2009

A Correspondence-Theoretic Approach to Dynamic Optimization

C. D. Aliprantis
Purdue University

Gabriele Camera
Chapman University, camera@chapman.edu

Recommended Citation

Aliprantis, C.D. and G. Camera (2009). A Correspondence-theoretic approach to dynamic optimization. *Macroeconomic Dynamics,* 13, 97–117. doi: 10.1017/S1365100509080134

A CORRESPONDENCE-THEORETIC APPROACH TO DYNAMIC OPTIMIZATION★

C. D. ALIPRANTIS † AND G. CAMERA ‡

† Department of Economics, Krannert School of Management, Purdue University

‡ Department of Economics, Tippie School of Business, University of Iowa

Abstract. This paper introduces a method to optimization in infinite-horizon economies based on the theory of correspondences. The proposed approach allow us to study time-separable and non time-separable dynamic economic models without resorting to fixed point theorems or transversality conditions. When our technique is applied to the standard time-separable model it provides an alternative and straightforward way to deriving the common recursive formulation of these models by means of Bellman equations.

JEL classification: E00, C61

Keywords: optimal plans, policy function, value function.

1. Introduction

This paper introduces a method to study existence of optima in infinite-horizon representative agent economies. The proposed theory relies neither on a variational approach and the use of transversality conditions, nor on the usual dynamic programming techniques that employ fixed point arguments; see for instance [4, 7, 8, 10]. Our focus is on infinitely-lived agents. Instead, our approach is based on the theory of correspondences and applies two classical fundamental theorems of mathematical analysis, Tychonoff's Product Theorem and Berge's Maximum Theorem.

★ This research is supported in part by the NSF Grants SES-0128039, DMS-0437210, and ACI-0325846.

The basic idea originates from the simple observation that in an infinite-horizon economy the set of all feasible plans defines a correspondence. This set-valued function maps the collection of all possible initial states of the economy into some vector space, which is simply the collection of all time-sequences that represent all current and future plans for consumption and savings. We name this correspondence the *plan correspondence*, and it is the building block of the proposed approach. Indeed, the starting point of our analysis is the study of the basic topological properties of the plan correspondence. In particular, we establish its continuity and compact valuedness. This allows us to prove existence of optimal plans and, given bounded and continuous preferences, of a well-defined value function. Subsequently, we demonstrate that one can easily characterize the main features of the value function including its continuity and concavity.

The approach we propose is of general applicability and, in particular, can accommodate various specifications of preferences, such as time non-separability. To our knowledge the approach is novel.[1] To illustrate its use, we apply our correspondence-theoretic approach to a planning problem for a standard time-separable infinite-horizon economy. For the particular case of geometric discounting we demonstrate how the proposed correspondence-theoretic approach provides an alternative and straightforward way to obtain a recursive representation. We offer a simple and direct proof for the fact that the value function is the unique solution of the Bellman equation. In particular, the proof does not invoke the Contraction Mapping Theorem, or any other fixed point argument for that matter. The approach we propose complements the infinite dimensional optimization literature; for example, see [2, 3, 11] who study equilibria in the overlapping generations dynamic setting of Samuelson [9].

[1]After completing this draft, we have been made aware of the related work [6], which also studies properties of value function and optimal policy correspondence by application of the Maximum theorem. The focus in that paper is to establish a version of the Maximum theorem with weaker continuity requirements. Such a theorem is applied to an intertemporal problem where standard continuity requirements of objective function and feasible plan correspondence are not satisfied.

The paper proceeds as follows. Section 2 presents a typical discrete-time dynamic model economy. Section 3 specializes to a time-separable model. Section 4 presents a textbook example of the classical one-sector growth model.

2. A Typical Discrete-Time Dynamic Economy

The typical dynamic framework in economics consists of an infinite horizon economy, where the representative agent can consume, produce, and save by accumulating some asset. We focus on a deterministic setting where time is discrete, i.e., there are countably many periods labeled $t = 0, 1, 2, \dots$, and the agent is infinitely-lived. Since we are interested in developing a method for determining optimal paths for consumption and savings, and we are not concerned about prices, we will concentrate on a planning problem. For our purposes, the planner corresponds to the representative consumer who faces a resource constraint, which is defined by the assumed production technology.

In each period t the agent must make a choice from a given time-invariant opportunity set X, the elements of which define the **states** of the economy. Elements of X can be interpreted as stocks of real assets (e.g., capital) available in a given period. In making this choice the agent faces a limitation that is described by a nonempty-valued feasibility constraint correspondence

$$\Gamma \colon X \twoheadrightarrow X \,.$$

We emphasize that Γ is also assumed to be time-invariant; however, our main results are valid if Γ varies over time. Given that the state at the beginning of a period is x, the set $\Gamma(x)$ contains all feasible states at the beginning of the following period. For example, if $f(x)$ is output produced today with $x \geq 0$ capital, then $\Gamma(x) = [0, f(x)]$. As usual the graph of Γ is denoted by G_Γ, i.e.,

$$G_\Gamma = \big\{ (x, y) \in X \times X \colon \; y \in \Gamma(x) \big\} \,.$$

It is assumed that the state of the economy evolves deterministically according to the choice of the agent. Specifically, given that the state of the economy at the beginning of period t is x_t the state of the economy at the beginning of period $t + 1$ is x_{t+1}, which is chosen by the agent in period t. Consequently, we define a

feasible plan, or simply a **plan**, starting with $x_0 \in X$ as an infinite sequence of states

$$\boldsymbol{x} = (x_0, x_1, x_2, \ldots)$$

such that $x_{t+1} \in \Gamma(x_t)$ holds for each $t = 0, 1, 2, \ldots$. It is important to recognize that for each $t \geq 0$ the *tail* or *continuation sequence* $(x_t, x_{t+1}, x_{t+2}, \ldots)$ of a plan $\boldsymbol{x}$, is also a plan starting with $x_t \in X$.

The collection of all feasible plans starting with x_0 is denoted $\Pi(x_0)$. That is,

$$\Pi(x_0) = \left\{ \boldsymbol{x} = (x_0, x_1, x_2, \ldots) \in X^{\{0,1,2,\ldots\}} : \ x_{t+1} \in \Gamma(x_t) \ \text{for all} \ t = 0, 1, 2, \ldots \right\}.$$

Since Γ is nonempty-valued, the set of all feasible plans $\Pi(x_0)$ is nonempty for each $x_0 \in X$.

We now impose some conditions on X and Γ that capture common assumptions on underlying technologies of the typical economic model. Throughout this paper we assume the following topological properties for the state space X and the constraint correspondence Γ.

Condition C1: *The state space X is a nonempty closed subset of some metrizable topological vector space Θ.*

In practice, Θ is usually a Euclidean space. We let

$$\mathbb{X} = \Theta^{\{0,1,2,\ldots\}}.$$

That is, $\mathbb{X}$ consists of all sequences with entries in Θ. The vector space $\mathbb{X}$ will be assumed equipped with the product topology under which it is also a metrizable topological vector space; see [1, Theorems 3.36 and 5.2]. The product topology is also known as *the topology of pointwise convergence* on $\mathbb{X}$ since an arbitrary sequence $\{\boldsymbol{x}^n\}$ in $\mathbb{X}$, where $\boldsymbol{x}^n = (x_0^n, x_1^n, x_2^n, \ldots)$, satisfies $\boldsymbol{x}^n \to \boldsymbol{x} = (x_0, x_1, x_2, \ldots)$ in $\mathbb{X}$ if and only if $x_t^n \xrightarrow[n\to\infty]{} x_t$ holds in Θ for each $t = 0, 1, 2, \ldots$.

A second condition deals with desirable properties of the constraint correspondence.

Condition C2: *The constraint correspondence* $\Gamma\colon X \twoheadrightarrow X$ *has*

(a) *nonempty and compact values, and*

(b) *is continuous.*

A consequence of Conditions **C1** and **C2** involves a nice property of the graph G_Γ.

Lemma 2.1. *Under Conditions* **C1** *and* **C2***, the feasibility constraint correspondence* Γ *has a closed graph, i.e.,* G_Γ *is a closed subset of* $X \times X$*.*

Proof. A compact-valued correspondence with Hausdorff range that is upper hemicontinuous has a closed graph; see [1, Theorem 17.10, p. 561]. ■

Lemma 2.1 will be important in proving certain desirable properties of the collections of all feasible plans which is the basic building block of our approach to dynamic optimization.

2.1. **The plan correspondence.** As mentioned in the introduction, the central idea in our approach to dynamic optimization is the use of the theory of correspondences. This idea originates from the simple observation that the set of all feasible plans starting with x_0 defines automatically a correspondence

$$\Pi\colon X \twoheadrightarrow \mathbb{X}\,.$$

That is, Π maps the set of *all* possible initial states x_0 into the space of all possible sequences. We call the correspondence Π the **plan correspondence**. Since the constraint correspondence Γ is nonempty-valued, it follows immediately that Π is likewise nonempty-valued.

The rest of the discussion in this subsection is devoted to investigating the fundamental topological properties of the plan correspondence Π. As we will see, Tychonoff' classical Product Theorem (see for instance [1, Theorem 2.61, p. 52]) will play a key role in establishing the properties of Π. For example, since Π is in essence a constraint on the plans, it is desirable to establish that Π is compact-valued and continuous. Tychonoff's Theorem is handy in this respect since it states that an arbitrary product of compact topological spaces with the product topology is itself a compact topological space.

Theorem 2.2. *Under Conditions* **C1** *and* **C2***, the nonempty-valued plan correspondence*

(1) *has closed graph, i.e., G_Π is a closed subset of $X \times \mathbb{X}$, and*
(2) *is compact-valued.*

Proof. To see that G_Π is a closed subset of $X \times \mathbb{X}$ assume that a sequence $\{(x_0^n, \boldsymbol{x}^n)\}$ in G_Π satisfies $(x_0^n, \boldsymbol{x}^n) \to (x_0, \boldsymbol{x})$ in $X \times \mathbb{X}$, where $\boldsymbol{x}^n = (x_0^n, x_1^n, x_2^n, \ldots)$. Thus, $x_t^n \xrightarrow[n\to\infty]{} x_t$ holds in Θ for each $t \geq 0$. Since X is closed, we get $x_t \in X$ for each $t \geq 0$.

Now fix $t \geq 0$. From $x_{t+1}^n \in \Gamma(x_t^n)$, we obtain $(x_t^n, x_{t+1}^n) \in G_\Gamma$ for each n. Since $(x_t^n, x_{t+1}^n) \to (x_t, x_{t+1})$ in $X \times X$ and the graph of Γ is closed (see Lemma 2.1), we get $(x_t, x_{t+1}) \in G_\Gamma$ or $x_{t+1} \in \Gamma(x_t)$ for all $t \geq 0$. Thus, $(x_0, \boldsymbol{x}) \in G_\Pi$ and so G_Π is a closed subset of $X \times \mathbb{X}$.

Next, we prove that Π is compact-valued. So, fix $x_0 \in X$. Since Π has a closed graph, it should be clear that $\Pi(x_0)$ is a closed subset of $\mathbb{X}$. To prove that $\Pi(x_0)$ is a compact subset of $\mathbb{X}$, it suffices to show that $\Pi(x_0)$ is included in a compact subset of $\mathbb{X}$.

To this end, we start by noticing that since Γ is compact-valued and upper hemicontinuous, it carries compact subsets of X to compact subsets of X. That is, if A is a compact subset of X, then $\Gamma(A) := \bigcup_{a\in A} \Gamma(a)$ is also a compact subset of X; see [1, Theorem 17.8, p. 560]. Now recursively define the sets

$$A_0 = \{x_0\} \quad \text{and} \quad A_{t+1} = \Gamma(A_t) \quad \text{for} \quad t = 0, 1, 2, \ldots \;.$$

Using an easy inductive argument, we see that each A_t is a compact subset of X (and so a compact subset of Θ). By Tychonoff's Product Theorem, the set

$$A = A_0 \times A_1 \times A_2 \times \cdots$$

is a compact subset of $\mathbb{X}$. To complete the proof, notice that $\Pi(x_0) \subseteq A$. ■

We emphasize that a pair $(x_0, \boldsymbol{x})$ belongs to the graph of Π if and only if $\boldsymbol{x} \in \Pi(x_0)$, which is also equivalent to saying that the initial state of $\boldsymbol{x}$ is x_0. This characterization will be used throughout the discussion that follows.

Theorem 2.3. *Under Conditions* **C1** *and* **C2***, the plan correspondence* Π *is continuous.*

Proof. We first prove that Π is upper hemicontinuous. To this end, assume that $x_0^n \to x_0$ holds in X and $\boldsymbol{x}^n \in \Pi(x_0^n)$ for each n. It suffices to show that there exists a subsequence $\{\boldsymbol{y}^n\}$ of $\{\boldsymbol{x}^n\}$ that converges to some point in $\Pi(x_0)$; see [1, Theorem 17.20, p. 565].

Let $A_0 = \{x_0, x_0^1, x_0^2, x_0^3, \ldots\}$ and note that A_0 is a compact subset of X and, of course, of Θ; see [1, Theorem 2.38, p. 42]. Now, as in the last part of the proof of Theorem 2.2, recursively define the sequence $\{A_t\}$ of compact subsets of X by

$$A^{t+1} = \Gamma(A_t)\,, \;\; t = 0, 1, 2, \ldots\ .$$

If we let $A = A_0 \times A_1 \times A_2 \times \cdots$, then A is (by Tychonoff's Product Theorem) a compact subset of $\mathbb{X}$ and clearly $\Pi(x_0^n) \subseteq A$ holds for each $n = 1, 2, \ldots$. But then $\{\boldsymbol{x}^n\}$, as a sequence in A, has a convergent subsequence. Let $\{\boldsymbol{y}^n\}$ be a subsequence of $\{\boldsymbol{x}^n\}$ satisfying $\boldsymbol{y}^n \to \boldsymbol{y}$ in $\mathbb{X}$. Now notice that the sequence $\{(y_0^n, \boldsymbol{y}^n)\} \subseteq G_\Pi$ satisfies $(y_0^n, \boldsymbol{y}^n) \to (x_0, \boldsymbol{y})$ in $X \times \mathbb{X}$. Since Π has a closed graph, we get $(x_0, \boldsymbol{y}) \in G_\Pi$ or $\boldsymbol{y} \in \Pi(x_0)$, as desired.

Next, we establish the lower hemicontinuity of Π. To this end, fix $x_0 \in X$ and assume that some open subset $\mathcal{O}$ of $\mathbb{X} = \Theta^{\{0,1,2,\ldots\}}$ satisfies $\Pi(x_0) \cap \mathcal{O} \neq \emptyset$. We must show that there exists a neighborhood N of x_0 in Θ such that $\Pi(z_0) \cap \mathcal{O} \neq \emptyset$ for each $z_0 \in N \cap X$.

Start by observing that (according to the definition of the product topology), we can suppose without loss of generality that $\mathcal{O}$ is of the form

$$\mathcal{O} = O_0 \times O_1 \times O_2 \times \cdots \times O_k \times \Theta \times \Theta \times \Theta \times \cdots\,, \tag{2.1}$$

where $k \geq 0$ and O_i is a nonempty open subset of Θ for each $i = 0, 1, \ldots, k$. The proof of the existence of the desired neighborhood N will be done by induction on k.

So, consider first the case $k = 0$. If

$$\mathcal{O} = O_0 \times \Theta \times \Theta \times \Theta \times \cdots\,,$$

then for all $z_0 \in O_0 \cap X$ we have $\Pi(z_0) \subseteq \mathcal{O}$. This implies $\Pi(z_0) \cap \mathcal{O} = \Pi(z_0) \neq \emptyset$ for each $z_0 \in O_0 \cap X$. Since O_0 is a neighborhood of x_0 in Θ, our conclusion is trivially true for $k = 0$.

For the inductive step, assume that for some $k \geq 0$ the following is true. If an open set of the form (2.1) satisfies $\Pi(x_0) \cap \mathcal{O} \neq \emptyset$, then there exists some neighborhood N of x_0 in Θ such that $\Pi(z_0) \cap \mathcal{O} \neq \emptyset$ for each $z_0 \in N \cap X$. Now suppose that an open set of $\mathbb{X}$ of the form

$$\mathcal{O} = O_0 \times O_1 \times O_2 \times \cdots \times O_k \times O_{k+1} \times \Theta \times \Theta \times \Theta \times \cdots$$

satisfies $\Pi(x_0) \cap \mathcal{O} \neq \emptyset$. To complete the inductive proof, we must demonstrate the existence of a neighborhood N of x_0 in Θ such that $z_0 \in N \cap X$ implies $\Pi(z_0) \cap \mathcal{O} \neq \emptyset$.

To this end, start by picking some plan $\boldsymbol{x} = (x_0, x_1, x_2, \ldots) \in \Pi(x_0) \cap \mathcal{O}$. In particular, note that $x_{k+1} \in \Gamma(x_k) \cap O_{k+1}$. Since $x_k \in O_k$, the lower hemicontinuity of $\Gamma\colon X \twoheadrightarrow X$ at x_k guarantees the existence of an open neighborhood $O'_k \subseteq O_k$ of x_k such that $z \in O'_k \cap X$ implies $\Gamma(z) \cap O_{k+1} \neq \emptyset$.[2] Now consider the open set

$$\mathcal{O}' = O_0 \times O_1 \times O_2 \times \cdots \times O_{k-1} \times O'_k \times \Theta \times \Theta \times \Theta \times \cdots\,,$$

and note that $\boldsymbol{x} \in \Pi(x_0) \cap \mathcal{O}'$. But then by our induction hypothesis there exists a neighborhood N of x_0 in Θ such that $z_0 \in N \cap X$ implies $\Pi(z_0) \cap \mathcal{O}' \neq \emptyset$.

To complete the proof, we must show that this neighborhood N of x_0 in Θ satisfies the desired property. To see this, fix $z_0 \in N \cap X$ and then choose

$$\boldsymbol{z} = (z_0, z_1, z_2, \ldots, z_k, z_{k+1}, \ldots) \in \Pi(z_0) \cap \mathcal{O}'\,.$$

Clearly, $z_k \in O'_k \cap X$ and consequently, by the choice of O'_k, there exists some $z'_{k+1} \in \Gamma(z_k) \cap O_{k+1}$. Now if we choose states $z'_{k+2}, z'_{k+3}, \ldots$ in X such that

$$z'_{t+1} \in \Gamma(z'_t) \quad \text{for all} \quad t = k+1, k+2, \ldots\,,$$

then the plan $\boldsymbol{z'} = (z_0, z_1, \ldots, z_k, z'_{k+1}, z'_{k+2}, \ldots)$ satisfies $\boldsymbol{z'} \in \Pi(z_0) \cap \mathcal{O}$. Thus, $z_0 \in N \cap X$ implies $\Pi(z_0) \cap \mathcal{O} \neq \emptyset$, and the proof is finished. ■

[2] Notice that here is the only place we use the lower hemicontinuity of Γ.

The preceding result is important since, as we will see, it allows us to employ Berge's Maximum Theorem to study optima. As a consequence, we can characterize the set of optima and derive the fundamental properties of the value function in a direct and very simple manner. However, before doing so, we need to take one more step. Namely, we need to establish the convexity of the plan correspondence Π.

For convenience denote by $\mathcal{P}$ the collection of all feasible plans for all possible initial states in X. That is,

$$\mathcal{P} = \bigsqcup_{x \in X} \Pi(x)\,.$$

As usual, the symbol $A = \bigsqcup A_{i\in I}$ means $A = \bigcup A_{i\in I}$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. In this case, note that if $x \neq x'$ then any plan $\boldsymbol{x}$ starting with x, i.e., $\boldsymbol{x} \in \Pi(x)$, cannot lie in $\Pi(x')$ because the first coordinate of $\boldsymbol{x}$ is not x'. That is, $\Pi(x) \cap \Pi(x') = \emptyset$.

To study convexity properties of the plan correspondence, we need one more assumption.

Condition C3: *The state space X is a convex subset of Θ and the constraint correspondence $\Gamma\colon X \twoheadrightarrow X$ has a convex graph.*

This condition guarantees the convexity of the graph of the plan correspondence. As we will see, this is fundamental in establishing the concavity of the value function.

Theorem 2.4. *Under Condition* **C3***, the plan correspondence Π has a convex graph, and so Π is also convex-valued.*

Proof. Let $(x_0, \boldsymbol{x}), (y_0, \boldsymbol{y}) \in G_\Pi$ and $0 \leq \alpha \leq 1$. Notice that $x_{t+1} \in \Gamma(x_t)$ and $y_{t+1} \in \Gamma(y_t)$ imply $(x_t, x_{t+1}), (y_t, y_{t+1}) \in G_\Gamma$ for all $t \geq 0$. Since Γ has a convex graph, it follows that for each $t \geq 0$ the convex combination of (x_t, x_{t+1}) and (y_t, y_{t+1}) satisfies

$$\big(\alpha x_t + (1-\alpha) y_t, \alpha x_{t+1} + (1-\alpha) y_{t+1}\big) = \big[\,\alpha(x_t, x_{t+1}) + (1-\alpha)(y_t, y_{t+1})\,\big] \in G_\Gamma\,.$$

In other words, $[\alpha x_{t+1} + (1-\alpha) y_{t+1}] \in \Gamma(\alpha x_t + (1-\alpha) y_t)$ holds true for all $t \geq 0$. This implies $[\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}] \in \Pi(\alpha x_0 + (1-\alpha) y_0)$ or

$$\alpha(x_0, \boldsymbol{x}) + (1-\alpha)(y_0, \boldsymbol{y}) = (\alpha x_0 + (1-\alpha) y_0, \alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \in G_\Pi \,.$$

So, the graph of the correspondence Π is a convex set, i.e., a convex subset of $X \times \mathbb{X}$. ■

Now that we have discussed the basic properties of the plan correspondence, we are ready to introduce a notion of preferences

2.2. **The lifetime utility function.** We take the preferences of the representative agent over plans as a primitive notion. As usual, we refer to these preferences as the "lifetime utility" of the representative agent. We also assume the agent is infinitely-lived. However, we will depart from the usual way of defining preferences for a given initial state. As a matter of fact, since we are working with the plan correspondence, it is useful to define preferences over all possible plans for *any* initial state. That is, we work with preferences defined on the graph of the plan correspondence.

Condition C4: *The* ***lifetime utility function*** *is a continuous function*

$$\widehat{U} \colon G_\Pi \to \mathbb{R} \,.$$

Given the lifetime utility function $\widehat{U}$, we associate a new function $U \colon \mathcal{P} \to \mathbb{R}$ defined for each $\boldsymbol{x} \in \Pi(x)$ by

$$U(\boldsymbol{x}) = \widehat{U}(x, \boldsymbol{x}) \,.$$

When U is restricted to a specific upper section $\Pi(x)$ of the plan correspondence Π, then we call $U \colon \Pi(x) \to \mathbb{R}$ the **state-contingent lifetime utility function**. In other words, U is simply the function $\widehat{U}$ restricted to the upper section of Π at the initial state $x \in X$.

The following result should be obvious.

Lemma 2.5. *For each initial state $x \in X$ the function $U \colon \Pi(x) \to \mathbb{R}$ is continuous.*

We now wish to discuss monotonicity of the lifetime utility function. In order to do so, we need to introduce an order relation on the graph of the correspondence Π. Recall first that a set A equipped with a reflexive, anti-symmetric and transitive relation $\geq$ is referred to as an ordered set or a partially ordered set.[3] We employ the following notation: If $\boldsymbol{z} = (z_0, z_1, z_2, \ldots) \in \mathbb{X}$, then $\boldsymbol{z}_{-0}$ denotes the sequence $\boldsymbol{z}$ without its first term, i.e., $\boldsymbol{z}_{-0} = (z_1, z_2, z_3, \ldots)$. Now assume that the state space X is an ordered set, ordered by $\geq$. By means of this order relation $\geq$ we can introduce an order relation $\succeq$ on G_Π by defining $(x_0, \boldsymbol{x}) \succeq (y_0, \boldsymbol{y})$ in G_Π to mean:

(1) $x_0 \geq y_0$ in X, and

(2) $\boldsymbol{x}_{-0} = \boldsymbol{y}_{-0}$, i.e., $x_t = y_t$ for all $t \geq 1$.

Notice that $\succeq$ is indeed an order relation on the graph G_Π of Π, so the order $\succeq$ allows us to define the monotonicity of the lifetime utility function.

Definition 2.6. *If the state space X is an ordered set, then the lifetime utility function $\widehat{U}$ is called* ***monotone*** *[resp.* ***strictly monotone****] if $(x_0, \boldsymbol{x}) \succ (y_0, \boldsymbol{y})$ in G_Π implies $\widehat{U}(x_0, \boldsymbol{x}) \geq \widehat{U}(y_0, \boldsymbol{y})$ in $\mathbb{R}$ [resp. $\widehat{U}(x_0, \boldsymbol{x}) > \widehat{U}(y_0, \boldsymbol{y})$].*

The monotonicity of the lifetime utility function expresses the following notion. Whenever the initial element x_0 of a plan $\boldsymbol{x}$ dominates the corresponding initial element y_0 of another plan $\boldsymbol{y}$, and the other elements of $\boldsymbol{x}$ and $\boldsymbol{y}$ are the same, then the lifetime utility generated by plan $\boldsymbol{x}$ must be at least as large as the lifetime utility generated by plan $\boldsymbol{y}$.

We are now ready to discuss optimal plans.

2.3. **The value function.** Given an initial state $x_0 \in X$ the representative agent maximizes his state-contingent lifetime utility function. That is, he solves the following optimization problem:

$$\begin{aligned} \text{Maximize:} &\quad U(\boldsymbol{x}) \\ \text{Subject to:} &\quad \boldsymbol{x} \in \Pi(x_0)\,. \end{aligned}$$

[3] As usual, for an order relation on a set A, the symbol $a > b$ in A means $a \geq b$ and $a \neq b$. We also write interchangeably $b \leq a$ instead of $a \geq b$ and $b < a$ for $a > b$.

A glance at Lemma 2.5 guarantees that, under Conditions **C1** and **C2**, this optimization problem has a solution. Therefore, one defines a real-valued function $v\colon X \to \mathbb{R}$ for each $x_0 \in X$ by

$$v(x_0) = \sup_{\boldsymbol{x}\in\Pi(x_0)} U(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\Pi(x_0)} U(\boldsymbol{x})\,.$$

The above function v is called the **value function**. Any feasible plan $\boldsymbol{x} \in \Pi(x_0)$ that satisfies $U(\boldsymbol{x}) = v(x_0)$ is called an **optimal plan**. The existence of optimal plans is a straightforward consequence of our approach to dynamic optimization.

Lemma 2.7. *Under Conditions* **C1**, **C2**, **C3**, *and* **C4**, *for a given initial state* $x_0 \in X$*:*

(a) *there exists at least one optimal plan,*

(b) *if the lifetime utility function* $\widehat{U}$ *is strictly concave, then there is exactly one optimal plan.*

Proof. By Theorem 2.2 the set $\Pi(x_0)$ is compact for each $x_0 \in X$. Now notice that, by Lemma 2.5, U is a continuous function when restricted to any $\Pi(x_0)$.

For property (b) notice that strict concavity of $\widehat{U}$ implies the strict concavity of $U\colon \Pi(x_0) \to \mathbb{R}$, and the uniqueness of the optimal plan follows. ■

In the remainder of this section, we apply our approach to dynamic optimization to obtain in a direct manner three key properties of the value function; continuity, monotonicity, and concavity. We start by discussing the continuity of the value function.

Theorem 2.8. *Under Conditions* **C1**, **C2**, *and* **C4** *the value function* v *is continuous.*

Proof. We know from Theorem 2.2 that the plan correspondence $\Pi\colon X \twoheadrightarrow \mathbb{X}$ is nonempty- and compact-valued. Also, according to Theorem 2.3, Π is continuous. Now apply Berge's classical Maximum Theorem (see for instance [1, Theorem 17.31, p. 570]) to infer that the maximum function $m\colon X \to \mathbb{R}$, defined for each $x_0 \in X$ by

$$m(x_0) = \max_{\boldsymbol{x}\in\Pi(x_0)} U(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\Pi(x_0)} \widehat{U}(x_0, \boldsymbol{x})\,,$$

is a continuous function. Now notice that v coincides with m. ■

Now that we have established the continuity of the value function, we turn our attention to its monotonicity. Recall that the constraint correspondence Γ is said to be monotone if X is an ordered set, and $y_0 < x_0$ in X implies $\Gamma(y_0) \subseteq \Gamma(x_0)$.

Theorem 2.9. *Under Conditions* **C1**, **C2**, *and* **C4**, *if the state space X is an ordered set, the constraint correspondence Γ is monotone, and the lifetime utility function $\widehat{U}$ is monotone (resp. strictly monotone), then the value function v is monotone (resp. strictly monotone).*

Proof. Assume that $\widehat{U}$ is strictly monotone and let $y_0 < x_0$ in X. The monotonicity of Γ implies $\Gamma(y_0) \subseteq \Gamma(x_0)$. By Lemma 2.7 there exists a plan $\boldsymbol{y} \in \Pi(y_0)$ such that $v(y_0) = U(\boldsymbol{y})$. From $\Gamma(y_0) \subseteq \Gamma(x_0)$, the sequence $\boldsymbol{x} = (x_0, y_1, y_2, \ldots)$ satisfies $\boldsymbol{x} \in \Pi(x_0)$, i.e., $(x_0, \boldsymbol{x}) \in G_\Pi$. So, $(y_0, \boldsymbol{y}), (x_0, \boldsymbol{x}) \in G_\Pi$ and $(x_0, \boldsymbol{x}) \succ (y_0, \boldsymbol{y})$. Since $\widehat{U}$ is strictly monotone, we see that

$$v(y_0) = U(\boldsymbol{y}) = \widehat{U}(y_0, \boldsymbol{y}) < \widehat{U}(x_0, \boldsymbol{x}) = U(\boldsymbol{x}) \leq v(x_0)\,.$$

Thus, $v(y_0) < v(x_0)$ proving that, in this case, v is strictly monotone.

If $\widehat{U}$ is monotone, then similar arguments show $v(y_0) \leq v(x_0)$, so that the value function v is monotone. ■

Our next result deals with concavity of the value function.

Theorem 2.10. *Under Conditions* **C1**, **C2**, **C3**, *and* **C4**, *if the lifetime utility function $\widehat{U}$ is concave (resp. strictly concave), then the value function v is likewise concave (resp. strictly concave).*

Proof. Fix two elements $x_0, y_0 \in X$ with $x_0 \neq y_0$ and let $0 < \alpha < 1$. Next, pick plans $\boldsymbol{x} \in \Pi(x_0)$ and $\boldsymbol{y} \in \Pi(y_0)$ such that $v(x_0) = U(\boldsymbol{x})$ and $v(y_0) = U(\boldsymbol{y})$. Clearly, $[\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}] \in \Pi(\alpha x_0 + (1-\alpha)y_0)$. Now taking into account that

$(x_0, \boldsymbol{x}) \neq (y_0, \boldsymbol{y})$, the strict concavity of $\widehat{U}$ yields

$$\begin{aligned} v(\alpha x_0 + (1-\alpha) y_0) &= \max_{\boldsymbol{z} \in \Pi(\alpha x_0 + (1-\alpha) y_0)} U(\boldsymbol{z}) \\ &\geq U(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \\ &= \widehat{U}(\alpha(x_0, \boldsymbol{x}) + (1-\alpha)(y_0, \boldsymbol{y})) \\ &> \alpha \widehat{U}(x_0, \boldsymbol{x}) + (1-\alpha)\widehat{U}(y_0, \boldsymbol{y}) \\ &= \alpha U(\boldsymbol{x}) + (1-\alpha) U(\boldsymbol{y}) \\ &= \alpha v(x_0) + (1-\alpha) v(y_0)\,. \end{aligned}$$

Hence, $v(\alpha x_0 + (1-\alpha) y_0) > \alpha v(x_0) + (1-\alpha) v(y_0)$, proving that v is strictly concave. ■

Of course, if the lifetime utility function $\widehat{U}$ is not strictly concave, then for a given initial state x_0 we can have a multiplicity of optimal plans, say $\boldsymbol{\pi}(x_0)$.[4] That is, we obtain a correspondence $\boldsymbol{\pi}\colon X \twoheadrightarrow \mathbb{X}$, defined by

$$\boldsymbol{\pi}(x_0) = \left\{ \boldsymbol{x} \in \Pi(x_0)\colon\ U(\boldsymbol{x}) = v(x_0) = \max_{\boldsymbol{y} \in \Pi(x_0)} \widehat{U}(x_0, \boldsymbol{y}) \right\}.$$

We call $\boldsymbol{\pi}$ the **optimal plan correspondence**, which satisfies the following properties.

Theorem 2.11. *If Conditions* **C1**, **C2** *and* **C4** *are true, then the optimal plan correspondence* $\boldsymbol{\pi}$ *is nonempty- and compact-valued, and upper hemicontinuous.*

Moreover, when Condition **C3** *is also true, we have the following additional properties:*

(a) *If the lifetime utility function* $\widehat{U}$ *is concave, then* $\boldsymbol{\pi}$ *is convex-valued.*
(b) *If the lifetime utility function* $\widehat{U}$ *is strictly concave, then* $\boldsymbol{\pi}$ *is a continuous function.*

---

[4] This means there is indeterminacy because several distinct plans are optimal. Of course, outside of a planning problem such as the one we consider, there can be other reasons for indeterminacy, even with quite standard preferences and technologies. For example see the discussion of externalities in [5].

Proof. The optimal plan correspondence coincides with the "argmax" correspondence

$$\boldsymbol{\pi}(x_0) = \left\{ \boldsymbol{x} \in \Pi(x_0) \colon \; \widehat{U}(x_0, \boldsymbol{x}) = \max_{\boldsymbol{y} \in \Pi(x_0)} \widehat{U}(x_0, \boldsymbol{y}) \right\}.$$

By Berge's Maximum Theorem, $\boldsymbol{\pi}$ is nonempty- and compact-valued and upper hemicontinuous.

For (a) note that $\Pi(x_0)$ is a convex and compact subset of $\mathbb{X}$. Since the function $U \colon \Pi(x_0) \to \mathbb{R}$ is continuous and concave, it follows that the nonempty set $\boldsymbol{\pi}(x_0)$ of maximizers of U over $\Pi(x_0)$ is convex.

For (b), observe that under Conditions **C1**, **C2**, **C3**, and **C4** and strict concavity of $\widehat{U}$, it follows from Lemma 2.7 that for each initial state $x_0 \in X$ there exists exactly one optimal plan in $\Pi(x_0)$. That is, in this case, $\boldsymbol{\pi}$ is a function. Since $\boldsymbol{\pi}$ as a correspondence is upper hemicontinuous, it is automatically a continuous function. ■

In the following sections we apply our optimization technique to a standard dynamic model from the macroeconomic literature.

3. A Time-Separable Model

An important class of dynamic models in economics are the ones characterized by time-separable lifetime utility functions. Under certain conditions, this type of preferences gives rise to recursive formulations for the value functions in terms of the Bellman equation; see [10]. In this section, we apply our framework to establish the existence of optima and derive the basic properties of the value functions in these models. To do so, throughout this section, we assume that Conditions **C1**, **C2** are valid.

We will say that the lifetime utility function $\widehat{U}$ is time-separable, if there exists a bounded continuous function $F \colon G_\Gamma \to \mathbb{R}$ (commonly referred to as the *return* function or as the *period utility* function) such that for each point $(x_0, \boldsymbol{x}) \in G_\Pi$ or, equivalently, $\boldsymbol{x} = (x_0, x_1, x_2, \ldots) \in \Pi(x_0)$, we have

$$\widehat{U}(x_0, \boldsymbol{x}) = U(\boldsymbol{x}) = \sum_{t=0}^{\infty} \beta(t) F(x_t, x_{t+1}). \tag{3.1}$$

Here $\beta(t) > 0$ is interpreted as the discount factor at period $t \geq 0$ and it is assumed that $\sum_{t=0}^{\infty} \beta(t) < \infty$. Clearly, the boundedness of F coupled with the condition on discounting, guarantees that $\widehat{U}$ as given by (3.1) is a real-valued function.[5] We refer to any dynamic model with preferences given by (3.1) as a **time-separable model**.

We start by establishing the continuity of time-separable preferences.

Lemma 3.1. *Any time-separable utility function* $\widehat{U}\colon G_\Pi \to \mathbb{R}$ *is continuous, i.e., it satisfies Condition* **C4**.

Proof. Assume that $(x_0^n, \boldsymbol{x}^n) \to (x_0, \boldsymbol{x})$ in G_Π. That is, $x_0^n \to x_0$ in X and

$$\boldsymbol{x}^n = (x_0^n, x_1^n, x_2^n, \ldots) \longrightarrow \boldsymbol{x} = (x_0, x_1, x_2, \ldots)$$

in $\mathbb{X}$. In other words, for each $t \geq 0$ we have $x_t^n \underset{n\to\infty}{\longrightarrow} x_t$ in X. We must establish that $\widehat{U}(x_0^n, \boldsymbol{x}^n) \to \widehat{U}(x_0, \boldsymbol{x})$ holds true in $\mathbb{R}$.

To this end, fix $\epsilon > 0$. Start by choosing some $M > 0$ such that $|F(x,y)| \leq M$ holds for all $(x,y) \in G_\Gamma$ and then pick some $\tau > 0$ such that $2M\sum_{t=\tau+1}^{\infty} \beta(t) < \frac{\epsilon}{2}$. Using the continuity of F, we see that there exists some n_0 such that $n \geq n_0$ implies $\big|\sum_{t=0}^{\tau} \beta(t)\big[F(x_t^n, x_{t+1}^n) - F(x_t, x_{t+1})\big]\big| < \frac{\epsilon}{2}$. But then for each $n \geq n_0$ we have

$$\begin{aligned}
&\big|\widehat{U}(x_0^n, \boldsymbol{x}^n) - \widehat{U}(x_0, \boldsymbol{x})\big| = \big|U(\boldsymbol{x}^n) - U(\boldsymbol{x})\big| \\
&\quad \leq \Big|\sum_{t=0}^{\tau} \beta(t)\big[F(x_t^n, x_{t+1}^n) - F(x_t, x_{t+1})\big]\Big| + \Big|\sum_{t=\tau+1}^{\infty} \beta(t)\big[F(x_t^n, x_{t+1}^n) - F(x_t, x_{t+1})\big]\Big| \\
&\quad < \tfrac{\epsilon}{2} + 2M\sum_{t=\tau+1}^{\infty} \beta(t) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.
\end{aligned}$$

This shows that $\widehat{U}(x_0^n, \boldsymbol{x}^n) \to \widehat{U}(x_0, \boldsymbol{x})$ holds in $\mathbb{R}$, as desired. ∎

We now proceed to establish the monotonicity property of $\widehat{U}$, i.e., of the lifetime utility defined on the graph of the plan correspondence. Recall that if the state space X is an ordered set, then the return function $F\colon G_\Gamma \to \mathbb{R}$ is said to

[5] Of course, the most common formulation assumes $\beta(t) = \beta^t$ for all $t \geq 0$, where $0 < \beta < 1$.

be monotone (resp. strictly monotone) in x whenever $(x_0, y), (y_0, y) \in G_\Gamma$ with $x_0 > y_0$ implies $F(x_0, y) \geq F(y_0, y)$ (resp. $F(x_0, y) > F(y_0, y)$) in $\mathbb{R}$.

Theorem 3.2. *Under Conditions* **C1** *and* **C2**, *if the state space* X *is ordered and the period utility function* F *is monotone* (*resp. strictly monotone*) *in* x, *then the lifetime utility function* $\widehat{U}$ *is monotone* (*resp. strictly monotone*).

Proof. Assume $(x_0, \boldsymbol{x}) \succ (y_0, \boldsymbol{y})$ in G_Π. That is, $x_0 > y_0$ in X and $\boldsymbol{x}_{-0} = \boldsymbol{y}_{-0}$. Now note that the monotonicity of F in x implies $F(y_0, x_1) \leq F(x_0, x_1)$ (with strict inequality if F is strictly monotone). But then in view of $x_t = y_t$ for all $t \geq 1$, we have

$$\begin{aligned}
\widehat{U}(y_0, \boldsymbol{y}) &= U(\boldsymbol{y}) = \beta(0)F(y_0, y_1) + \sum_{t=1}^{\infty} \beta(t)F(y_t, y_{t+1}) \\
&\leq \beta(0)F(x_0, y_1) + \sum_{t=1}^{\infty} \beta(t)F(y_t, y_{t+1}) \\
&= \beta(0)F(x_0, x_1) + \sum_{t=1}^{\infty} \beta(t)F(x_t, x_{t+1}) \\
&= U(\boldsymbol{x}) = \widehat{U}(x_0, \boldsymbol{x})\,.
\end{aligned}$$

Therefore, $\widehat{U}(y_0, \boldsymbol{y}) \leq \widehat{U}(x_0, \boldsymbol{x})$ so that $\widehat{U}$ is monotone. Strict monotonicity of $\widehat{U}$ follows by observing that the weak inequality in the displayed formula above is strict. ■

Now that we have established continuity and monotonicity of the lifetime utility function, we are ready to study some basic properties of the value function. The next result indicates that the value function is continuous.

Theorem 3.3. *Under Conditions* **C1** *and* **C2**, *the value function* v *of a time-separable model is bounded and continuous.*

Proof. The continuity of v is a simple consequence of Theorem 2.8. To see that v is bounded, assume that $|F(x, y)| \leq M$ holds for each $(x, y) \in G_\Gamma$. Now if

$\boldsymbol{x} \in \Pi(x_0)$ satisfies $v(x_0) = U(\boldsymbol{x})$, then we have

$$|v(x_0)| = |U(\boldsymbol{x})| = \Big|\sum_{t=0}^{\infty} \beta(t) F(x_t, x_{t+1})\Big| \leq \sum_{t=0}^{\infty} \beta(t) |F(x_t, x_{t+1})| \leq M \sum_{t=0}^{\infty} \beta(t) < \infty\,.$$

Since $x_0 \in X$ is arbitrary, we see that that v is a bounded function. ■

The next result deals with monotonicity properties of the value function.

Theorem 3.4. *Assume that Conditions* **C1**, **C2**, *and* **C3** *are valid, the state space* X *is an ordered set, the constraint correspondence* Γ *is monotone, and the return function is monotone* (*resp. strictly monotone*) *in* x. *Then the value function* v *is monotone* (*resp. strictly monotone*).

Proof. This follows immediately from Theorems 3.2 and 2.9. ■

Next, we discuss the concavity of the value function v.

Lemma 3.5. *Under Conditions* **C1**, **C2**, *and* **C3**, *if* F *is concave* (*resp. strictly concave*), *then the lifetime utility function* $\widehat{U}$ *is concave* (*resp. strictly concave*).

Proof. We assume that F is strictly concave on the convex set G_Γ, and we prove that $\widehat{U}$ is strictly concave on the convex set G_Π. (Concavity can be proved in a similar manner.)

To this end, let $(x_0, \boldsymbol{x}), (y_0, \boldsymbol{y}) \in G_\Pi$ satisfy $(x_0, \boldsymbol{x}) \neq (y_0, \boldsymbol{y})$. This means that for some $k \geq 0$ we have $x_k \neq y_k$. Now fix $0 < \alpha < 1$. The strict concavity of F yields

$$\begin{aligned} F(\alpha(x_t, x_{t+1}) + (1-\alpha)(y_t, y_{t+1})) &\geq \alpha F(x_t, x_{t+1}) + (1-\alpha) F(y_t, y_{t+1}) \ \forall\, t \neq k, \\ F(\alpha(x_k, x_{k+1}) + (1-\alpha)(y_k, y_{k+1})) &> \alpha F(x_k, x_{k+1}) + (1-\alpha) F(y_k, y_{k+1})\,. \end{aligned}$$

Consequently, we have

$$\begin{aligned}
\widehat{U}\big(\alpha(x_0,\boldsymbol{x})+(1-\alpha)(y_0,\boldsymbol{y})\big) &= U(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y}) \\
&= \sum_{t\neq k}\beta(t)F(\alpha(x_t,x_{t+1})+(1-\alpha)(y_t,y_{t+1})) \\
&\quad\; +\beta(k)F(\alpha(x_k,x_{k+1})+(1-\alpha)(y_k,y_{k+1})) \\
&> \sum_{t\neq k}\beta(t)\big[\alpha F(x_t,x_{t+1})+(1-\alpha)F(y_t,y_{t+1})\big] \\
&\quad\; +\beta(k)\big[\alpha F(x_k,x_{k+1})+(1-\alpha)F(y_k,y_{k+1})\big] \\
&= \alpha\sum_{t=0}^{\infty}\beta(t)F(x_t,x_{t+1})+(1-\alpha)\sum_{t=0}^{\infty}\beta(t)F(y_t,y_{t+1}) \\
&= \alpha U(\boldsymbol{x})+(1-\alpha)U(\boldsymbol{y}) \\
&= \alpha\widehat{U}(x_0,\boldsymbol{x})+(1-\alpha)\widehat{U}(y_0,\boldsymbol{y})\,.
\end{aligned}$$

Thus $\widehat{U}(\alpha(x_0,\boldsymbol{x})+(1-\alpha)(y_0,\boldsymbol{y})) > \alpha\widehat{U}(x_0,\boldsymbol{x})+(1-\alpha)\widehat{U}(y_0,\boldsymbol{y})$, so that $\widehat{U}$ is a strictly concave function. ■

Strict concavity is a common assumption in economic models since it guarantees the uniqueness of optimal plans.

Corollary 3.6. *If F is strictly concave, then for each initial state $x_0 \in X$ there exists exactly one optimal plan, i.e., there exists a unique plan $\boldsymbol{x} \in \Pi(x_0)$ such that $v(x_0) = U(\boldsymbol{x})$.*

We now show that our optimization method can be used to derive very easily the recursive properties of the value function in the typical case of geometric discounting.

3.1. **Geometric discounting and the Bellman equation.** We now consider the most common time-separable model, where $\beta(t) = \beta^t$ and $0 < \beta < 1$ is a fixed discount factor. In this case, one can establish that the value function has an additional important property. Namely, it satisfies the classical Bellman equation, i.e., one can describe the dynamic model in a recursive manner.

Theorem 3.7. *Under Conditions* **C1** *and* **C2***, the value function* $v\colon X \to \mathbb{R}$ *is the one and only bounded function that satisfies the Bellman functional equation, i.e., for each* $x_0 \in X$ *we have*

$$v(x_0) = \sup_{y \in \Gamma(x_0)} \big[F(x_0, y) + \beta v(y)\big] .$$

Proof. We verify first that v satisfies the Bellman equation. So, fix $x_0 \in X$ and let

$$m = \sup_{y \in \Gamma(x_0)} \big[F(x_0, y) + \beta v(y)\big] . \tag{3.2}$$

Since $F(x_0, \cdot)$ and v are both continuous functions on the set $\Gamma(x_0)$, the function $F(x_0, \cdot) + v(\cdot)$ is likewise continuous on $\Gamma(x_0)$. Taking into account that $\Gamma(x_0)$ is a compact set, we see that the supremum over $\Gamma(x_0)$ in (3.2) is attained. So, there is some $y_0 \in \Gamma(x_0)$ such that $m = F(x_0, y_0) + \beta v(y_0)$. Now, according to Lemma 2.7, there exists an optimal plan $\boldsymbol{y} = (y_0, y_1, y_2, \ldots) \in \Pi(y_0)$ such that $v(y_0) = U(\boldsymbol{y})$. Clearly, $\boldsymbol{x} = (x_0, y_0, y_1, y_2, \ldots)$ is a plan in $\Pi(x_0)$. This implies

$$m = F(x_0, y_0) + \beta v(y_0) = F(x_0, y_0) + \beta U(\boldsymbol{y}) = U(\boldsymbol{x}) \le v(x_0) . \tag{3.3}$$

Use once more Lemma 2.7 to select an optimal plan $\boldsymbol{z} = (x_0, z_1, z_2, \ldots) \in \Pi(x_0)$ such that $v(x_0) = U(\boldsymbol{z})$. Clearly, $\boldsymbol{z}_{-0} = (z_1, z_2, \ldots) \in \Pi(z_1)$, and from this and $z_1 \in \Gamma(x_0)$ we see that

$$v(x_0) = U(\boldsymbol{z}) = F(x_0, z_1) + \beta U(\boldsymbol{z}_{-0}) \le F(x_0, z_1) + \beta v(z_1) \le m . \tag{3.4}$$

From (3.3) and (3.4), we infer that $v(x_0) = m$, as desired.

Next, we prove that v is the only bounded function that satisfies the Bellman equation. To see this, let $w\colon X \to \mathbb{R}$ be a bounded function that satisfies the Bellman equation, i.e., for each $x \in X$ we have

$$w(x) = \sup_{y \in \Gamma(x)} \big[F(x, y) + \beta w(y)\big] . \tag{3.5}$$

Fix $x_0 \in X$. Now let $\epsilon > 0$. We claim that there exists a plan $\boldsymbol{x} \in \Pi(x_0)$ such that

$$w(x_t) < F(x_t, x_{t+1}) + \beta w(x_{t+1}) + \epsilon$$

holds for all $t \geq 0$. The construction is done by induction. If the element $x_t \in X$ has been selected, then we use (3.5) to select some $x_{t+1} \in \Gamma(x_t)$ such that $w(x_t) < F(x_t, x_{t+1}) + \beta w(x_{t+1}) + \epsilon$. Another easy inductive argument shows that for each $\tau \geq 0$ we have

$$w(x_0) \leq \sum_{t=0}^{\tau} \beta^t F(x_t, x_{t+1}) + \beta^{\tau+1} w(x_{\tau+1}) + \epsilon \sum_{t=0}^{\tau} \beta^t \,. \tag{3.6}$$

Taking into account that the boundedness of w implies $\lim_{\tau\to\infty} \beta^{\tau+1} w(x_{\tau+1}) = 0$, by letting $\tau \to \infty$ in (3.6), we get

$$w(x_0) \leq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) + \tfrac{\epsilon}{1-\beta} = U(\boldsymbol{x}) + \tfrac{\epsilon}{1-\beta} \leq v(x_0) + \tfrac{\epsilon}{1-\beta}$$

for all $\epsilon > 0$. This implies $w(x_0) \leq v(x_0)$.

Finally, for the reverse inequality, fix a plan $\boldsymbol{x} \in \Pi(x_0)$ such that $v(x_0) = U(\boldsymbol{x})$. An easy inductive argument shows that for each $\tau \geq 0$ we have

$$w(x_0) \geq \sum_{t=0}^{\tau} \beta^t F(x_t, x_{t+1}) + \beta^{\tau+1} w(x_{\tau+1}) \,.$$

Letting $\tau \to \infty$ yields $w(x_0) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = U(\boldsymbol{x}) = v(x_0)$. Thus, $w(x_0) \geq v(x_0)$ is also true, so that $w(x_0) = v(x_0)$ for each $x_0 \in X$. ■

The above result, though well known, offers a new contribution to the literature on dynamic optimization because it provides another way of proving the recursive property of the value function for this class of models. The key contribution is that to prove that the value function is the unique solution of the Bellman equation, we do not invoke the Contraction Mapping Theorem or for that matter any fixed point argument, as is commonly done. Theorem 3.7 offers a simple and direct proof of this fact by taking advantage of the correspondence-theoretic approach that we have developed. For instance, notice that the proof of Theorem 3.7 relies on the continuity of the value function, which in turn is established by a simple application of Berge's Maximum Theorem. The key ingredient is the introduction of the plan correspondence concept and the demonstration of its continuity.

We conclude our study in this section by discussing the policy function for economies with geometric discounting. To do this, assume that Conditions **C1** and **C2** are valid, and, of course, from Lemma 3.1 Condition **C4** is true. A glance at Theorem 3.7 shows that the value function v is continuous and that it satisfies the Bellman equation, i.e.,

$$v(x) = \max_{y \in \Gamma(x)} [F(x,y) + \beta v(y)]$$

holds true for each $x \in X$.

This recursive property allows the definition of a correspondence $g\colon X \twoheadrightarrow X$ given by

$$g(x) = \{y \in \Gamma(x)\colon\; F(x,y) + \beta v(y) = v(x)\}\,.$$

This "argmax" correspondence is called the policy correspondence (see [10]). The choices in $g(x)$ are the best choices that the agent can make on any date that starts with the state x, given the constraint imposed on the agent by $\Gamma(x)$. We emphasize that in general there can be more than one choice $y \in \Gamma(x)$ that is optimal. That is, in general g is a multi-valued function. When g is a function, i.e., when the optimal choice is unique for each $x \in X$, then g is called the *policy function*.

Now we can establish the following well-known result by a simple application of the technique we have developed in the proof of Theorem 2.11.

Theorem 3.8. *The policy correspondence $g\colon X \twoheadrightarrow X$ is nonempty- and compact-valued, and upper hemicontinuous. Moreover,*

(a) *if the function F is concave, then g is also convex-valued, and*
(b) *if the function F is strictly concave, then g is a continuous function.*

Given that the return function F is bounded, it is not difficult to see (and is well known) that a plan $\boldsymbol{x} = (x_0, x_1, x_2, \ldots)$ is optimal if and only if it satisfies Bellman's Principle of Optimality. That is, for each $t \geq 0$ we have

$$v(x_t) = F(x_t, x_{t+1}) + \beta v(x_{t+1})\,. \tag{3.7}$$

In other words, a plan $\boldsymbol{x} \in \Pi(x_0)$ is optimal if and only if $x_{t+1} \in g(x_t)$ holds for all $t \geq 0$. This observation in connection with the preceding discussion yields the following.

Theorem 3.9. *If a time-separable model with geometric discounting satisfies Conditions* **C1**, **C2** *and* **C3** *and has a strictly concave return function, then for each initial state* $x_0 \in X$ *there exists a unique optimal plan* $\boldsymbol{x} = (x_0, x_1, \ldots)$ *that is given by the recursive formula*

$$x_{t+1} = g(x_t) = g^{t+1}(x_0) \quad \textit{for each} \quad t = 0, 1, 2, \ldots \ .$$

4. An Example: The One-Sector Growth Model

The textbook formulation of the one-sector growth model runs something like this.There is a single commodity which is used as capital, along with labor, to produce output. In the simplest formulation, labor is presumed to be supplied in fixed amounts and there is a representative agent. In each period $t = 0, 1, 2, \ldots$ a part c_t of the output is consumed and a part x_{t+1} is saved as capital for next period, which fully depreciates after its use. The quantities c_t and x_{t+1} satisfy the feasibility constraint

$$c_t + x_{t+1} = f(x_t)\,,$$

where $f\colon [0, \infty) \to [0, \infty)$ is the production function and x_0, the initial capital stock, is given. The function f is assumed to satisfy the Inada conditions. In particular, f is strictly increasing and strictly concave.

We now verify that the model satisfies the conditions necessary to apply our approach:

(i) the metrizable topological vector space is $\Theta = \mathbb{R}$,

(ii) the state space is the ordered, closed, and convex subset $X = [0, \infty)$ of Θ, and

(iii) the constraint correspondence $\Gamma\colon X \twoheadrightarrow X$ defined by $\Gamma(x) = [0, f(x)]$, is

  (a) nonempty- and compact-valued,

  (b) monotone,

(c) continuous (according to Theorem 5.1), and
(d) has a closed convex graph.

With each plan $\boldsymbol{x} \in \Pi(x_0)$ we associate the consumption plan $\boldsymbol{c_x} = (c_0, c_1, c_2, \ldots)$ that is defined for each $t = 0, 1, 2, \ldots$ by

$$c_t = f(x_t) - x_{t+1}\,.$$

Clearly, $0 \le c_t \le f(x_t)$ for each $t = 0, 1, 2, \ldots$.

The objective here is to find a plan $\boldsymbol{x} \in \Pi(x_0)$ that maximizes the lifetime utility function $U\colon \Pi(x_0) \to \mathbb{X}$, defined by

$$U(\boldsymbol{x}) = \sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t u\big(f(x_t) - x_{t+1}\big)\,, \tag{4.1}$$

where, as usual, $u\colon [0,\infty) \to [0,\infty)$ is a bounded function satisfying the Inada conditions (and hence u is strictly concave) and normalized so that $u(0) = 0$. Since u is bounded, U is a well-defined real-valued function.

Now we define the return function $F\colon G_\Gamma \to \mathbb{R}$ by $F(x,y) = u\big(f(x) - y\big)$. It is not difficult to see that the return function F is continuous, strictly increasing in x, and strictly concave. Clearly, this model is a special case of the general recursive dynamic model, where $U(\boldsymbol{x}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$. In particular, the state-contingent lifetime utility function is now given for each $(x, \boldsymbol{x}) \in G_\Pi$ by $\widehat{U}(x, \boldsymbol{x}) = U(\boldsymbol{x})$.

Consequently, from the results obtained in Section 3, Conditions **C1**, **C2**, **C3**, and **C4** are true and $\widehat{U}$ is strictly monotone and strictly concave. Thus, we have established the following well-known result.

Theorem 4.1. *In the one-sector growth model, the value function v is:*

(a) *bounded,*
(b) *continuous,*
(c) *strictly concave,*
(d) *strictly increasing, and*

(e) *the only bounded solution of the Bellman equation, i.e.,* $v\colon X \to \mathbb{R}$ *is the only bounded function that for each* $x \geq 0$ *satisfies*

$$v(x) = \sup_{0 \leq y \leq f(x)} [F(x,y) + \beta v(y)]$$

Also, the policy function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$*, defined by* $v(x) = F(x, g(x)) + \beta v(g(x))$*,*

(i) *is continuous, and*

(ii) *for each* $x_0 \in X$ *the unique optimal plan* $\boldsymbol{x} = (x_0, x_1, \ldots)$ *is given by the recursive formula* $x_{t+1} = g(x_t) = g^{t+1}(x_0)$ *for each* $t = 0, 1, 2, \ldots$.

5. Concluding remarks

We have introduced a method to study existence of optima in dynamic economies that relies neither on a variational approach and the use of transversality conditions, nor on the usual dynamic programming techniques that employ fixed point arguments. Instead, our approach is based on the theory of correspondences and applies two classical fundamental theorems of mathematical analysis, Tychonoff's Product Theorem and Berge's Maximum Theorem.

The basic ingredient is the study of the properties of what we call the plan correspondence. This set-valued function maps the collection of all possible initial states of the economy into the collection of all time-sequences representing plans for consumption and savings. If it can be established that this correspondence is continuous, convex- and compact-valued, then one can easily prove existence of optimal plans given bounded and continuous preferences over plans. In addition, once can easily characterize the main features of the associated value function, and in particular its continuity and concavity. Our approach to proving existence of optima can easily accommodate preferences that are not time-separable. Given standard time-separable preferences, it also offers a straightforward way to obtain a recursive representation by means of a Bellman equation.

APPENDIX

We state a result about correspondences, not readily available in the literature.

Theorem 5.1. *Let X be a topological space and let $f, h\colon X \to \mathbb{R}$ be two continuous functions. Then the correspondence $\Gamma\colon X \twoheadrightarrow \mathbb{R}$, defined by letting*

$$\Gamma(x) = \text{the closed subinterval of } \mathbb{R} \text{ with endpoints } f(x) \text{ and } h(x)\,,$$

is continuous.[6]

Proof. Replacing f by $\max\{f, h\}$ and h by $\min\{f, h\}$, we can suppose without loss of generality that $h(x) \le f(x)$ holds true for each $x \in X$. Now fix some $x_0 \in X$.

We first show that Γ is upper hemicontinuous at x_0. To this end, assume that $\mathcal{O}$ is an open subset of $\mathbb{R}$ satisfying $\Gamma(x_0) = [h(x_0), f(x_0)] \subseteq \mathcal{O}$. We must show that there exists a neighborhood N of x_0 such that $x \in N$ implies $\Gamma(x) \subseteq \mathcal{O}$.

Since $\mathcal{O}$ is open and $h(x_0), f(x_0) \in \mathcal{O}$, there exists some real number $\epsilon > 0$ such that $(h(x_0)-\epsilon, h(x_0)+\epsilon) \subseteq \mathcal{O}$ and $(f(x_0)-\epsilon, f(x_0)+\epsilon) \subseteq \mathcal{O}$. In particular, observe that $(h(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq \mathcal{O}$. Next pick a neighborhood N of x_0 such that $x \in N$ implies

$$h(x_0) - \epsilon < h(x) < h(x_0) + \epsilon \quad \text{and} \quad f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon\,.$$

Now notice that $x \in N$ implies $\Gamma(x) = [h(x), f(x)] \subseteq (h(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq \mathcal{O}$, and so N is the desired neighborhood.

Next, we prove that Γ is lower hemicontinuous at x_0. To see this, assume that for some open subset $\mathcal{O}$ of $\mathbb{R}$ we have $\Gamma(x_0) \cap \mathcal{O} \neq \emptyset$ or $[h(x_0), f(x_0)] \cap \mathcal{O} \neq \emptyset$. We must show that there exists a neighborhood N of x_0 such that $x \in N$ implies $\Gamma(x) \cap \mathcal{O} \neq \emptyset$.

We start by fixing some $y \in \mathcal{O}$ such that $h(x_0) \le y \le f(x_0)$. Next, we pick some $\epsilon > 0$ such that $(y - \epsilon, y + \epsilon) \subseteq \mathcal{O}$. Now we distinguish three cases.

Case I: $h(x_0) = f(x_0) = y$.

In this case, by the continuity of h and f at x_0 there exists some neighborhood N of x_0 such that $x \in N$ implies $h(x), f(x) \in (y - \epsilon, y + \epsilon)$ and so

$$\Gamma(x) = [h(x), f(x)] \subseteq (y - \epsilon, y + \epsilon) \subseteq \mathcal{O}\,.$$

[6] Keep in mind that, as usual, the closed interval $[a, a]$ is simply the singleton $\{a\}$. Clearly, the correspondence Γ is also nonempty-, convex- and compact-valued.

Consequently, $x \in N$ yields $\Gamma(x) \subseteq \mathcal{O}$ so that $\Gamma(x) \cap \mathcal{O} = \Gamma(x) \neq \emptyset$. This proves that, in this case, N is a desired neighborhood.

Case II: $h(x_0) < y < f(x_0)$.

Here, the continuity of h and f at x_0 guarantees the existence of some neighborhood N of x_0 such that $h(x) < y$ and $y < f(x)$ hold for all $x \in N$. But then $x \in N$ implies $y \in \Gamma(x) \cap \mathcal{O}$ proving that $\Gamma(x) \cap \mathcal{O} \neq \emptyset$ for all $x \in N$.

Case III: $h(x_0) = y < f(x_0)$ or $h(x_0) < y = f(x_0)$.

Since $(y - \epsilon, y + \epsilon) \subseteq \mathcal{O}$ notice that in both possibilities there exists some $z \in (y - \epsilon, y + \epsilon)$ (and so $z \in \mathcal{O}$) satisfying $h(x_0) < z < f(x_0)$, and the desired conclusion follows from Case II above. ■

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, 3^{rd} Edition, Springer–Verlag, New York & London, 2006.

[2] Y. Balasko, D. Cass and K. Shell, Existence of competitive equilibrium in a general overlapping-generations model, *Journal of Economic Theory* **23** (1980), 307–322.

[3] Y. Balasko and K. Shell, The overlapping-generations model I: The case of pure exchange without money, *Journal of Economic Theory* **23** (1980), 281–306.

[4] R. A. Becker and J. H. Boyd III, *Capital Theory, Equilibrium Analysis and Recursive Utility*, Blackwell Publishers, Malden, MA, 1997.

[5] M. Boldrin and A. Rustichini, Growth and indeterminacy in dynamic models with externalities, *Econometrica* **62** (1994), 323–343.

[6] P. K. Dutta and T. Mitra, Maximum theorems for convex structures with an application to the theory of optimal intertemporal allocation, *Journal of Mathematical Economics* **18** (1989), 77–86.

[7] M. Harris, *Dynamic Economic Analysis*, Oxford University Press, London and New York, 1987.

[8] M. Majumdar, T. Mitra, and K. Nishimura, Eds., *Optimization and Chaos*, Studies in Economic Theory, Vol. 11, Springer–Verlag, Berlin & Heidelberg, 2000.

[9] P. A. Samuelson, An exact consumption-loan model of interest with or without the social contrivance of money, *Journal of Political Economy* **66** (1958), 467–482.

[10] N. Stokey, R. E. Lucas, Jr., and E. C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge, MA, 1989.

[11] C. A. Wilson, Equilibrium in dynamic models with an infinity of agents, *Journal of Economic Theory* **24** (1981), 95–111.