q[Y_2] + \mathrm{cov}_t^Q[Y_1, Y_2]\right)\times \left(E_t^Q[X] + \mathrm{cov}_t^Q[X, Y]\right)\right\}\mathrm{d}s \tag{3}$$

and π_0 is π_j evaluated at $j = 0$. Therefore, $V(t, T_N)$ may be expressed as

$$V(t,T_N) = 1_{\{\tau > t\}}\left\{ \sum_{j=1}^{N-1} \tilde{B}(t,T_j)\left[\pi_j + C(1-t_s)(1-t_g)\right] + \left[\pi_0 + \tilde{B}(t,T_N)\left(100 + C(1-t_s)(1-t_g)\right)\right]\right\} \tag{4}$$

This pricing formula in equation (4) is very general and independent of any model specifications. To derive a pricing model for implementation, we need to specify the interest rate as well as default intensity processes. The default intensity process is assumed to depend on the stock market index. Following Jarrow and Turnbull (2000), we that assume the default-free short rate, stock market index, and default intensity evolve according to the following stochastic processes[1]

$$\mathrm{d}r(t) = a\left[\bar{r}(t) - r(t)\right]\mathrm{d}t + \sigma_r \mathrm{d}W_r(t) \tag{5}$$

$$\mathrm{d}M(t) = r(t)M(t)\mathrm{d}t + \sigma_m M(t)\mathrm{d}W_m(t) \tag{6}$$

$$\lambda(t) = \lambda_0 + \lambda_1 r(t) + \lambda_2 W_m(t) \tag{7}$$

where $W_r(t)$ and $W_m(t)$ are risk-neutral standard Brownian motions with the following correlation

$$E_t^Q\left[\mathrm{d}W_r(t)\mathrm{d}W_m(t)\right] = \rho\mathrm{d}t \tag{8}$$

The taxable defaultable bond price $V(t, T_N)$ is then an explicit function of the parameters in equations (5) to (8). For the Vasicek default-free short rate model in equation (5), Heath, Jarrow and Morton (1992) prove the relation between $\bar{r}(t)$ and forward rates equals

$$\bar{r}(t) = f(0,t) + \frac{\dfrac{\partial f(0,t)}{\partial t} + \dfrac{\sigma_r^2(1-\mathrm{e}^{-2at})}{2a}}{a}$$

[1] With this specification, there exists the possibility that $\lambda(t)$ may become negative. However, we are careful in our later empirical analysis to prevent instances where this occurs for either process by actually considering $\lambda^{+}(t) = \max\{\lambda(t), 0\}$. These processes are also used in Janosi, Jarrow and Yildirim (2002) and we adopt them to facilitate a comparison with our empirical results.

where $f(t,u)$ is a default-free instantaneous forward rate contracted at t and applicable at u. As in the existing literature, we specify the volatility function as an exponential time decay function, $\sigma_r e^{-a(u-t)}$, for all $u \geq t$. This results in the instantaneous default-free spot rate being

$$r(u) = f(t,u) + \frac{\sigma_r^2 (1 - e^{-a(u-t)})^2}{2a^2} + \int_t^u \sigma_r e^{-a(u-v)} dW_r(v) \tag{9}$$

Next, we show that $\tilde{B}(t,T_j)$ in equation (4) can be expressed in terms of the default-free bond price $B(t,T_j)$. Let $A_j = -\int_t^{T_j} r(u)du$ and $B_j = -\int_t^{T_j} \lambda(u)du$. Thus, we obtain

$$\begin{aligned}\tilde{B}(t,T_j) &= E_t^Q\left[e^{-\int_t^{T_j}[r(u)+\lambda(u)]du}\right] \\ &= \exp\left[E_t^Q(A_j) + \tfrac{1}{2}\text{var}_t^Q + E_t^Q(B_j) + \tfrac{1}{2}\text{var}_t^Q(B_j) + \text{cov}_t^Q(A_j,B_j)\right] \\ &= B(t,T_j)\exp\left[E_t^Q(B_j) + \tfrac{1}{2}\text{var}_t^Q(B_j) + \text{cov}_t^Q(A_j,B_j)\right] \end{aligned} \tag{10}$$

Using equations (7) and (9) yields

$$\begin{aligned} E_t^Q(B_j) = &-\left(\lambda_0 + \lambda_2 W_m(t)\right)(T_j - t) - \lambda_1 \int_t^{T_j} f(t,u)du \\ &- \frac{\lambda_1 \sigma_r^2}{2a^2}\left((T_j - t) + \frac{1 - e^{-2a(T_j - t)}}{2a} - 2\frac{1 - e^{-a(T_j - t)}}{a}\right)\end{aligned}$$

$$\begin{aligned}\text{var}_t^Q(B_j) = &\frac{\lambda_2^2 (T_j - t)^3}{3} + \frac{\lambda_1^2 \sigma_r^2}{a^2} \times \\ &\left[(T_j - t) + \frac{1 - e^{-2a(T_j - t)}}{2a} - 2\frac{1 - e^{-a(T_j - t)}}{a}\right] \\ &+ \frac{2\rho\sigma_r\lambda_1\lambda_2}{a}\left[\frac{(T_j - t)^2}{2} + \frac{e^{-a(T_j - t)}}{a}(T_j - t) - \frac{1 - e^{-a(T_j - t)}}{a^2}\right]\end{aligned}$$

$$\begin{aligned}\text{cov}_t^Q(A_j,B_j) = &\frac{\rho\sigma_r\lambda_2}{a}\left[\frac{(T_j - t)^2}{2} + \frac{e^{-a(T_j - t)}}{a}(T_j - t) - \frac{1 - e^{-a(T_j - t)}}{a^2}\right] \\ &+ \frac{\lambda_1 \sigma_r^2}{a^2}\left[(T_j - t) + \frac{1 - e^{-2a(T_j - t)}}{2a} - 2\frac{1 - e^{-a(T_j - t)}}{a}\right]\end{aligned}$$

The terms π_j and π_0 in equation (4) are expressed in terms of $E_t^Q[X] + \text{cov}_t^Q[X, Y]$ and $E_t^Q[Y_2] + \frac{1}{2}\text{var}_t^Q[Y_2] + \text{cov}_t^Q[Y_1, Y_2]$. This is evident from equation (3) with details of the derivations provided in Appendix C. This completes our theoretical pricing model. Although the mathematical expression appears complicated, numerically implementing the model may be done very efficiently.

3 Empirical tests

In this section, we empirically test the pricing model derived in Section 2. The significant impact of taxation on the pricing of corporate debt, which increases with maturity and the coupon rate, is demonstrated. Significance tests of the model's parameters are undertaken to gauge the model's performance and verify its usefulness. A measure of in-sample fit (R^2) also demonstrates the model's ability to explain the pricing of corporate debt.

3.1 Data description

The corporate bonds data in this study are from the Fixed Income Security Database (FISD) at the University of Houston. Detailed information on this database is contained in Warga (1999). This database consists of monthly bid prices for US Treasuries as well as US corporate debt and their credit ratings. As seen in Table 1, we choose the same sampling period, May 1991 - March 1997, and the same sample of 20 firms as in Janosi, Jarrow and Yildirim (2002) to facilitate a comparison with our results. This benchmark is important to ascertain the marginal contribution of incorporating taxation versus liquidity. However, our sample is more heterogeneous as we include senior and subordinated bonds, provided they are investment grade. Table 1 contains the credit ratings of our sample.

Consistent with previous empirical studies, we exclude all debt issues with embedded options and only employ quoted, not matrix (model), prices.

The 20 firms are selected to cover different industry sectors: financial, food and beverages, petroleum, airlines, utilities, department stores and technology. For each firm, at least three bonds are traded in any month during the sampling period. For the US Treasuries, we include all outstanding bills, notes, and bonds during the sample period contained in the Fixed Income Security (FISD) Database. Those with obvious data errors, such as very large coupons, yields, outstanding issues, or maturities are excluded.[2] The proxy for the equity market index is the S&P500 index, and for the default-free short rate, the three-month Treasury yield obtained from Datastream is chosen.

3.2 Parameter estimation

We recall that the theoretical value of taxable defaultable debt derived in equation (4) is a function of the bond specifics C and T_N, as well as the tax rates t_s and t_g.

[2] One bond of Security Pacific is excluded in comparison to the dataset of Janosi, Jarrow and Yildirim (2002).

TABLE 1 Summary of companies, the number of issuances, credit ratings, and time period under investigation.

Company name	First date	Last date	Number of bonds	Moody	S&P
FINANCIALS					
Security Pacific Corp	12/31/1991	07/31/1994	6	A3	A
Fleet Financial Group	12/31/1991	10/31/1996	3	Baa2	BBB+
Bankers Trust NY	01/31/1994	04/30/1994	4	A1	AA
Merrill Lynch & Co	12/31/1991	03/31/1997	15	A2	A
FOOD AND BEVERAGES					
Pepsico Inc	12/31/1991	03/31/1997	8	A1	A
Coca-Cola Enterprises Inc	12/31/1991	06/30/1994	4	A2	AA–
AIRLINES					
AMR Corporation	02/29/1992	08/31/1994	10	Baa1	BBB+
Southwest Airlines Co	05/31/1992	03/31/1997	3	Baa1	A–
UTILITIES					
Carolina Power & Light	08/31/1992	01/31/1993	5	A2	A
Texas Utilities Ele Co	04/30/1994	03/31/1997	14	Baa2	BBB
PETROLEUM					
Mobil Corp	12/31/1991	02/29/1996	5	Aa2	AA
Union Oil of California	12/31/1991	03/31/1997	8	Baa1	BBB
Shell Oil Co	03/31/1992	02/28/1995	7	Aaa	AAA
DEPARTMENT STORES					
Sears Roebuck & Co	12/31/1991	08/31/1996	10	A2	A
Dayton Hudson Corp	04/30/1993	03/31/1997	15	A3	A
Wal-Mart Stores, Inc	12/31/1991	03/31/1997	3	Aa3	AA
TECHNOLOGY					
Eastman Kodak Company	01/31/1992	09/30/1994	6	A2	A–
Xerox Corp	12/31/1991	03/31/1997	4	A2	A
Texas Instruments	10/31/1992	03/31/1997	3	A3	A
Intl Business Machines	01/31/1994	03/31/1997	5	A1	A

The column "Number of bonds" is the number of different debt issues outstanding on the first date. Moody and S&P refer to the company's debt ratings on the first date. The companies and sample period in this study match those in Janosi, Jarrow and Yildirim (2002).

It is also a function of $\tilde{B}(t, T_j)$, π_j, and π_0. The latter three terms are functions of the parameters R, a, σ_r, ρ, λ_0, λ_1, and λ_2, and the state variable $W_m(t)$ at time t. The integration of the forward rate $f(t, u)$ is also required as seen in the expression $E_t^Q(Y_2)$ of Appendix C. The Nelson and Siegel (1987) model parameterizes the forward rate curve, implying additional parameters β_0, β_1, β_2, and β_3 are to be estimated. Thus, we may express $V(t, T_N)$ as a function of these parameters along with bond and tax rate specifics. It is generally not possible to simultaneously estimate all the underlying parameters of $V(t, T_N)$ and test such a model in the classical approach.

Therefore, we follow Janosi, Jarrow and Yildirim (2002) and implement the model using a multi-step approach. We first estimate the parameters ρ, σ_m and thus the state variable $W_m(t)$. Secondly, we estimate β_0, β_1, β_2, and β_3 followed by a and σ_r, and finally λ_0, λ_1, and λ_2. Given the parameter estimates of a, σ_r, ρ, σ_m, $W_m(t)$, β_0, β_1, β_2, and β_3 derived from observable price histories, we test the model using the null hypothesis $H_0{:}\lambda_0 = \lambda_1 = \lambda_2 = 0$.

The multiple steps are executed as follows. The correlation ρ between the default-free short rate and stock market index is estimated as

$$\hat{\rho} = \text{corr}\left[\frac{M(t) - M(t-1)}{M(t-1)},\ r(t) - r(t-1)\right]$$

For each day t at the end of the month from May 24, 1991 to March 31, 1997, one ρ is estimated. The price histories used for this estimate consist of observed stock market prices and short rates over 365 trading days just prior to day t at the end of each month.

The stock market index annualized volatility is estimated as

$$\hat{\sigma}_m = \sqrt{365\,\text{var}\left[\frac{M(t) - M(t-1)}{M(t-1)}\right]}$$

For each day t at the month's end, one σ_m is estimated based on prices for the previous 365 trading days.

Given a parameter estimate $\hat{\sigma}_m$ at $t - 1$, the state variable $W_m(t)$ is estimated as

$$\hat{W}_m(t) = W_m(t-1) + \frac{\log\left[\frac{M(t)}{M(t-1)}\right] - \frac{r(t-1)}{365} + \frac{1}{2}\frac{1}{365}\hat{\sigma}_m^2(t-1)}{\hat{\sigma}_m(t-1)}$$

with $W_m(0) = 0$. We need to estimate $\hat{\sigma}_m(t-1)$ in order to compute $\hat{W}_m(t)$.

Next, we estimate a and σ_r underlying the short rate process. From equation (9), it is clear that this default-free short rate process depends on the default-free instantaneous forward rate curve. However, there are typically an inadequate number of default-free treasury bonds to evaluate the forward rate curve. To estimate all forward rates, we employ Nelson and Siegel (1987)'s parametric forward rate function

$$f(t,u) = \beta_0 + \beta_1 \exp\left(-\frac{u-t}{\beta_3}\right) + \beta_2 \frac{u-t}{\beta_3} \exp\left(-\frac{u-t}{\beta_3}\right)$$

where β_0, β_1, β_2, and β_3 are estimated using default-free bond prices. The parameters β_0 and β_3 are restricted to be positive. Elton, Gruber, Agrawal and Mann (2001) also apply this numerical technique in their empirical study. The default-free instantaneous forward rate curve is determined by these four parameters. Estimation of β_0, β_1, β_2, and β_3 is conducted by parameterizing default-free bond prices in these four unknowns, and then minimizing the sum of squares between the theoretical bond prices and observed market prices. Once these are estimated for each t, the forward curve is completely parameterized, allowing the parameters a and σ_r to be estimated using a similar history of default-free bond prices.

Up to this point, monthly estimates: $\hat{\rho}$, $\hat{W}_m$, $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\beta}_3$, $\hat{a}$ and $\hat{\sigma}_r$ are obtained. The theoretical price of a taxable defaultable coupon bond in equation (4) may be expressed as a non-linear function $g(\cdot)$

$$V(t, T_N) = g\left(\lambda_0, \lambda_1, \lambda_2 \,\middle|\, \hat{\rho}, \hat{W}_m, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{a}, \hat{\sigma}_r; C, t_s, t_g, R, T_1, \ldots, T_N\right)$$

For each company or issuer, available bond prices for different maturities T_N are used to estimate its default intensity parameters λ_0, λ_1, and λ_2 that could not be estimated directly using any price histories. This is similar to the idea of implied volatilities in option pricing. However, at any t, the number of traded bonds for each company is often quite small. To reduce estimation error, we augment the number of bonds at t by pooling observed bond prices of the company during the previous seven months. By including the past seven months, a larger pool of bonds are available for estimation, as seen in Table 2.

A non-linear regression at t is performed on the objective function

$$\min_{\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2} \sum_{i=1}^{K} \left(\bar{V}(t_i, T_{N_i}) - g_i\left(\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2 \,\middle|\, \Phi\right) \right)^2$$

where Φ is the set of relevant estimates as seen in the above non-linear function $g(\cdot)$, and bar $\bar{V}(t, T_N)$ is the observed bond price at t_i. Within Φ we assume a recovery rate R of 0.6 for all firms. Consistent with Elton, Gruber, Agrawal and Mann (2001), we also assume state taxes t_s of 0.075 and a federal tax rate t_g of 0.35. This minimization yields consistent estimates in the non-linear regression model

$$\bar{V}(t_i, T_{N_i}) = g_i\left(\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2 \,\middle|\, \Phi\right) + \varepsilon_i$$

where ε_i is a mean zero stationary residual noise for bond i at time t_i. The F-test of the restrictions $H_0 : \lambda_0 = \lambda_1 = \lambda_2 = 0$ for the non-linear regression is also computed.

For each company, simultaneous estimation of λ_0, λ_1, λ_2, and testing of H_0 are carried out for each separate regression corresponding to each month. In the interest of brevity, only the averages of $\hat{\lambda}_0$, $\hat{\lambda}_1$, $\hat{\lambda}_2$, their p-values, and F-statistics across the regressions for each company are reported in Table 2.

3.3 Tests of non-linear model

Significance tests for the three estimated parameters parallel those of Janosi, Jarrow and Yildirim (2002). About 50% of the regressions produce λ_0 estimates

TABLE 2 Summary of default parameters

Company name	No. of bonds	No. of regressions	λ_0	λ_1	λ_2	*F*-test
FINANCIALS						
Security Pacific Corp	45	32	0.0076*	0.0095*	–0.0005	0.0109
Fleet Financial Group	21	57	0.0121*	0.0115	–0.0101*	0.0110
Bankers Trust Ny	32	4	0.0005	0.0151*	0.0012	0.0010
Merrill Lynch	154	64	0.0085*	0.0142*	–0.0002	0.0021
FOOD AND BEVERAGES						
Pepsico Inc	68	64	0.0004	0.0186*	–0.0004	0.0003
Coca-Cola Enterprises Inc	57	31	0.0003	0.0143*	–0.0056	0.0016
AIRLINES						
AMR Corporation	102	31	0.0015*	0.0152*	0.0003	0.0301
Southwest Airlines Co	34	59	0.0181	–0.0012	0.0002*	0.1000
UTILITIES						
Carolina Power & Light	40	6	0.0132*	–0.0002	0.0011*	0.0123
Texas Utilities Ele Co	61	36	0.0120*	–0.0031*	–0.0002	0.0001
PETROLEUM						
Mobil Corp	45	51	0.0005	0.0122*	–0.0006	0.0308
Union Oil of California	51	64	0.0032*	0.0002*	0.0013	0.0009
Shell Oil Co	57	36	0.0005*	0.0120*	–0.0005	0.0310
DEPARTMENT STORES						
Sears Roebuck & Co	64	57	0.0031	0.0236*	–0.0008	0.1020
Dayton Hudson Corp	126	48	0.0101*	0.0002*	–0.0021	0.0543
Wal-Mart Stores, Inc	76	64	0.0056	–0.0021*	0.0012*	0.0001
TECHNOLOGY						
Eastman Kodak Company	53	33	0.0020	0.0110*	–0.0001	0.0033
Xerox Corp	41	64	0.0110*	–0.0023*	–0.0001	0.0092
Texas Instruments	27	54	0.0001	0.01039*	0.0006	0.0004
Intl Business Machines	46	39	0.0039	0.0112*	–0.0001	0.0300

In each month of the sample period, a pool of the company's bonds across maturities for that month and for the past seven months is formed. The "Number of bonds" refers to the number in this pool, which is consistent throughout. Otherwise, the particular pool for the month is deleted. Non-linear regression is then performed to obtain the λ-estimates. The "Number of regressions" column corresponds to the number of months where such regressions are performed or the number of distinct regressions that are conducted over the sample period. The reported estimates in the columns $\hat{\lambda}_0$, $\hat{\lambda}_1$, and $\hat{\lambda}_2$ as well as the *F*-test for each company are the average values across all the regressions. The *F*-test is based on the null hypothesis H_0: $\lambda_0 = \lambda_1 = \lambda_2 = 0$. An asterix (*) denotes significance of the parameter estimate at the 10% level in terms of the average p-value. The results are comparable to Janosi, Jarrow and Yildirim (2002), indicating our parsimonious model with tax effects offers a similar fit for corporate debt.

that are significantly different from zero at the 10% significance level. About 85% of the regressions yield significant estimates of λ_1 while only 20% generate significant estimates of λ_2. The latter results are similar to the findings of Janosi, Jarrow and Yildirim (2002) where only five out of the 20 estimated values for λ_2 are significant without liquidity and none are significant when liquidity is considered. The properties of λ_1 and λ_2 which represent the sensitivity of default to the default-free short rate and the equity market index are of particular interest. For example, the sign of λ_1 in Table 2 is generally positive. Therefore, high interest rates are consistent with higher default risk. In addition, λ_2 is generally negative implying that higher market indices are consistent with lower default risk.

The dynamics of λ_1 are consistent with our economic intuition. Higher interest rates increase the burden of interest payments and therefore the default likelihood. As in Janosi, Jarrow and Yildirim (2002), market risk does not appear to exert much influence on default risk although they argue it would be inappropriate to conclude the association between stock market returns and default is inconsequential. As they suggest, an industry index rather than a market index may lead to stronger results. The results of the joint F-test of the hypothesis that $\lambda_0 = \lambda_1 = \lambda_2 = 0$ in Table 2 indicate that on average, for 19 out of 20 firms, this hypothesis may be rejected at the 10% significance level. These results are similar to those of Janosi, Jarrow and Yildirim (2002).

Table 3 contains the R^2 figures for our model with tax effects as well as those reported in Janosi, Jarrow and Yildirim (2002) for each firm.[3]

As seen in Table 3, the regression R^2 are high with a minimum value of 0.75 for Bankers Trust and a maximum value of 0.90 for Eastman Kodak. These high R^2 values attest to the theoretical model's excellent in-sample fit.[4] Overall, on average, the in-sample fit of our model with taxes is similar to Janosi, Jarrow and Yildirim (2002).[5]

The default intensities and corresponding default probabilities are recorded in Table 4. In addition, the default intensity parameters are also presented for com-

[3] We also computed augmented Dickey Fuller statistics for the parameters in an attempt to verify their stationarity, as in Janosi, Jarrow and Yildirim (2002). Despite favorable results with 10, 14, and 9 out of 18 respective $\hat{\lambda}_0$, $\hat{\lambda}_1$, and $\hat{\lambda}_2$ estimates being stationary at the 10% level, both our methodologies employ lagged observations in the non-linear parameter estimation. Therefore, the stationarity test procedure is not valid. We thank the editor for discovering this flaw in a previous version.

[4] We also compute the average RMSE (root mean squared error) for the 20 companies used in the regressions. They are all very small, eg, 2% to 4% relative to the average bond prices. These values are similar to those of Janosi, Jarrow and Yildirim (2002) who report their *average* values across all 20 firms.

[5] Janosi, Jarrow and Yildirim (2002) also compute a generalized cross validation (GCV) statistic to enable a comparison across their five models which have different numbers of parameters. In contrast, there is only one version of our pricing model with tax effects. Moreover, the GCV statistic is related to R^2 as GCV equals

$$(1 - R^2)\frac{\text{Total sum of squares}}{n\left(1 - \frac{1}{n}\text{tr}(A)\right)^2}$$

where tr(A) is the trace of the matrix $A = X(X'X)^{-1}X'$ for X denoting the Jacobian in the non-linear least squares procedure.

TABLE 3 R^2 statistics for the proposed model with tax effects as well as, for comparative purposes, this quantity for the first two liquidity models reported in Janosi, Jarrow and Yildirim (2002, Table 2).

Company name	**R^2 with tax effects**	**R^2 for JJY Model 1**	**R^2 for JJY Model 2**
FINANCIALS			
Security Pacific Corp	0.8308	0.8120	0.8366
Fleet Financial Group	0.8507	0.4143	0.4501
Bankers Trust Ny	0.7543	0.9504	0.9576
Merrill Lynch	0.8806	0.8918	0.9000
FOOD AND BEVERAGES			
Pepsico Inc	0.8402	0.8554	0.8616
Coca-Cola Enterprises Inc	0.8098	0.8142	0.8783
AIRLINES			
AMR Corporation	0.8568	0.9216	0.9346
Southwest Airlines Co	0.8203	0.8432	0.8513
UTILITIES			
Carolina Power & Light	0.8104	0.8559	0.8598
Texas Utilities Ele Co	0.8012	0.8329	0.8370
PETROLEUM			
Mobil Corp	0.8451	0.9787	0.9824
Union Oil of California	0.8560	0.9219	0.9255
Shell Oil Co	0.8403	0.8309	0.8556
DEPARTMENT STORES			
Sears Roebuck & Co	0.8074	0.7245	0.7338
Dayton Hudson Corp	0.8864	0.8859	0.9219
Wal-Mart Stores, Inc	0.8901	0.9415	0.9430
TECHNOLOGY			
Eastman Kodak Company	0.9012	0.9257	0.9379
Xerox Corp	0.8561	0.9210	0.9235
Texas Instruments	0.8394	0.8947	0.9142
Intl Business Machines	0.8802	0.8942	0.9134
Average over all firms	0.8429	0.8555	0.8709

Despite our study being conducted on more heterogeneous data, average R^2 statistics in the last row indicate comparable performance between the tax and liquidity frameworks. Recall that we include more investment grade debt issues in our sample while Janosi, Jarrow and Yildirim (2002) only consider senior debt. Note that implementation of our model incorporates a 60% recovery rate, to be consistent with Elton, Gruber, Agrawal and Mann (2001), while Janosi, Jarrow and Yildirim (2002) utilize a 50% recovery rate.

parison with the results of Janosi, Jarrow and Yildirim (2002). Table 4 indicates that our tax-based model provides reasonable estimates of default probabilities. Furthermore, our coefficient estimates underlying the default intensities are similar to those of Janosi, Jarrow and Yildirim (2002).

TABLE 4 Summary of estimated parameters and resulting default intensities for proposed tax-based model as well as models 1 and 2 of Janosi, Jarrow and Yildirim (2002).

	Model			
Variable	**With taxes**	**JJY 1**	**JJY 2**	**RiskMetrics**
Number of significantly positive λ_1	11	6	6	–
Number of significantly negative λ_1	3	4	14	–
Number of significantly positive λ_2	1	5	0	–
Number of significantly negative λ_2	3	0	0	–
Estimate of default intensity (AAA)	0.001	0.007	0.018	0.0000
Estimate of default intensity (AA)	0.007	0.008	0.012	0.0003
Estimate of default intensity (A)	0.018	0.014	0.019	0.0001
Estimate of default intensity (BBB)	0.014	0.019	0.023	0.0160
Average default intensity	0.014	0.013	0.018	0.0041

Of the five models they implement, model 2 is considered to be superior while model 1 has the fewest number of parameters. Observe that our model with taxes is more sensitive to the spot interest rate than Janosi, Jarrow and Yildirim (2002). However, both frameworks indicate a weak relationship between the market index and default risk. Despite being larger, the default intensities for our model are closer to those provided by Riskmetrics for AAA and AA class debt than model 2 of Janosi, Jarrow and Yildirim (2002). The liquidity and tax models nearly coincide for BBB rated debt, although both models overestimate default risk for higher quality debt. Note that implementation of our model incorporates a 60% recovery rate, to be consistent with Elton, Gruber, Agrawal and Mann (2001), while Janosi, Jarrow and Yildirim (2002) utilize a 50% recovery rate.

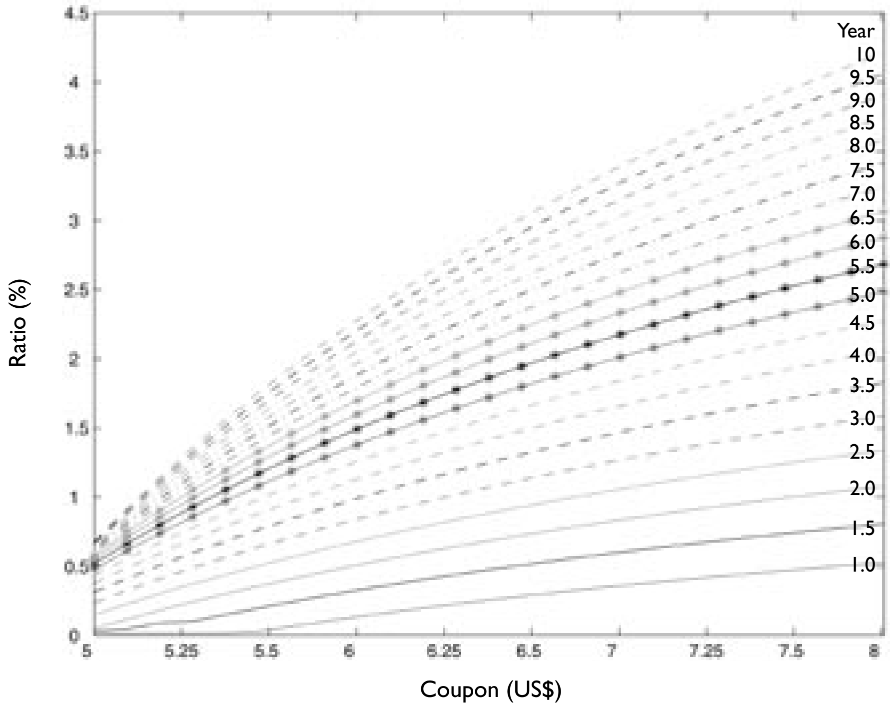
3.4 Economic impact of taxation

To illustrate the importance of taxes on the pricing of defaultable coupon bond prices, we perform the following analyses. Given a set of the average estimated values of $\hat{a}$, $\hat{\sigma}_r$, $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\beta}_3$, or 0.0324, 0.0135, 0.0960, –0.0187, –0.0181, and 2.2818 respectively, the default-free bond price $B(t, T_N) = V_G$ is computed for different maturities $T_N = 1.0, 1.5, \ldots, 9.5, 10$ years, and different coupon rates from 5% (or 5) to 8%. Then average estimated values of $\hat{\rho}$, $\hat{W}_m(t)$, $\hat{\lambda}_0$, $\hat{\lambda}_1$, and $\hat{\lambda}_2$, or –0.0985, 1.6155, 0.0076, 0.0095, and –0.0005 together with $R = 0.6$ and $t_s = t_g = 0$ are used to calculate a defaultable coupon bond price without tax, $\tilde{B}(t, T_N) = V_C$. We then add non-zero tax rates $t_s = 0.075$ and $t_g = 0.35$, as in Elton, Gruber, Agrawal and Mann (2001), to calculate a taxable defaultable coupon bond price $V(t, T_N) = V_{CT}$. For both V_C and V_{CT}, the same maturities and coupon rates are investigated.

The ratio $(V_G - V_C)/(V_C - V_{CT})$ for different maturities and coupon rates is then constructed. Either a small ratio or a large ratio that deviates from one indicates a significant tax impact that causes taxable defaultable debt to be under-priced or over-priced respectively.

The results are plotted in Figure 1 with each line representing a different maturity, the bottom line being one year and the top line corresponding to 10 years.

FIGURE 1 Tax impact for different coupons and maturities.



The ratio is defined as $(V_G - V_C)/(V_G - V_{CT})$, where V_G are default-free government bond prices and V_{CT} and V_C are defaultable coupon bond prices with and without taxes. The ratio is plotted for bonds with different coupons ranging from 5% (or US$5 per par US$100) to 8% and for bonds whose maturities range from one year to 10 years in increments of six months.

The experiment demonstrates the important role taxation plays in the pricing of taxable defaultable coupon bonds. The presence of tax reduces the defaultable bond's value as the maturity and the coupon rate increases. This result stems from longer streams of coupon payments and higher coupon payments producing larger taxable gains. However, for lower coupon and shorter maturity defaultable bonds, the ratios are less than one, indicating the advantage of a tax rebate generated by the capital loss in the event of default. As seen in Figure 1, the price difference between default-free and defaultable bonds is three times the price difference when taxes are considered with a 6.5% coupon. Thus, the tax impact is economically significant, suggesting a lesser role for default. This implication is consistent with the results of Elton, Gruber, Agrawal and Mann (2001).

4 Conclusions

This paper develops a theoretical model for pricing defaultable debt by incorporating taxation into the reduced-form credit risk framework. Empirical evidence indicates that the proposed model provides an excellent in-sample fit. Indeed, its performance is comparable to Janosi, Jarrow and Yildirim (2002) in terms of explaining the variation in corporate debt prices over time. Thus, by incorporating the effects of taxation, we are able to enhance existing credit risk models while retaining model parsimony. Furthermore, a basis for empirical observations regarding the importance of taxes on the pricing of corporate debt is established, with the impact of taxation increasing as a function of the debt's maturity and coupon rate.

We also find additional evidence that the default likelihood is correlated with the default-free interest rate, a result consistent with previous empirical studies. However, the association between default risk and the return of the S&P500 appears to be rather weak. Further research is needed to address this issue.

Appendix A – Lemmas

Lando (1998) proves the following lemmas. Where necessary, the usual regularity conditions are assumed.

LEMMA 1: Consider a contingent claim that pays a random amount X at time T provided default has not occurred, and zero otherwise. The time t value of this claim is

$$E_t^Q\left[\exp\left(-\int_t^T r(s)\mathrm{d}s\right)X1_{\{\tau>T\}}\right] = 1_{\{\tau>t\}}E_t^Q\left[\exp\left(-\int_t^T [r(s)+\lambda(s)]\mathrm{d}s\right)X\right]$$

LEMMA 2: Consider a security that pays a cash flow $Y(s)$ per unit time at time s provided default has not occurred, and zero otherwise. The time t value of this security is

$$E_t^Q\left[\int_t^T Y(s)1_{\{\tau>s\}}\exp\left(-\int_t^s r(u)\mathrm{d}u\right)\mathrm{d}s\right]$$
$$= 1_{\{\tau>t\}}E_t^Q\left[\int_t^T Y(s)\exp\left(-\int_t^s [r(u)+\lambda(u)]\mathrm{d}u\right)\mathrm{d}s\right]$$

LEMMA 3: Consider a security that pays $Z(\tau)$ if default occurs at time τ, and zero otherwise. The time t value of the security is

$$E_t^Q\left[\exp\left(-\int_t^\tau r(s)\mathrm{d}s\right)Z(\tau)\right]$$
$$= 1_{\{\tau>t\}}E_t^Q\left[\int_t^T Z(s)\lambda(s)\exp\left(-\int_t^s [r(u)+\lambda(u)]\mathrm{d}u\right)\mathrm{d}s\right]$$

Appendix B – Derivation of $V_1(t, T_N)$

From equation (2),

$$v_j(t) = 1_{\{\tau > t\}} E_t^Q \left\{ \int_{T_{j-1}}^{T_j} f_j(s)\lambda(s)\exp\left(-\int_t^s [r(u)+\lambda(u)]\mathrm{d}u\right)\mathrm{d}s \right\}$$

Assuming the usual technical conditions for interchanging integral operators,

$$v_j(t) = 1_{\{\tau > t\}} \int_{T_{j-1}}^{T_j} f_j(s) E_t^Q\{X\exp(Y)\}\mathrm{d}s$$

using the definitions of $X = \lambda(s)$ and $Y = -\int_t^s [r(u) + \lambda(u)]\mathrm{d}u$.

X and Y are Gaussian processes implying

$$E_t^Q[X\mathrm{e}^Y] = E_t^Q[\mathrm{e}^Y]E_t^Q[X] + \mathrm{cov}_t^Q[X, \mathrm{e}^Y]$$

The second term on the right is

$$E_t^Q[\mathrm{e}^Y]\mathrm{cov}_t^Q[X,Y]\left[\frac{\exp\left(\mathrm{var}_t^Q[Y]-1\right)}{\mathrm{var}_t^Q[Y]}\right]^{\frac{1}{2}}$$

For small $\mathrm{var}_t(Y)$, the last factor may be taken as one to yield

$$E_t^Q[X\mathrm{e}^Y] = E_t^Q[\mathrm{e}^Y]\left(E_t^Q[X] + \mathrm{cov}_t^Q[X,Y]\right)$$

Next we decompose Y into $Y_1 + Y_2$ where

$$Y_1 = -\int_t^{T_{j-1}} [r(u)+\lambda(u)]\mathrm{d}u$$

$$Y_2 = -\int_{T_{j-1}}^{s} [r(u)+\lambda(u)]\mathrm{d}u$$

Then,

$$\begin{aligned} E_t^Q[\mathrm{e}^Y] &= E_t^Q[\mathrm{e}^{Y_1+Y_2}] \\ &= \exp\left\{\sum_{i=1}^{2}\left[E_t^Q(Y_i) + \tfrac{1}{2}\mathrm{var}_t^Q(Y_i) + \mathrm{cov}_t^Q[Y_1, Y_2]\right]\right\} \\ &= \tilde{B}(t, T_{j-1})\exp\left[E_t^Q(Y_2) + \tfrac{1}{2}\mathrm{var}_t^Q(Y_2) + \mathrm{cov}_t^Q[Y_1, Y_2]\right] \end{aligned}$$

Thus,

$$V_1(t,T_N)=1_{\{\tau>t\}}\sum_{j=1}^{N}\tilde{B}(t,T_{j-1})\int_{T_{j-1}}^{T_j}\Big[f_j(s)\exp\Big(E_t^Q[Y_2]+\tfrac{1}{2}\mathrm{var}_t^Q[Y_2]+\mathrm{cov}_t^Q[Y_1,Y_2]\Big)\Big(E_t^Q[X]+\mathrm{cov}_t^Q[X,Y]\Big)\Big]\mathrm{d}s$$

$$v_j(t)=1_{\{\tau>t\}}\tilde{B}(t,T_{j-1})\int_{T_{j-1}}^{T_j}\Big\{f_j(s)\exp\Big[E_t^Q[Y_2]+\tfrac{1}{2}\mathrm{var}_t^Q[Y_2]+\mathrm{cov}_t^Q[Y_1,Y_2]\Big]\Big[E_t^Q[X]+\mathrm{cov}_t^Q[X,Y]\Big]\Big\}\mathrm{d}s$$

Hence, $V_1(t,T_N)=\sum_{j=1}^{T_N}v_j(t)$ is shown.

Appendix C – Derivation of $\boldsymbol{\pi_j}$

In what follows, equation (9) is used repeatedly to substitute $f(t,s)$ and the stochastic integrals of $\mathrm{d}W_r(u)$ for $r(s)$. Let

$$\begin{aligned}X&=\lambda(s)\\&=\left[\lambda_0+\lambda_1 f(t,s)+\frac{\sigma_r^2\lambda_1}{2a^2}\left(1-\mathrm{e}^{-a(s-t)}\right)^2+\lambda_2 W_m(t)\right]\\&\quad+\int_t^s\lambda_1\sigma_r\mathrm{e}^{-a(s-u)}\mathrm{d}W_r(u)+\lambda_2\int_t^s\mathrm{d}W_m(u)\end{aligned}$$

and

$$\begin{aligned}Y&=-\int_t^s[r(u)+\lambda(u)]\mathrm{d}u\\&=-\left[\frac{(1+\lambda_1)\sigma_r^2}{2a^2}+\lambda_0\right](s-t)-\frac{(1+\lambda_1)\sigma_r^2}{2a^2}\left[\frac{1-\mathrm{e}^{-2a(s-t)}}{2a}-2\frac{1-\mathrm{e}^{-a(s-t)}}{a}\right]\\&\quad-(1+\lambda_1)\int_t^s f(t,u)\mathrm{d}u-\lambda_2 W_m(t)(s-t)\\&\quad-\int_t^s\lambda_2(s-u)\mathrm{d}W_m(u)-\int_t^s\frac{(1+\lambda_1)\sigma_r}{a}\left[1-\mathrm{e}^{-a(s-u)}\right]\mathrm{d}W_r(u)\end{aligned}$$

Then,

$$E_t^Q[X] = \left[\lambda_0 + \lambda_1 f(t,s) + \frac{\sigma_r^2 \lambda_1}{2a^2}\left(1 - \mathrm{e}^{-a(s-t)}\right)^2 + \lambda_2 W_m(t)\right]$$

and

$$\begin{aligned}\operatorname{cov}_t^Q[X,Y] = {} & \frac{\lambda_1(1+\lambda_1)\sigma_r^2}{a}\left[\frac{1-\mathrm{e}^{-2a(s-t)}}{2a} - \frac{1-\mathrm{e}^{-a(s-t)}}{a}\right] \\ & + \lambda_1\lambda_2\sigma_r\rho\left[(s-t)\frac{\mathrm{e}^{-a(s-t)}}{a} - \frac{1-\mathrm{e}^{-a(s-t)}}{a^2}\right] \\ & + \frac{\lambda_2(1+\lambda_1)\sigma_r\rho}{a}\left[\frac{1-\mathrm{e}^{-a(s-t)}}{a} - (s-t)\right] - \frac{\lambda_2^2}{2}(s-t)^2\end{aligned}$$

Therefore,

$$\begin{aligned}E_t^Q[X] + \operatorname{cov}_t^Q[X,Y] = {} & \left[\lambda_0 + \frac{2\rho\lambda_2\sigma_r - \lambda_1^2\sigma_r^2}{2a^2} + \lambda_1 f(t,s) + \lambda_2 W_m(t)\right] \\ & - \frac{\lambda_2^2}{2}(s-t)^2 - \frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a}(s-t) \\ & + \mathrm{e}^{-a(s-t)}\left[\frac{\lambda_1^2\sigma_r^2 - \rho\lambda_2\sigma_r}{a^2}\right] \\ & + \mathrm{e}^{-2a(s-t)}\left[\frac{-\lambda_1^2\sigma_r^2}{2a^2}\right] + \frac{\rho\lambda_1\lambda_2\sigma_r}{a}(s-t)\mathrm{e}^{-a(s-t)}\end{aligned}$$

Now,

$$\begin{aligned}Y_1 = {} & -\int_t^{T_{j-1}} [r(u) + \lambda(u)]\,\mathrm{d}u \\ = {} & -\frac{(1+\lambda_1)\sigma_r^2}{2a^2}\left[\frac{1-\mathrm{e}^{-2a(T_{j-1}-t)}}{2a} - 2\frac{1-\mathrm{e}^{-a(T_{j-1}-t)}}{a}\right] \\ & - \left[\frac{(1+\lambda_1)\sigma_r^2}{2a^2} + \lambda_0 + \lambda_2 W_m(t)\right](T_{j-1}-t) - (1+\lambda_1)\int_t^{T_{j-1}} f(t,u)\,\mathrm{d}u \\ & - \int_t^{T_{j-1}} \lambda_2(T_{j-1}-u)\,\mathrm{d}W_m(u) \\ & - \int_t^{T_{j-1}} \frac{(1+\lambda_1)\sigma_r}{a}\left[1-\mathrm{e}^{-a(T_{j-1}-u)}\right]\mathrm{d}W_r(u)\end{aligned}$$

$$
\begin{aligned}
Y_2 = &-\int_{T_{j-1}}^{s}\left[r(u)+\lambda(u)\right]\mathrm{d}u \\
&-\left[\frac{(1+\lambda_1)\sigma_r^2}{2a^2}+\lambda_0+\lambda_2 W_m(t)\right](s-T_{j-1})-(1+\lambda_1)\int_{T_{j-1}}^{s} f(t,u)\mathrm{d}u \\
&-\frac{(1+\lambda_1)\sigma_r^2}{2a^2}\left[\frac{\mathrm{e}^{-2a(T_{j-1}-t)}-\mathrm{e}^{-2a(s-t)}}{2a}-2\frac{\mathrm{e}^{-a(T_{j-1}-t)}-\mathrm{e}^{-a(s-t)}}{a}\right] \\
&-\int_{T_{j-1}}^{s}\lambda_2(s-u)\mathrm{d}W_m(u)-\int_{t}^{T_{j-1}}\lambda_2(s-T_{j-1})\mathrm{d}W_m(u) \\
&-\int_{T_{j-1}}^{s}\frac{(1+\lambda_1)\sigma_r}{a}\left[1-\mathrm{e}^{-a(s-u)}\right]\mathrm{d}W_r(u) \\
&-\int_{t}^{T_{j-1}}\frac{(1+\lambda_1)\sigma_r}{a}\left[\mathrm{e}^{-a(T_{j-1}-u)}-\mathrm{e}^{-a(s-u)}\right]\mathrm{d}W_r(u)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
E_t^Q\left[Y_2\right] = &-\frac{(1+\lambda_1)\sigma_r^2}{2a^2}\left[\frac{\mathrm{e}^{-2a(T_{j-1}-t)}}{2a}-2\frac{\mathrm{e}^{-a(T_{j-1}-t)}}{a}\right] \\
&-\left[\frac{(1+\lambda_1)\sigma_r^2}{2a^2}+\lambda_0+\lambda_2 W_m(t)\right](s-T_{j-1})-(1+\lambda_1)\int_{T_{j-1}}^{s} f(t,u)\mathrm{d}u \\
&+\mathrm{e}^{-a(s-t)}\left[-\frac{(1+\lambda_1)\sigma_r^2}{a^3}\right]+\mathrm{e}^{-2a(s-t)}\left[\frac{(1+\lambda_1)\sigma_r^2}{4a^3}\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var}_t^Q\left[Y_2\right] = &\frac{\lambda_2^2}{3}(s-T_{j-1})^3 \\
&+\lambda_2^2(s-T_{j-1})^2(T_{j-1}-t) \\
&+\frac{(1+\lambda_1)^2\sigma_r^2}{a^2}\left[(s-T_{j-1})+\frac{1-\mathrm{e}^{-2a(s-T_{j-1})}}{2a}-2\frac{1-\mathrm{e}^{-a(s-T_{j-1})}}{a}\right] \\
&+\frac{(1+\lambda_1)^2\sigma_r^2}{a^2}\left[\frac{1-\mathrm{e}^{-2a(T_{j-1}-t)}}{2a}+\frac{\mathrm{e}^{-2a(s-T_{j-1})}-\mathrm{e}^{-2a(s-t)}}{2a}\right. \\
&\left.\qquad-\frac{\mathrm{e}^{-a(s-T_{j-1})}-\mathrm{e}^{-a(T_{j-1}+s-2t)}}{a}\right]
\end{aligned}
$$

$$+\frac{\lambda_2(1+\lambda_1)\sigma_r\rho}{a}\left[\frac{(s-T_{j-1})^2}{2}-\frac{1-e^{-a(s-T_{j-1})}}{a^2}+(s-T_{j-1})\frac{e^{-a(s-T_{j-1})}}{a}\right]$$
$$+\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a}(s-T_{j-1})\left[\frac{1-e^{-a(T_{j-1}-t)}}{a}-\frac{e^{-a(s-T_{j-1})}-e^{-a(s-t)}}{a}\right]$$

and

$$\begin{aligned}\mathrm{cov}_t^Q[Y_1,Y_2]&=\frac{1}{2}\lambda_1\lambda_2(s-T_{j-1})(T_{j-1}-t)^2\\&+\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a}\left[\frac{1-e^{-a(T_{j-1}-t)}}{a^2}-(T_{j-1}-t)\frac{e^{-a(T_{j-1}-t)}}{a}\right]\\&-\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a}\left[\frac{e^{-a(s-T_{j-1})}-e^{-a(s-t)}}{a^2}-(s-t)\frac{e^{-a(s-t)}}{a}\right]\\&+\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a}\left[(T_{j-1}-t)-\frac{1-e^{-a(T_{j-1}-t)}}{a}\right](s-T_{j-1})\\&+\frac{(1+\lambda_1)^2\sigma_r^2}{a^2}\left[\frac{\left(1-e^{-a(T_{j-1}-t)}\right)^2}{2a}-\frac{e^{-a(s-T_{j-1})}}{2a}\right.\\&\qquad\left.+\frac{e^{-a(s-t)}}{a}-\frac{e^{-a(T_{j-1}+s-2t)}}{2a}\right]\end{aligned}$$

Adding terms together and simplifying yields

$$\begin{aligned}&E_t^Q[Y_2]+\tfrac{1}{2}\mathrm{var}_t^Q[Y_2]+\mathrm{cov}_t^Q[Y_1,Y_2]\\&=\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{2a^3}\\&-\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a^2}(T_{j-1}-t)e^{-a(T_{j-1}-t)}\\&-\frac{(1+\lambda_1)(\lambda_1\sigma_r+\rho\lambda_2)\sigma_r}{a^3}e^{-a(T_{j-1}-t)}\\&+\frac{\lambda_1(1+\lambda_1)\sigma_r^2}{4a^3}e^{-2a(T_{j-1}-t)}\\&-(1+\lambda_1)\int_{T_{j-1}}^{s}f(t,u)\,du+\frac{\lambda_2^2}{6}(s-T_{j-1})^3\end{aligned}$$

$$+\left[\frac{\lambda_2^2}{2}(T_{j-1}-t)+\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{4a}\right](s-T_{j-1})^2$$
$$+\left[\frac{\lambda_1(1+\lambda_1)\sigma_r^2}{2a^2}-\lambda_0-\lambda_2 W_m(t)+\frac{\lambda_1\lambda_2}{2}(T_{j-1}-t)^2\right](s-T_{j-1})$$
$$+\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a}\left[(T_{j-1}-t)-\frac{1-e^{-a(T_{j-1}-t)}}{2a}\right](s-T_{j-1})$$

$$+\frac{(1+\lambda_1)(\lambda_1\sigma_r+\rho\lambda_2)\sigma_r}{a^3}e^{-a(s-t)}-\frac{\lambda_1(1+\lambda_1)\sigma_r^2}{4a^3}e^{-2a(s-t)}$$
$$-\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{2a^3}e^{-a(s-T_{j-1})}+\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a^2}(s-t)e^{-a(s-t)}$$
$$+\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{2a^2}(s-T_{j-1})e^{-a(s-t)}$$

For small a, we can employ Taylor series expansion on the above to yield

$$E_t^Q[Y_2]+\tfrac{1}{2}\operatorname{var}_t^Q[Y_2]+\operatorname{cov}_t^Q[Y_1,Y_2]$$
$$=\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{2a^3}$$
$$-\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a^2}(T_{j-1}-t)e^{-a(T_{j-1}-t)}$$
$$-\frac{(1+\lambda_1)(\lambda_1\sigma_r+\rho\lambda_2)\sigma_r}{a^3}e^{-a(T_{j-1}-t)}$$
$$+\frac{\lambda_1(1+\lambda_1)\sigma_r^2}{4a^3}e^{-2a(T_{j-1}-t)}$$
$$-(1+\lambda_1)\int_{T_{j-1}}^{s}f(t,u)\,du+\frac{\lambda_2^2}{6}(s-T_{j-1})^3$$

$$+\left[\frac{\lambda_2^2}{2}(T_{j-1}-t)+\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{4a}\right](s-T_{j-1})^2$$
$$+\left[\frac{\lambda_1(1+\lambda_1)\sigma_r^2}{2a^2}-\lambda_0-\lambda_2 W_m(t)+\frac{\lambda_1\lambda_2}{2}(T_{j-1}-t)^2\right](s-T_{j-1})$$
$$+\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a}\left[(T_{j-1}-t)-\frac{1-e^{-a(T_{j-1}-t)}}{2a}\right](s-T_{j-1})$$

$$+\left[\frac{(1+\lambda_1)\left[\lambda_1\sigma_r+\rho\lambda_2(1+aT_{j-1}-at)\right]\sigma_r e^{-a(T_{j-1}-t)}}{a^3}-\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{2a^3}\right]\times$$

$$\left[1-a(s-T_{j-1})\right]-\frac{\lambda_1(1+\lambda_1)\sigma_r^2 e^{-2a(T_{j-1}-t)}}{4a^3}\left[1-2a(s-T_{j-1})\right]$$

$$+\frac{3\rho\lambda_2(1+\lambda_1)\sigma_r e^{-a(T_{j-1}-t)}}{2a^2}(s-T_{j-1})\left[1-a(s-T_{j-1})\right]$$

In the empirical tests of Janosi, Jarrow and Yildirim (2002) and this paper, a lies within the interval (0.01, 0.05). Here $(s - T_{j-1}) \in [0, 0.5]$ because the integration is from T_{j-1} to T_j with respect to s. Suppose $a = 0.05$, $s - T_{j-1} = 0.5$, when the largest error occurs. For the first order approximation of $\exp(-a(s - T_{j-1}))$ as $1 - a(s - T_{j-1})$ (and terms involving s in the exponent), the relative and absolute errors are

$$\frac{e^{-a(s-T_{j-1})}}{1-a(s-T_{j-1})}=1.000317$$

$$e^{-a(s-T_{j-1})}-\left[1-a(s-T_{j-1})\right]=3.099120\times10^{-4}$$

while for the second order approximation, the relative and absolute errors are

$$\frac{e^{-a(s-T_{j-1})}}{1-a(s-T_{j-1})+\frac{1}{2}a^2(s-T_{j-1})^2}=0.999997$$

$$e^{-a(s-T_{j-1})}-\left[1-a(s-T_{j-1})+\frac{1}{2}a^2(s-T_{j-1})^2\right]=-2.587971\times10^{-6}$$

Both the relative and absolute errors are too small to produce a significant truncation error. Therefore, it is reasonable to omit the higher order terms from the numerical approximation.

Collecting terms, $E_t^Q[Y_2] + ½\text{var}_t^Q[Y_2] + \text{cov}_t^Q[Y_1, Y_2]$ is simplified to

$$-(1+\lambda_1)\int_{T_{j-1}}^{s} f(t,u)\,du + H_{j-1}(s-T_{j-1})+I_{j-1}(s-T_{j-1})^2$$

where

$$I_{j-1}=\frac{\lambda_2^2}{2}(T_{j-1}-t)+\rho\lambda_2(1+\lambda_1)\sigma_r\left[\frac{1}{4a}-\frac{3e^{-a(T_{j-1}-t)}}{2a}\right]$$

$$H_{j-1} = \frac{\lambda_1(1+\lambda_1)\sigma_r^2}{2a^2} - \lambda_0 - \lambda_2 W_m(t) + \frac{\lambda_1\lambda_2}{2}(T_{j-1}-t)^2$$
$$+\frac{\lambda_1(1+\lambda_1)\sigma_r^2}{2a^2}\mathrm{e}^{-2a(T_{j-1}-t)} + \frac{\rho\lambda_2(1+\lambda_1)}{a}(T_{j-1}-t)$$
$$-(1+\lambda_1)\left[\frac{\lambda_1\sigma_r^2}{a^2} + \frac{\rho\lambda_2(T_{j-1}-t)\sigma_r}{a} + \frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a^2}\right]\mathrm{e}^{-a(T_{j-1}-t)}$$

We are left with the default-free instantaneous forward rate $f(t,u)$ in the expression. We employ the parametric methodology of Nelson and Siegel (1987) to estimate the forward rate curve

$$\int_{T_{j-1}}^{s} f(t,u)\mathrm{d}u \approx \frac{\beta_2 \mathrm{e}^{-\left((T_{j-1}-t)/\beta_3\right)}}{\beta_3}(s-T_{j-1})^2 + N_{j-1}(s-T_{j-1})$$

where

$$N_{j-1} = \beta_0 + \left[\beta_1 + \frac{\beta_2}{\beta_3}(T_{j-1}-t)\right]\mathrm{e}^{-\left((T_{j-1}-t)/\beta_3\right)}$$

Therefore,

$$E_t^Q[Y_2] + \tfrac{1}{2}\mathrm{var}_t^Q[Y_2] + \mathrm{cov}_t^Q[Y_1, Y_2]$$
$$= \left[-(1+\lambda_1)N_{j-1} + H_{j-1}\right](s-T_{j-1})$$
$$+\left[\frac{\beta_2(1+\lambda_1)\mathrm{e}^{-\left((T_{j-1}-t)/\beta_3\right)}}{\beta_3} + I_{j-1}\right](s-T_{j-1})^2$$

with H_{j-1}, I_{j-1} and N_{j-1} defined earlier.

Thus, in equations (3) and (4), π_j may be explicitly evaluated as

$$\pi_j = \int_{T_j}^{T_{j+1}} \Big\{ f_j(s)\exp[E_t^Q[Y_2] + \tfrac{1}{2}\mathrm{var}_t^Q[Y_2] + \mathrm{cov}_t^Q[Y_1, Y_2]] \times [E_t^Q[X] + \mathrm{cov}_t^Q[X, Y]] \Big\}\,\mathrm{d}s$$

$$= \mathrm{e}^{-\frac{(K_j - 1/\beta_3)^2}{4L_j} - \frac{(T_j - t)}{\beta_3}} \left\{ \lambda_1 \left[F - RC \frac{K_j - 1/\beta_3}{2L_j} \right] \times \right.$$

$$\left. \left[\frac{\beta_2}{\beta_3} \left(T_j - t - \frac{K_j - 1/\beta_3}{2L_j} \right) + \beta_1 \right] - \frac{\lambda_1 \beta_2 RC}{2L_j \beta_3} \right\} W_j \left(\frac{K_j - 1/\beta_3}{2L_j} \right)$$

$$- \left[F - RC \frac{K_j - 2a}{2L_j} \right] \frac{\lambda_1^2 \sigma_r^2}{2a^2} \mathrm{e}^{-2a(T_{j-1} - t) - \frac{(K_j - 2a)^2}{4L_j}} W_j \left(\frac{K_j - 2a}{2L_j} \right)$$

$$+ \mathrm{e}^{-a(T_j - t) - \frac{(K_j - a)^2}{4L_j}} \left[F - RC \frac{K_j - a}{2L_j} \right] \left[\frac{\lambda_1^2 \sigma_r^2 - \rho \lambda_2 \sigma_r}{a^2} + \frac{\rho \lambda_1 \lambda_2 \sigma_r}{a} \times \right.$$

$$\left. \left(T_j - t - \frac{K_j - a}{2L_j} \right) \right] W_j \left(\frac{K_j - a}{2L_j} \right)$$

$$- \mathrm{e}^{-a(T - t) - \frac{(K_j - a)^2}{4L_j}} \frac{\rho \lambda_1 \lambda_2 \sigma_r RC}{2L_j a} W_j \left(\frac{K_j - a}{2L_j} \right)$$

$$+ \mathrm{e}^{-\frac{K_j^2}{4L_j}} \left\{ \left[F - RC \frac{K_j}{2L_j} \right] \left[J + \lambda_1 \beta_0 + \frac{\lambda_2^2}{4L_j} - \left(T_j - t - \frac{K_j}{2L_j} \right) \times \right. \right.$$

$$\left. \left. \times \left(\frac{\lambda_2^2}{2} \left(T_j - t - \frac{K_j}{2L_j} \right) + \frac{\rho \lambda_2 (1 + \lambda_1) \sigma_r}{a} \right) \right] \right\} W_j \left(\frac{K_j}{2L_j} \right)$$

$$+ \mathrm{e}^{-\frac{K_j^2}{4L_j}} \left\{ \frac{RC}{2L_j} \left[\lambda_2^2 \left(T_j - t - \frac{K_j}{2L_j} \right) + \frac{\rho \lambda_2 (1 + \lambda_1) \sigma_r}{a} \right] \right\} W_j \left(\frac{K_j}{2L_j} \right)$$

$$+ \mathrm{e}^{-\frac{(K_j - 1/\beta_3)^2}{4L_j} - \frac{(T_j - t)}{\beta_3}} \frac{\lambda_1 \beta_2 RC}{2L_j \beta_3} G1_j \left(\frac{K_j - 1/\beta_3}{2L_j} \right)$$

$$+ \frac{\lambda_1}{2L_j} \mathrm{e}^{-\frac{(K_j - 1/\beta_3)^2}{4L_j} - \frac{(T_j - t)}{\beta_3}} \left\{ \beta_1 RC + \frac{\beta_2}{\beta_3} \left[F + RC \left(T_j - t - \frac{K_j - 1/\beta_3}{L_j} \right) \right] \right\} \times$$

$$G0_j \left(\frac{K_j - 1/\beta_3}{2L_j} \right)$$

$$
\begin{aligned}
&- \frac{\lambda_1^2 RC\sigma_r^2}{4L_j a^2}\, e^{-2a(T_j-t)-\frac{(K_j-2a)^2}{4L_j}}\, G0_j\left(\frac{K_j-2a}{2L_j}\right)\\
&+ \frac{\rho\lambda_1\lambda_2 RC\sigma_r}{2L_j\, a}\, e^{-a(T_j-t)-\frac{(K_j-2a)^2}{4L_j}}\, G1_j\left(\frac{K_j-2a}{2L_j}\right)\\
&+ \frac{1}{2L_j a}\, e^{-a(T_j-t)-\frac{(K_j-a)^2}{4L_j}} \left\{\frac{RC(\lambda_1^2\sigma_r^2-\rho\lambda_2\sigma_r)}{a} + \rho\lambda_1\lambda_2\sigma_r \times \right.\\
&\qquad \left.\left[F + RC\left(T_j - t - \frac{K_j-a}{L_j}\right)\right]\right\} G0_j\left(\frac{K_j-a}{2L_j}\right)\\
&- e^{-K_j^2/4L_j}\, \frac{RC\lambda_2^2}{4L_j} G2_j\left(\frac{K_j}{2L_j}\right)\\
&- \frac{e^{-K_j^2/4L_j}}{2L_j}\left[\frac{\lambda_2^2}{2}\left(F - RC\frac{K_j}{2L_j}\right) + RC\left(\frac{\rho\lambda_2(1+\lambda_1)\sigma_r)}{a} + \lambda_2^2 \times \right.\right.\\
&\qquad \left.\left.\left(T_j - t - \frac{K_j}{2L_j}\right)\right)\right] G1_j\left(\frac{K_j}{2L_j}\right)\\
&+ \frac{e^{-K_j^2/4L_j}}{2L_j}\left\{\lambda_1\beta_0 RC + \frac{RC\lambda_2^2}{2L_j} + JRC\right\} G0_j\left(\frac{K_j}{2L_j}\right)\\
&- \frac{e^{-K_j^2/4L_j}}{2L_j}\left[F - RC\frac{K_j}{2L_j}\right]\times\\
&\qquad \left[\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a} + \lambda_2^2\left(T_j - \frac{K_j}{2L_j} - t\right)\right] G0_j\left(\frac{K_j}{2L_j}\right)\\
&- \frac{e^{-K_j^2/4L_j}}{2L_j} RC\left(T_j - \frac{K_j}{2L_j} - t\right)\left[\frac{\lambda_2^2}{2}\left(T_j - t - \frac{K_j}{2L_j} - t\right) + \right.\\
&\qquad \left.\frac{\rho\lambda_2(1+\lambda_1)\sigma_r}{a}\right] G0_j\left(\frac{K_j}{2L_j}\right)
\end{aligned}
$$

with the functions $G0_{j-1}(a)$, $G1_{j-1}(a)$, $G2_{j-1}(a)$ and $W_{j-1}(a)$ defined as

$$G0_{j-1}(a) = \mathrm{e}^{L_{j-1}y^2}\Big|_{y=a}^{y=0.5+a}$$

$$G1_{j-1}(a) = \mathrm{e}^{L_{j-1}y^2}y\Big|_{y=a}^{y=0.5+a}$$

$$G2_{j-1}(a) = \mathrm{e}^{L_{j-1}y^2}y^2\Big|_{y=a}^{y=0.5+a}$$

$$W_{j-1}(a) = \int_a^{0.5+a} \mathrm{e}^{L_{j-1}y^2}\,\mathrm{d}y$$

and the variables K_j, L_j, F, and J defined as

$$K_j = H_j - (1+\lambda_1)N_j$$

$$L_j = I_j + \frac{\beta_2(1+\lambda_1)\mathrm{e}^{-\frac{T_j - t}{\beta_3}}}{\beta_3}$$

$$F = 100R + 100(1-R)t_s(1-t_g) + 0.5RC$$

$$J = \lambda_0 + \frac{2\rho\lambda_2\sigma_r - \lambda_1^2\sigma_r^2}{2a^2} + \lambda_2 W_m(t)$$

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