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Gabriele Camera

Chapman University, camera@chapman.edu

Robert R. Reed

University of Alabama

Christopher Waller

University of Notre Dame

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Jack of All Trades or a Master of One? Specialization, Trade and Money¹

Gabriele Camera	Robert Reed	Christopher Waller
Purdue University	University of Kentucky	University of Kentucky, ZEI-Bonn
Gcamera@mgmt.purdue.edu	Rrreed@pop.uky.edu	Cjwall@pop.uky.edu

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Abstract

We consider a model of decentralized exchange where individuals choose the set of goods they produce. Specialization involves producing a smaller set of goods and doing it more proficiently. In doing so, agents reduce production costs, but also reduce the ease of trading their output. We derive the equilibrium degree of specialization and examine how it is affected by underlying fundamentals. Due to the existence of a hold-up problem, individuals specialize too little relative to the social optimum. Introducing money leads to more specialization relative to barter and increases welfare.

1. Introduction

As Adam Smith recognized long ago, by specializing in production, agents become more proficient and thereby lower production costs. As a result, welfare is improved by having individuals specialize in the production of goods and then trade with each other. If exchange is centralized agents can produce a small set of goods to lower production costs, then go to the market and trade. However, if trade is decentralized, then individuals must be matched with acceptable partners before trade can occur. In a world of barter, trading generally requires a double coincidence of wants, which is more likely to occur if agents can produce or store a wider variety of goods. Since specialization of

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production reduces the set of goods an individual has to offer, it reduces the probability of a double coincidence of wants and trading in a given period. As a result, decentralization makes the issue of how much to specialize more complicated since agents must trade-off efficiency of production against the frequency of trade. In short, an agent must decide whether to be “a jack of all trades or a master of one.” Our objective in this paper is to analyze an individual’s specialization decision in a decentralized trading environment.

The decision of how much to specialize is further complicated by the existence of money. As Smith recognized, specialization in production leads to the use of money in trading. However, as pointed out by Kiyotaki and Wright (1993), the use of money lowers trading risk by allowing trade to occur in single coincidence matches. Since trading risk is reduced by the use of money, it seems intuitive that agents would respond to this by specializing more on the margin in order to lower costs of production. But if so, under what conditions? In addition, once money is introduced, does an increase in the steady state money stock have any further effects on specialization? And finally, does the introduction of money improve welfare?

In this paper we study agents’ specialization choices in both a barter and a monetary economy. We then determine if, and under what conditions, the existence of money leads to greater specialization in production. We use a random matching model with divisible goods and indivisible money as in Trejos and Wright (1995) and Shi (1995). We allow agents to choose which goods they produce, ranging from a single good to the entire set of goods in the economy. Specialization is defined to mean a reduction in the set of goods an agent can produce. It is assumed that the total and marginal cost of producing a given quantity of output declines as agents specialize. Prices and terms of trade are determined via a simple match-dependent bargaining protocol: Nash bargaining in barter matches, and buyer-take-all in monetary matches. Finally, we compare the individual’s choice of specialization to that of a social planner.

Our work complements the studies of Kiyotaki and Wright (1993), Shi (1997) and Reed (1998), who have examined how specialization affects market participation in search-theoretic models of money.² On the contrary, our model focuses on the relationship between specialization and produc-

²In Kiyotaki and Wright (1993) production is stochastic; by choosing a narrower production set (‘specializing’), the frequency of production rises. In Shi (1997) the agent can costly produce his desired consumption good or costlessly produce a good to be traded for his desired consumption good (‘specialization’). In Reed (1998) an agent who specializes more spends less time maintaining her production skills, and more time in the market trading. While

tion efficiency. Furthermore, we expand the scope of analysis relative to earlier work by relaxing the assumption of indivisible goods and fixed terms of trade. In this way, we can examine how specialization affects individual output, marginal costs of production, terms of trade and prices. We also study how risk aversion affects specialization.

To summarize our results, we show existence and uniqueness of the optimal degree of specialization in both barter and monetary economies. In a barter economy we show that agents specialize more as search frictions fall and risk aversion falls. We find that the individual choice of specialization is generally inefficient because of the existence of a trading externality and a pricing distortion. When money is introduced into the economy, specialization increases lowering the probability of barter matches. While agents take into account their surplus loss from less frequent barter matches, they do not consider the surplus loss of their trading partner. Furthermore, because terms of trade are determined by bilateral bargaining, the individual's choice of specialization creates a hold-up problem and thus a pricing distortion. By specializing less than average, an individual can appropriate more of the surplus from trade. As a result of these two effects individual specialization choices are not socially optimal.

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 describes pricing protocols and value functions. Section 4 studies equilibria in a barter and a monetary economy. Section 5 concludes. All proofs are in the Appendix.

2. Environment

Time is continuous with consumption and production occurring at discrete points. There is a $[0,1]$ continuum of infinitely-lived agents and perishable good types both of which are uniformly distributed along a unit circle. In the ensuing discussion a point along the circle is taken to represent a particular good type or a particular agent type. Individuals discount future utility at rate $r > 0$.

Agents have specialized preferences. A representative agent i derives utility from consumption of those commodities lying along an arc going from point i (her location along the circle) to $i + x$, where $x \in (0,1) \forall i$. Consumption of quantity $q_j > 0$ of good j provides utility $u(q_j) > 0$ to individual i only if j lies in her consumption set and if it is not her own production. It yields zero

we focus on agents' decisions to produce a variety of goods, an alternative would be to acquire a broader set of goods to increase the frequency of exchange, an idea explored by Shevchenko (1999).

utility otherwise.³ Additionally $u'(q_j) > 0$, $u''(q_j) < 0$, $u'''(q_j) > 0$, and $u(0) = 0$. For tractability we consider $u(q) = q^\sigma$, $\sigma \in (0, 1)$.

2.1 Endowments, Production, and the Benefits of Specialization

Agent i is initially assigned a production location k on the circle (corresponding to good k) randomly and independently of her type. Following that, i may choose to expand her production set to an interval of length y_i on an arc centered around point k . The resulting production set is $[k - y_i/2, k + y_i/2]$ where $y_i \in [0, 1]$. A larger y_i implies a broader production set meaning individual i specializes less. For instance in standard search models agents are assumed to be able to produce only one type of good, which corresponds to $y_i = 0$ in our model. We assume that this specialization choice is made at the beginning of life, and cannot be changed. The agent chooses the length of the interval, y_i , to maximize her expected discounted lifetime utility from consumption.

Production is costly and is associated with a utility loss that depends on the extent of specialization, y_i , and the quantity produced, q . It is assumed that for a given y_i , the disutility associated with the production of q units of any good is $c(q, y_i) \geq 0$, a twice continuously differentiable function such that $c(0, y_i) = 0$, $c_q(q, y_i) > 0$, $c_{qq}(q, y_i) \geq 0$. We further assume that for all $y_i \in [0, 1]$, there exists $0 < \bar{q} < \infty$ such that $u(\bar{q}) = c(\bar{q}, y_i)$, and $u'(0) > c_q(0, y_i)$. That is, a single (or double) coincidence match can always generate some positive surplus, no matter how specialized the economy; furthermore, the amount produced and exchanged in every trade match is finite.

We assume that $c_{y_i}(q, y_i) > 0$, $c_{y_i y_i}(q, y_i) \geq 0$, $c_{y_i}(0, y_i) = 0$, and $c_{q y_i}(q, y_i) > 0$. Our assumptions on the cost function capture the idea of proficiency in production – the first inequality shows that increasing the set of production goods raises the total cost of any good produced by agent i while the second inequality assumes that the marginal cost of producing any good is also increasing in y_i . For tractability we consider $c(q, y) = (1 + y)q$.

A fraction $M \in [0, 1]$ of individuals initially receives one unit of indivisible fiat money which can be freely discarded. We assume that individuals cannot store more than one unit of money but we do allow for barter trade to occur in *all* double coincidence matches. That is, regardless of an agent's money holdings, individuals may produce and trade goods. In this way, introducing money

³In search models of money it is generally assumed that individuals cannot consume own production to motivate the need for exchange. In our environment, this creates additional complications in deriving the probabilities of exchange. To avoid them we normalize to zero the utility from autarkic consumption.

expands the set of trading opportunities.⁴

2.2 Exchange

Agents meet bilaterally and randomly according to a Poisson process with arrival rate α , and we let $\alpha = 1$ by normalizing the time interval. Only one transaction per date can be carried out. Trading histories are private information but types and actions within each match are observable.

Given her initial production location, individual i 's optimal choice of specialization, y_i , has implications for her probability of exchange. Since we study stationary and symmetric equilibria, let Y denote the length of all other agents' production sets, taken as given by agent i .

Now consider the probability that someone is able to produce something that agent i likes. This depends on both x and Y as follows. Any producer whose location is inside the arc $[i, i + x]$ can produce for agent i . Furthermore, producers located on the arcs extending to the left of point i , $[i - Y/2, i]$, and to the right of point $i + x$, $[i + x, i + x + Y/2]$ can also produce for i . Combining these three sets of producers, the proportion of agents able to produce for i is $x + Y$. Since a coincidence of wants will always occur for all $Y \geq 1 - x$ the ex-ante probability that a randomly encountered agent can produce i 's consumption good is

$$p(Y) = \begin{cases} x + Y & \text{if } Y \leq 1 - x \\ 1 & \text{otherwise.} \end{cases} \quad (1)$$

Similarly, the probability that a randomly encountered agent will want to consume a good produced by agent i is

$$p(y_i) = \begin{cases} x + y_i & \text{if } y_i \leq 1 - x \\ 1 & \text{otherwise.} \end{cases} \quad (2)$$

Given that production and consumption locations are randomly, independently and uniformly distributed across individuals, the ex-ante probability of double coincidence of wants in a match is $p(Y)p(y_i)$.

The sequence of events for a representative trader is as follows. At the beginning of time she chooses the extent of specialization. Subsequently she begins her search process, meeting other agents pairwise and randomly over time. Contingent on a match, she bargains. If this leads to trade, production and consumption take place after which she searches anew.

⁴This is unlike most earlier search models where money 'crowds out' valuable barter trades thereby creating an artificial cost that makes money less valuable. See Rupert et al. (2001) for a discussion of the different approaches.

3. Symmetric Stationary Equilibria

We study stationary rational expectations equilibria, where symmetric Nash strategies are adopted, and identical agents use identical time-invariant strategies. Thus, without loss of generality, in what follows we omit the subscript i in the specialization choice of agent i . Each individual takes as given the extent of specialization of all others, Y , when considering her choice of y . Furthermore, the beliefs over strategies are identical across individuals, and agents correctly evaluate the surplus derived in each possible transaction. In what follows we study a barter economy and a monetary economy. When analyzing monetary equilibria, we focus on equilibria where money is fully acceptable.

3.1 Bargaining

Consider a match where trade is feasible. The two agents can choose to barter or to engage in monetary exchange depending on their holdings of money and the existence of a single or double coincidence of wants. It is assumed that the terms of trade in a stationary equilibrium must satisfy the generalized Nash bargaining protocol. In matches where barter takes place both parties have an equal bargaining weight. In matches where money is exchanged we assume buyers make take-it-or-leave-it offers to sellers. This follows much of the literature, although one could use generalized Nash bargaining instead (see Rupert et al. (2001)).

The assumed bargaining protocols have the virtue of making the analysis quite tractable because they have two important implications for the equilibrium pattern of exchange. First, take-it-or-leave-it offers imply that the buyer's degree of specialization cannot affect the quantity she expects to buy with money, denoted by Q_m . Her specialization choice can only affect the quantity she expects to sell for money, denoted by q_m . Second, since surplus is earned by both parties only when barter occurs, monetary exchange will take place in equilibrium only in single coincidence matches. Consequently, barter will be chosen whenever feasible.⁵

Consider a double coincidence match. The quantity of output received by the representative agent in the match, Q , and the quantity she produces q , are determined via a Nash bargaining process. Since specialization choices are made at the beginning of life, y and Y are taken as given when two agents bargain. Recall also that both parties in the match take as given the value functions

⁵This is almost certainly true for any bargaining weights not just buyer-take-all (see the proof in Rupert et al. (2001)).

and the quantities exchanged in all other matches, Q . Under this conjecture, the quantities traded must satisfy⁶

$$\begin{aligned} & \max_{q,Q} [u(q) - c(Q, Y)][u(Q) - c(q, y)] \\ & \text{s.t. } u(q) > c(Q, Y) \text{ and } u(Q) > c(q, y). \end{aligned}$$

Thus, the equilibrium quantities must satisfy the first order conditions:

$$\begin{aligned} u'(q)[u(Q) - c(q, y)] - c_q(q, y)[u(q) - c(Q, Y)] &= 0 \\ u'(Q)[u(q) - c(Q, Y)] - c_Q(Q, Y)[u(Q) - c(q, y)] &= 0. \end{aligned} \tag{3}$$

In equilibrium these quantities depend on y and Y so we denote them by $Q(y, Y)$ and $q(y, Y)$, omitting the arguments when understood. Under the assumed properties of $u(Q)$ and $c(Q, Y)$, it is easy to show that if $y > Y$ then $Q > q$, and vice versa.

Under the assumed preferences and cost functions (3) implies:

$$\begin{aligned} q(y, Y) &= Q(Y) \cdot \left(\frac{1+Y}{1+y}\right)^{\frac{1}{1-\sigma^2}} \\ Q(y, Y) &= q(y, Y) \cdot \left(\frac{1+Y}{1+y}\right)^{\frac{-1}{1+\sigma}} \end{aligned} \tag{4}$$

where

$$Q(Y) = \left(\frac{\sigma}{1+Y}\right)^{\frac{1}{1-\sigma}} \tag{5}$$

and satisfies

$$u'(Q) - c_Q(Q, Y) = 0. \tag{6}$$

In a symmetric equilibrium, $y = Y$, $q = Q = Q(Y)$, thus barter trades are efficient. The symmetric equilibrium $Q(Y)$ is unique, bounded above (by a positive number) and below (by zero) and it is a decreasing function of Y (we omit the argument, when understood). As the degree of specialization rises (Y falls) a higher quantity of goods is traded because marginal production costs are lower. A change in σ affects the marginal utility of consumption and since the Nash bargaining protocol implies that marginal utility must be a constant for our functional forms, $u'(Q) = 1 + Y$, then Q must also change.

Now consider a single coincidence match where an agent with money is the buyer and an agent without is the seller. Let $V_m(y, Y)$ and $V_0(y, Y)$ denote the stationary expected lifetime utility of

⁶We assume that the threat point is equal to the continuation value. However, other threat points could be used.

an agent who holds money and an agent who does not. The assumed bargaining protocol implies that the equilibrium quantities must satisfy:

$$\begin{aligned} V_m(Y, Y) - c(Q_m, Y) - V_0(Y, Y) &= 0 \\ V_m(y, Y) - c(q_m, y) - V_0(y, Y) &= 0 \end{aligned} \tag{7}$$

The first line determines the quantity agent i receives from an arbitrary seller for money while the second line determines how much she produces for money. In equilibrium these quantities depend on y and Y , and are denoted $Q_m(Y)$ and $q_m(y, Y)$, omitting the arguments when understood. We let $q_m = Q_m = 0$ correspond to a non-monetary equilibrium. In a symmetric equilibrium, $y = Y$, $q_m(Y) = Q_m(Y)$.⁷

3.2 Value Functions

Since all double coincidence matches lead to barter trade in equilibrium, the value functions must satisfy:

$$\begin{aligned} rV_0(y, Y) = & p(Y)p(y)[u(Q(y, Y)) - c(q(y, Y), y)] \\ & + M[1 - p(Y)]p(y)[V_m(y, Y) - c(q_m(y, Y), y) - V_0(y, Y)] \end{aligned} \tag{8}$$

$$\begin{aligned} rV_m(y, Y) = & p(Y)p(y)[u(Q(y, Y)) - c(q(y, Y), y)] \\ & + (1 - M)p(Y)[1 - p(y)][V_0(y, Y) + u(Q_m(Y)) - V_m(y, Y)]. \end{aligned} \tag{9}$$

Equation (8) shows that an individual without money will barter with probability $p(Y)p(y)$ and will sell for money in a single coincidence meeting with a money-holder who desires her good with probability $M[1 - p(Y)]p(y)$. Equation (9) is the expected flow return to a money holder. The first term in (9) is the expected payoff from engaging in a barter trade while the second term is the payoff from paying with cash in a single coincidence match. Note that greater individual specialization (lower value of y) increases the probability of a single coincidence match $(1 - M)p(Y)[1 - p(y)]$ but lowers the probability of double coincidence matches $p(Y)p(y)$.

⁷It can now be verified that in a symmetric equilibrium barter would be strictly preferred not only by the potential seller, but also by the potential buyer. This is because the equilibrium surplus guaranteed by barter, $u(Q) - c(Q, Y)$, is larger than the equilibrium surplus generated from purchasing with money, $u(Q_m) - (V_m - V_0)$, i.e. $u(Q_m) - c(Q_m, Y) < u(Q) - c(Q, Y) \forall Q_m \neq Q$. This is because $u(q) - c(q, Y)$ is strictly concave in q , reaching a maximum at $q = Q$.

Expressions (8)-(9) show why we explicitly denote the value from trading as a function of Y and y : the expected net payoff in trade matches, expectations regarding the frequency of trade, and the expectations regarding the quantities traded all depend on y and Y .

Note that money has no value in a fully diversified economy, since there are no single coincidence matches when $Y \geq 1 - x$, hence $V_m = V_0$. Similarly, money is valueless when $M = 1$, due to the inventory restrictions. Hence, when we study monetary equilibria, we consider $Y \in [0, 1 - x)$ and $M < 1$, without loss of generality. We next characterize the equilibrium quantity $Q_m(Y)$.

Lemma 1. *Suppose $y = Y$ is a symmetric monetary equilibrium. Then Q_m is unique and it must satisfy*

$$u(Q_m)\mu(Y) - c(Q_m, Y) = 0 \quad (10)$$

with $0 < \mu(Y) < 1$ and $u(Q_m) > c(Q_m, Y)$. Also, $\lim_{Y \rightarrow 1-x} Q_m = \lim_{M \rightarrow 1} Q_m = 0$, and if $c(Q, Y) = Q(1 + Y)$ then $Q_m(Y)$ is strictly concave in Y , with maximum at $Y = \max\{0, \frac{1}{2} - x\}$.

Lemma 1 has an interesting implication. Using (10) under the assumed functional forms:

$$Q_m(Y) = \left(\frac{\mu(Y)}{1 + Y} \right)^{\frac{1}{1-\sigma}} \quad (11)$$

where $\mu(y, Y) = \frac{(1-M)p(Y)[1-p(y)]}{r+(1-M)p(Y)[1-p(y)]}$ and $\mu(Y) \equiv \mu(y, Y)|_{y=Y}$.

3.3 The Specialization Choice of the Individual.

The representative agent must choose the extent of her specialization before the random distribution of money occurs. This initial choice is final. In doing so she takes as given the expected degree of specialization of all others. thus, the individual chooses y to maximize her ex-ante flow return from trade denoted by

$$W(y, Y) = (1 - M)rV_0(y, Y) + MrV_m(y, Y).$$

Upon substitution of (7), (8) and (9), and using Lemma 1:

$$W(y, Y) = p(Y)p(y)[u(Q(y, Y) - c(q(y, Y), y)] + rM\mu(y, Y)u(Q_m(Y)) \quad (12)$$

To study the individual's choice of y , it is necessary to differentiate $W(y, Y)$ with respect to y :

$$\begin{aligned}
W_y(y, Y) = & p(Y)p'(y) [u(Q(y, Y)) - c(q(y, Y), y)] - p(Y)p(y)c_y(q(y, Y), y) \\
& + p(Y)p(y) [u'(Q(y, Y))Q_y(y, Y) - c_q(q(y, Y), y)q_y(y, Y)] \\
& + rM\mu_y(y, Y)u(Q_m(Y))
\end{aligned} \tag{13}$$

where $Q_y(y, Y) = \frac{\partial Q(y, Y)}{\partial y}$, $q_y(y, Y) = \frac{\partial q(y, Y)}{\partial y}$. The effect of an increase in y on the agent's expected lifetime utility, given that everyone else is expected to choose Y , affects both the value of barter trades and money trades. The first three terms in (13) correspond to the effects on barter matches from being less specialized. The first term reflects the benefit of being able to produce more types of goods, which increases the probability of a double coincidence match, if $0 \leq y \leq 1 - x$. If $y > 1 - x$, however, this 'matching benefit' vanishes since $p'(y) = 0$, and therefore $W_y(y, Y)$ has a discontinuity at $y = 1 - x$. The second term measures the increased marginal cost of producing more types of goods.

The third term captures how the choice of specialization affects the expected terms of trade in each match via a hold-up problem. Recall that the initial choice of specialization is final, but it takes place before any transaction has occurred. Thus, the choice of y is effectively an irreversible investment in production technology. A hold-up problem occurs when agents are not able to appropriate the full rents of their investment, thereby creating a disincentive to invest. In our model y corresponds to the choice of investing in the efficiency of the agent's production technology, where higher y implies less investment in technology. Thus, a hold-up problem arises in barter matches. In monetary matches no hold-up problem arises because of the bargaining protocol selected. Because the buyer takes the whole surplus, Q_m does not depend on y , but only on Y . Sellers recognize that they will never earn any surplus, no matter what y they select, hence the holdup problem does not exist in monetary trades.

The introduction of money allows trade to occur in single coincidence matches, and the last line of (13) shows that increased diversification negatively affects the value of these trades, since $\mu_y < 0$. The reason is simple. While increasing y raises the probability of double coincidence matches, it also reduces the occurrence of single coincidence matches, and thus the value of holding money. This is the only relevant effect, since in equilibrium an increase in y does not affect the surplus obtained by acquiring money since it is always zero.

To study the individual's choice of specialization we need to consider the second derivative of

$W(y, Y)$. Under the assumed functional forms, we can prove a very useful lemma.

Lemma 2. *Let $u(q) = q^\sigma$ and $c(q, y) = q(1 + y)$. Given Y , (i) if $y > 1 - x$ then $W_y(y, Y) < 0$, (ii) if $W_y(0, Y) \leq 0$ then $W_y(y, Y) < 0 \forall y$, (iii) if $W_y(1 - x, Y) \geq 0$ then $W_y(y, Y) > 0 \forall y$, (iv) if $W_y(\hat{y}, Y) = 0$ for some $\hat{y} \in (0, 1 - x)$, then $W_{yy}(\hat{y}, Y) < 0$, and (v) $u'(Q)Q_y(y, Y) - c_q(q, y)q_y(y, Y) > 0 \forall y, Y$.*

This has three important implications. First, since $W_y(y, Y) < 0$ for $y > 1 - x$, we can focus on $W(y, Y)$ as a continuous function defined on $[0, 1 - x] \times [0, 1 - x]$, with no loss in generality when studying symmetric equilibria. This is because no individual benefit can be obtained by diversifying beyond $1 - x$. The second implication is that there is a unique y that maximizes $W(y, Y)$, $\forall Y$. Furthermore, the individual's optimal production set, denoted y^* , can be fully characterized by sole consideration of (13). Specifically, given some Y :

$$y^* = \begin{cases} 1 - x & \Leftrightarrow W_y(1 - x, Y) \geq 0 \\ \hat{y} & \Leftrightarrow W_y(\hat{y}, Y) = 0 \\ 0 & \Leftrightarrow W_y(0, Y) \leq 0 \end{cases} \quad (14)$$

where $\hat{y} \in (0, 1 - x)$. Finally, since $u'(Q)Q_y - c_q(q, y)q_y > 0$, the holdup problem gives agents an incentive to specialize less.

Definition of Equilibrium. *A symmetric, stationary, monetary equilibrium is a list $\{V_0, V_m, y, q, q_m\}$ such that (i) $\{V_0, V_m\}$ satisfy (8)-(9), (ii) $q = q(y, Y)$ and $Q = Q(y, Y)$ satisfy (3), $q_m = q_m(y, Y)$ and $Q_m = Q_m(Y)$ satisfy (7), $Q_m(Y) > 0$, and (iii) $y = Y = y^*$ satisfies (14).*

In the ensuing analysis we let Y_b and Y_m denote the symmetric equilibrium values of specialization in the barter equilibrium and the monetary equilibrium respectively, letting \hat{Y}_i , $i = b, m$, denote an element of $(0, 1 - x)$. We can study the barter economy by setting $M = 0$ in (12) and (13), which allows us to compare the choice of specialization in a barter economy to that chosen in a monetary economy.

3.4 The Specialization Choice of a Social Planner.

To discuss the efficiency of the decentralized equilibrium we consider the specialization choice of

a social planner who treats agents symmetrically and takes the matching technology and bargaining protocols as given. The planner chooses a common Y in order to maximize the expected lifetime utility of a representative individual. Define his objective function by:

$$\begin{aligned} W_s(Y) &= (1 - M)rV_0(Y) + MrV_m(Y) \\ &= p(Y)^2 [u(Q(Y)) - c(Q(Y), Y)] \\ &\quad + p(Y)(1 - p(Y))(1 - M)M[u(Q_m(Y)) - c(Q_m(Y), Y)] \end{aligned}$$

an expression equivalent to (12) when $y = Y$, where $Q(Y)$ satisfies (6), and $Q_m(Y)$ satisfies (10).⁸

The planner's equilibrium choice of Y must maximize $W_s(Y)$. The necessary first order condition is:

$$\begin{aligned} \frac{\partial W_s(Y)}{\partial Y} &= p(Y)2[u(Q(Y)) - c(Q(Y), Y)] - p(Y)^2 c_Y(Q, Y) \\ &\quad + (1 - M)M(1 - 2p(Y))[u(Q_m(Y)) - c(Q_m(Y), Y)] \\ &\quad + (1 - M)Mp(Y)(1 - p(Y))\frac{\partial [u(Q_m(Y)) - c(Q_m(Y), Y)]}{\partial Y} \end{aligned} \tag{15}$$

The first line in (15) corresponds to what the planner would face in a barter economy. The first term reflects the social benefit of less specialization, while the second term reflects its costs. This trade-off is essentially the same as that faced by the private individual. However, the social marginal benefit of a higher Y , $2[u(Q) - c(Q, Y)]$, is larger than the private marginal benefit, $u(Q) - c(Q, Y)$, because of a trading externality. When lowering her degree of specialization (raising y), the agent increases the probability of a double coincidence match and gains some surplus from additional trades. However, she ignores the gains everyone else receives from trading with her. Since the barter quantities are efficient, by the envelope theorem the planner's selection of Y does not affect the bartered quantities. Thus, the holdup problem arising from the choice of specialization creates a price distortion that the planner wants to avoid.

The remaining components of (15) correspond, respectively, to changes in the occurrence and the magnitude of the surplus in monetary transactions induced by an increase in Y . First, the planner recognizes that less specialization makes single coincidence matches, hence monetary exchange, less likely. Individuals do not take this into account since they expect to receive no surplus when they sell for money. The resulting inefficiency encourages over-specialization relative to the social

⁸We have substituted $p(Y)(1 - p(Y))(1 - M)M[u(Q_m(Y)) - c(Q_m(Y), Y)]$ for $rM\mu(Y)u(Q_m(Y))$ using Lemma 1.

optimum.⁹ Second, the planner can affect the surplus generated in monetary exchanges since the real price of money, $Q_m(Y)$, depends on the equilibrium specialization level. A buyer, however, cannot do so due to the nature of the bargaining protocol. The resulting inefficiency distorts the choice of specialization depending on (i) how Q_m is expected to differ from the surplus-maximizing quantity (above, equal, or below) and (ii) the parameter x , which affects how Q_m reacts to a change in Y (see Lemma 1).

In the ensuing analysis we will denote the planner's choice of specialization by Y_b^* for the case of a barter equilibrium and Y_m^* for the case of a monetary equilibrium, letting \hat{Y}_i^* , $i = b, m$, denote an element of $(0, 1 - x)$.

Using (15) with $M \in \{0, 1\}$ it is easy to show that in a barter economy:

$$Y_b^* = \begin{cases} 0 & \Leftrightarrow \frac{\partial W_s(0)}{\partial Y} \leq 0 \\ \hat{Y}_b^* & \Leftrightarrow \frac{\partial W_s(\hat{Y}_b^*)}{\partial Y} = 0 \\ 1 - x & \Leftrightarrow \frac{\partial W_s(1-x)}{\partial Y} \geq 0 \end{cases}$$

where for the assumed functional forms $\hat{Y}_b^* = \frac{\sigma(2+x)-2}{2-3\sigma}$, an element of $(0, 1 - x)$ if and only if $\sigma > \frac{4-2x}{5-2x}$.

For the monetary equilibrium with $M \in (0, 1)$ it is more difficult to provide a complete analytical characterization of Y_m^* because changes in specialization affect both prices and trading probabilities. It is easy to show, however, that if Y_m^* is an equilibrium, then $Y_m^* < Y_b^*$ for all $M \in (0, 1)$, whenever $Q_m(Y_b^*) \approx Q(Y_b^*)$, that is when the quantities traded in money matches are efficient.¹⁰ Hence, the planner would choose greater specialization in a monetary economy relative to a barter economy when money is priced efficiently.

4. Equilibria

In this section we examine the effect that the hold-up problem has on the equilibrium choice of specialization and how the introduction of money affects specialization, terms of trade and welfare.

4.1 Barter Equilibria

⁹The magnitude of this effect is clearly sensitive to the choice of bargaining protocol. It would be reduced if sellers received some trade surplus.

¹⁰Since r influences Y_m only through its effect on Q_m and Q , we can find a value of r such that $\mu(Y) = \sigma$ and $Q_m/Q = 1$.

Consider the benchmark case of a non-monetary economy by setting $M = 0$. Using (13):

$$W_y(y, Y) = p(Y) \{u(Q) - c(q, y) - p(y)c_y(q, y) + p(y) [u'(Q)Q_y - c_q(q, y)q_y]\} \quad (16)$$

The next Lemma discusses existence and uniqueness of a symmetric barter equilibrium.

Lemma 3. *Let $u(q) = q^\sigma$ and $c(q, y) = q(1 + y)$. A symmetric barter equilibrium always exists, and it is unique. Y_b can be interior depending on preferences and trading frictions. Specifically:*

$$Y_b = \begin{cases} 1 - x & \Leftrightarrow \sigma \leq \sqrt{\frac{2-x}{3-x}} \\ \hat{Y}_b & \Leftrightarrow \sigma \in \left(\sqrt{\frac{2-x}{3-x}}, \sqrt{\frac{1}{1+x}} \right) \\ 0 & \Leftrightarrow \sigma \geq \sqrt{\frac{1}{1+x}}. \end{cases} \quad (17)$$

where \hat{Y}_b falls in x .

The existence of corner or interior solutions depends on the relative magnitudes of σ and x . In particular, (5) and (16) imply $\hat{Y}_b = \frac{1-\sigma^2(1+x)}{2\sigma^2-1}$. As illustrated in Figure 1, expanding the desired set of consumption goods (increasing x) makes trade easier which entices individuals to specialize more. The parameter σ also affects Y_b because σ captures two distinct effects: it determines the degree of relative risk aversion, $1 - \sigma$, and the magnitude of the surplus from trade of a given quantity and level of specialization. It follows that an increase in σ is associated with a lower degree of relative risk aversion. Since specialization increases the risk of not trading, lower risk aversion induces agents to specialize more. At the same time, a higher σ lowers the surplus from trade because utility is lower for all Q (since $Q \leq 1$). Thus a smaller surplus from trade induces agents to specialize more to lower the costs of production and increase the surplus from trade.

For a given x , if agents are sufficiently risk averse (σ small) they will choose to fully diversify their production abilities, $Y_b = 1 - x$. This implies that trade takes place in every match but the quantities exchanged are small due to high marginal production costs. If risk aversion is low, agents specialize completely, $Y_b = 0$. Trade is less frequent but involves larger quantities. For moderate degrees of risk aversion some specialization occurs but it is not complete, $Y_b \in (0, 1 - x)$. Interestingly, if individuals are very particular about the goods they desire (x small), complete specialization will almost never occur. Finally, substituting Y_b in $Q(Y)$ we obtain the equilibrium quantity as a function of the parameters of the economy.

However, the private choice of specialization is socially inefficient for two reasons: (i) agents are unable to coordinate their actions and (ii) the existence of a holdup problem. In particular, the economy is *under-specialized* in equilibrium, relative to the socially desirable outcome. This is obvious by comparing \hat{Y}_b^* to \hat{Y}_b , and it is proved below.

Lemma 4. *Let $u(q) = q^\sigma$ and $c(q, y) = q(1 + y)$. The symmetric barter equilibrium is inefficient, and such that $Y_b \geq Y_b^*$.*

The intuition is simple. A social planner would not want to choose Y to distort the terms of trade. However, the individual wants to do so because by under-specializing she can improve the terms of trade she faces in every match to the detriment of her future trade partners. This is so because, as shown in Lemma 2, $u'(Q)Q_y - c_q(q, y)q_y > 0$. Note that there is still the trade externality discussed before, that tends to push Y_b below the efficient level. Although these distortions have opposing effects on the individual choice of specialization, the first dominates and leads to the inefficiency result.

4.2. Monetary Equilibria

We now let $M \in (0, 1)$ to study monetary equilibria and the effects of monetary trades on the equilibrium choice of specialization, welfare and compare it to the planner's outcome. Our main results are contained in the following Lemma and proposition:

Lemma 5. *Let $u(q) = q^\sigma$ and $c(q, y) = q(1 + y)$. A symmetric monetary equilibrium always exists if $M \in (0, 1)$ such that:*

$$Y_m = \begin{cases} 0 \text{ or } \hat{Y}_m & \text{if } \sigma < \sqrt{\frac{1}{1+x}} \\ 0 & \text{if } \sigma \geq \sqrt{\frac{1}{1+x}} \end{cases}$$

where $\hat{Y}_m \in (0, \hat{Y}_b)$ approaches \hat{Y}_b as $M \rightarrow 0, 1$. If $\sigma < \sqrt{\frac{1}{1+x}}$ and M is sufficiently small, then $Y_m = \hat{Y}_m$.

Proposition 1. *Let $u(q) = q^\sigma$ and $c(q, y) = q(1 + y)$. The barter economy cannot be more specialized than the monetary economy, i.e. $Y_m \leq Y_b$.*

Relative to the case of a barter economy, it is now more difficult to determine conditions under which an interior solution exists, or is unique. When the money supply is small, however, it is straightforward to prove that a unique monetary equilibrium always exists. The proof exploits Lemma 3 and makes use of the fact that $W(y, Y)$ is continuous in M . Moreover, if the equilibrium is interior, the degree of specialization falls in x as it does in the barter case.

The fundamental result is that when *some* diversification takes place in equilibrium, the degree of specialization in a monetary economy is always strictly greater than that of a barter economy, i.e. $Y_m < Y_b$. The main implication is that agents always specialize to some extent in any monetary equilibrium since $Y_m < 1 - x$. This makes sense because there are no trading frictions in a fully diversified economy. Hence money cannot be valued as a medium of exchange. Second, when liquidity is scarce, adding more liquidity improves the extent of specialization. Money encourages specialization, relative to a barter economy, because increased diversification reduces the probability of single coincidence matches and thus the usefulness of money. Consequently, diversification has an additional marginal cost in the monetary economy relative to a barter economy.

The incentive to specialize more in the monetary economy is the result of changes in the underlying trading uncertainty faced by individual agents. In the absence of money, by diversifying production an individual can self-insure against the intrinsic trading risk stemming from the imperfect matching technology. However, the ‘premium’ for this self-insurance is a higher marginal cost of production. The introduction of money alters the trading environment, which in turn affects the agent’s decision to self-insure on the margin. By allowing trade in single coincidence matches, money expands the set of trading opportunities thereby reducing individual trading risk. In short, money serves as a form of social trading insurance. The provision of this social trading insurance reduces the agent’s incentive to self-insure against trading risk, which on the margin takes the form of increased specialization in production. Thus, the introduction of money creates a classic moral hazard problem – when provided with insurance, agents reduce the extent to which they self-insure against the bad outcome.

With the introduction of money, as in the barter case, agents will tend to under-specialize relative to the social optimum.

Proposition 2. *Let $u(q) = q^\sigma$ and $c(q, y) = q(1 + y)$. The symmetric monetary equilibrium is generally inefficient, and such that $Y_m \geq Y_m^*$ if M is close to 0 or 1. Furthermore, for M close to 0 and 1, welfare is higher in the monetary economy than in the barter economy.*

Due to the analytical complexity of the model, the results contained in the proposition are proved for a subset of the possible initial money supplies, namely those that render the monetary economy sufficiently ‘similar’ to a barter economy. Alternatively, they can be proved for any parameterization in which the quantities produced in equilibrium are sufficiently similar across matches. The intuition developed in Proposition 2, however, carries over to more general parameterizations. This is illustrated in the following section, by means of numerical simulations. In it, we also illustrate how changes in the money stock affect the equilibrium choice of specialization, traded quantities and welfare.

4.3 A Numerical Illustration.

We provide an illustration of the workings of the model using Figures 2-4. We have parameterized preferences by setting $\sigma = 0.82$ and $x = 0.1$, so that the choice of specialization is interior in both the barter and the monetary economy. In doing so, we have also chosen r such that $Q_m(Y_m^*) = Q(Y_m^*)$ when $M = 0$. This allows us to see how the trading externality and the hold-up problem affect the ratio $Q_m(Y_m)/Q(Y_m)$ relative to the planner’s welfare maximizing ratio $Q_m(Y_m^*)/Q(Y_m^*) = 1$, when $M = 0$. Given our other parameter values, this led us to set $r = .0457$.

In Figure 2 we plot the equilibrium production sets for the planner, (Y_b^*, Y_m^*) , and for the agents, (Y_b, Y_m) . The equilibrium value of Y is always above the planner’s choice. The basic result of the paper, introducing money leads to an increase in specialization, is also evident for both the individual and the planner’s choices. Furthermore, as the economy becomes increasingly liquid, the equilibrium choice of specialization decreases and converges to the barter equilibrium choice as $M \rightarrow 1$.

Figure 3 shows how quantities traded in money and barter matches change as the initial liquidity in the economy changes. With exogenous production specialization, the quantity traded in barter matches will be the same regardless of the amount of liquidity in the economy and the quantity traded in money matches declines as the money stock increases. In contrast, we find that introducing money causes both Q and Q_m to rise due to the productivity gains from specialization. As $M \rightarrow 1$

Q falls back to the level in the barter equilibrium and Q_m eventually goes to zero as money's value as a medium of exchange falls. Despite the inefficiencies associated with the choice of specialization, we find the same patterns for different levels of liquidity when the choice of specialization is made by the social planner. Due to the hold-up problem, however, we find that the equilibrium quantities in both types of matches are much lower than in the planned economy. This occurs because individuals under-specialize relative to the planner's choice and therefore the level of productivity in equilibrium is too low.

Figure 4 shows the behavior of welfare. Despite the inefficiencies linked to the matching externality and the hold-up problem, we do find that introducing money raises welfare. This occurs in both the uncoordinated and planned economies. We also observe that welfare is higher when Y is selected by the planner rather than privately. Finally, these results confirm the typical finding of search-theoretic models that money promotes welfare even when specialization is endogenous.

5. Concluding Comments

By endogenizing the choice of production specialization we have found several interesting results. First, the typical assumption of complete production specialization used in most monetary search models may not be individually or socially optimal. Second, the individual choice of specialization is generally inefficient because of the existence of a trading externality and a pricing distortion arising from a hold-up problem. Finally, introducing money increases specialization and welfare.

We have demonstrated how the implementation of a simple trading institution (money) alters the incentives to specialize in production and consequently the level of output and terms of trade. Since an inability to coordinate exists in this economy, a natural question is how to amend it. Private agents may resolve the problem by developing alternative trading arrangements such as middlemen, stores, or production cooperatives. As these trading institutions develop, trading frictions fall which, according to our model, will entice agents to specialize further since they do not have to worry as much about trading. Consequently, society would further reap the fruits arising from the division of labor, as Adam Smith envisioned long ago.

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Appendix

Proof of Lemma 1. Equations (8)-(9) imply $V_m(y, Y) - V_0(y, Y) = u(Q_m) \mu(y, Y)$, where

$$\mu(y, Y) = \frac{(1-M)p(Y)[1-p(y)]}{r + (1-M)p(Y)[1-p(y)]} < 1.$$

Suppose $y = Y$ is a symmetric monetary equilibrium. Using (7), the equilibrium quantity, $Q_m(Y)$, must satisfy $u(Q_m) \mu(Y) - c(Q_m, Y) = 0$. It follows that (i) Q_m is unique since, given Y , $\frac{u(Q_m)}{c(Q_m, Y)} \in [0, \infty)$ monotonically decreasing in Q_m (intermediate value theorem); (ii) $u(Q_m) > c(Q_m, Y)$ since $\mu(Y) \in [0, 1)$; (iii) if $Y \geq 1-x$ or if $M = 1$ then $Q_m = 0$, since $\lim_{Y \rightarrow 1-x} \mu(Y) = \lim_{M \rightarrow 1} \mu(Y) = 0$ and $c(0, Y) = 0$.

The function $\mu(Y)$ falls in M , it is strictly concave in Y , on $Y \in [0, 1-x)$, with maximum at $Y = \max\{0, \frac{1}{2} - x\}$. When we let $c(Q, Y) = Q(1+Y)$, then $u(Q)/Q$ falls in Q since $u'(Q) - u(Q)/Q < 0$ (mean value theorem). Thus, Q_m must satisfy $\frac{u(Q_m)}{Q_m} - \frac{1+Y}{\mu(Y)} = 0$ for all equilibrium Y . It is easy to show that $\frac{1+Y}{\mu(Y)}$ is strictly convex in Y . It follows that $Q_m(Y)$ must be strictly concave in Y with maximum at $Y = \min\{0, \frac{1}{2} - x\}$. In particular, taking the total differential of $u(Q_m) \mu(Y) - c(Q_m, Y) = 0$, $\frac{dQ_m}{dY} = \frac{\mu'(Y)u(Q_m) - Q_m}{(1+Y)(1-\sigma)}$. ■

Proof of Lemma 2. Let $u(q) = q^\sigma$, $\sigma \in (0, 1)$, and $c(q, y) = q(1+y)$. An interior solution $\{Q(y, Y), q(y, Y)\}$ must satisfy the two first order conditions (sufficient and necessary since the product of surpluses is concave in q and Q) given by (3). Divide the first by the second equation and observe that $\{Q(y, Y), q(y, Y)\}$ must solve

$$u'(Q)u'(q) = c_q(q, y)c_Q(Q, Y) \tag{18}$$

together with either one of the equalities in (3) generates (4). We omit the arguments, when no confusion arises. Differentiating with respect to y :

$$\begin{aligned} q_y(y, Y) &= \frac{-q}{(1-\sigma^2)(1+y)} < 0 \\ Q_y(y, Y) &= q_y(y, Y) \cdot \sigma \left(\frac{1+y}{1+Y} \right)^{\frac{1}{1+\sigma}} < 0. \end{aligned} \tag{19}$$

Voluntary participation in bargaining requires: $u(Q) - c(q, y) \geq 0$ and $u(q) - c(Q, Y) \geq 0$. Using (4), it is easily seen that $u(Q) \geq c(q, y)$ and $u(q) \geq c(Q, Y)$ always hold (the condition being

$\sigma \leq 1$). From (4) it follows that in equilibrium:

$$\begin{aligned} y < Y &\Rightarrow q(y, Y) > Q(y, Y) > Q(Y) \\ y = Y &\Rightarrow Q(Y) = q(y, Y) = Q(y, Y) \\ y > Y &\Rightarrow Q(Y) > Q(y, Y) > q(y, Y) \end{aligned}$$

Case $M = 0, 1$: barter equilibrium. Let $y \in [0, 1 - x]$. Since $p(Y) > 0$, the sign of $W_y(y, Y) = p(Y)H(y, Y)$ depends solely on the sign of $H(y, Y)$, where:

$$H(y, Y) \equiv u(Q(y, Y)) - c(q(y, Y), y) - p(y)q(y, Y) + p(y) [u'(Q)Q_y - c_q(q, y)q_y] \quad (20)$$

Define $T(y, Y) \equiv u'(Q)Q_y - c_q(q, y)q_y$. Using (4)-(19):

$$\begin{aligned} T(y, Y) &= \frac{1}{1 + \sigma} \left[\frac{\sigma^{1+\sigma}}{(1+y)(1+Y)^\sigma} \right]^{\frac{1}{1-\sigma^2}} = \frac{q(y, Y)}{1 + \sigma} \\ T_y(y, Y) &= \frac{-T}{(1 - \sigma^2)(1 + y)} = \frac{q_y(y, Y)}{1 + \sigma} \end{aligned}$$

and therefore

$$W_y(y, Y) = p(Y)H(y, Y) = p(Y)T(y, Y) [(1 + y)(1 - \sigma^2) - \sigma^2(x + y)].$$

It is obvious that it is the sign of $(1 + y)(1 - \sigma^2) - \sigma^2(x + y)$, a linear function in y , that determines the sign of $W_y(y, Y)$. Note that $(1 + y)(1 - \sigma^2) - \sigma^2(x + y) \leq 0$ iff $\sigma \geq h(y) \equiv \sqrt{\frac{1+y}{1+x+2y}}$, where $h(y') < h(y)$ for $y' > y$.

Now let $y \in (1 - x, 1]$. It is easy to show, that $W_y(y, Y) < 0 \forall y$. This is so since $p'(y) = 0$ and $-p(y) [q(y, Y) - T(y, Y)] < 0$. Hence, the individual would never differentiate her production beyond $1 - x$.

Thus, from $M = 0, 1$ we will let $W_y(y, Y)$ be defined on $(y, Y) \in [0, 1 - x]^2$ with no loss in generality. It follows that, given Y :

$$\sigma \begin{cases} \leq h(1 - x) & \Rightarrow W_y(1 - x, Y) \geq 0 \Rightarrow W_y(y, Y) > 0 \forall y \\ \in (h(1 - x), h(0)) & \Rightarrow \exists \text{ a unique } \hat{y} \in (0, 1 - x) \text{ s.t. } W_y(y, Y) \begin{cases} > 0 \text{ if } y < \hat{y} \\ = 0 \text{ if } y = \hat{y} \\ < 0 \text{ if } y > \hat{y} \end{cases} \\ \geq h(0) & \Rightarrow W_y(0, Y) \leq 0 \Rightarrow W_y(y, Y) < 0 \forall y \end{cases} \quad (21)$$

We note that \hat{y} is a global maximum of $W(y, Y)$ since it is easily proved that $H(\hat{y}, Y) = 0 \Rightarrow H_y(y, Y) < 0 \forall y \leq \hat{y}$ because $H_y(y, Y) = \frac{T(y, Y)\sigma^2[-2(1+y)(1-\sigma^2)+x+y]}{(1-\sigma^2)(1+y)}$. Furthermore, one can show that $H_y(y, Y) < 0 \forall y$ when $\sigma \leq h(1-x)$.

Case $M \in (0, 1)$: monetary equilibrium. We must only consider $Y \in [0, 1-x]$, so that $Q_m > 0$. For $y \in [0, 1-x]$ equation (13) is:

$$W_y(y, Y) = p(Y) \left\{ H(y, Y) - M(1-M) \left[u(Q_m) - c(q_m, y) + (1-p(y)) \left((1+y) \frac{dq_m}{dy} + q_m \right) \right] \right\} \quad (22)$$

since in equilibrium $\frac{\partial[V_m(y, Y) - V_0(y, Y)]}{\partial y} \equiv \frac{\partial c(q_m(y, Y), y)}{\partial y} = (1+y) \frac{dq_m}{dy} + q_m$. Given Q_m and Y , totally differentiate (10) to obtain $\frac{dq_m}{dy} = \frac{u(Q_m)\mu_y(y, Y) - q_m}{1+y}$ where

$$\mu_y(y, Y) \equiv \frac{\partial \mu(y, Y)}{\partial y} = \frac{-r\mu(y, Y)}{[1-p(y)][r + (1-M)p(Y)(1-p(y))]}$$

Substituting $c(q_m, y) = u(Q_m)\mu(y, Y)$ and $\frac{dq_m}{dy}$ in (22) we obtain:

$$W_y(y, Y) = p(Y) [H(y, Y) - (1-M)MG(y, Y)] \quad (23)$$

where $G(y, Y) \equiv \frac{r[1-\mu(y, Y)]u(Q_m)}{r+(1-M)p(Y)(1-p(y))} > 0$. It is easily seen that $\frac{\partial G(y, Y)}{\partial y} \equiv G_y(y, Y) > 0 \forall y, Y$.

Now let $y \in (1-x, 1]$. It is easy to show that $W_y(y, Y) < 0 \forall y$ since (i) $H(y, Y) < 0$ and (ii) $G(y, Y) = u(Q_m)$ due to $\mu(y, Y) = 0$ for $y \geq 1-x$. Hence, the individual would never differentiate her production beyond $1-x$. Thus, even when $M > 0$ we can consider $W_y(y, Y)$ for $y \in [0, 1-x]$, with no loss in generality.

It is immediate that $W_y(y, Y)$ differs from the case of a barter economy only due to the negative term $-(1-M)MG(y, Y)$. It follows that

1. If $\sigma \geq h(0)$ then $H(y, Y) \leq 0 \forall y$. Since $G(y, Y) > 0$, then $W_y(y, Y) < 0 \forall y$.
2. If $h(1-x) < \sigma < h(0)$ then $H(\hat{y}, Y) = 0$ for a unique $\hat{y} \in (0, 1-x)$. Since $G_y(y, Y) > 0$, and $H_y(y, Y) < 0$ it follows that (i) if $W_y(0, Y) \leq 0$ then $W_y(y, Y) < 0 \forall y$, and (ii) if $W_y(0, Y) > 0$ then \exists a unique $\tilde{y} \in (0, \hat{y})$ such that $W_y(\tilde{y}, Y) = 0$ with $W_{yy}(\tilde{y}, Y) < 0$. Note that $\tilde{y} < \hat{y}$ since $G(y, Y) > 0$.
3. If $\sigma \leq h(1-x)$ then $H(y, Y) \geq 0 \forall y$. Since $G_y(y, Y) > 0$, and $H_y(y, Y) < 0$ it follows that (i) if $W_y(0, Y) \leq 0$ then $W_y(y, Y) < 0 \forall y$, (ii) if $W_y(0, Y) > 0$ and $W_y(1-x, Y) < 0$ then \exists a

unique $\tilde{y} \in (0, 1-x)$ such that $W_y(\tilde{y}, Y) = 0$, with $W_{yy}(\tilde{y}, Y) < 0$, (note that $\tilde{y} < y_b = 1-x$) and (iii) if $W_y(1-x, Y) \geq 0$ then $W_y(y, Y) > 0 \forall y$.

Consequently, for a barter equilibrium when $M = 0, 1$, and for a monetary equilibrium when $M \in (0, 1)$, we will let $W_y(y, Y)$ be defined on $(y, Y) \in [0, 1-x]^2$. ■

Proof of Lemma 3. Let $u(q) = q^\sigma$, $c(q, y) = q(1+y)$, $M \in \{0, 1\}$ and $y = Y$. Since $p(Y) > 0$, the sign of $W_y(y, Y) = p(Y)H(y, Y)$ depends solely on the sign of $H(y, Y)$, where:

$$H(y, Y) \equiv u(Q(y, Y)) - c(q(y, Y), y) - p(y)q(y, Y) + p(y) [u'(Q)Q_y - c_q(q, y)q_y]$$

Using (14) and $H(y, Y)$:

$$Y_b = \begin{cases} 1-x & \Leftrightarrow H(1-x) \geq 0 \Leftrightarrow \sigma \leq h(1-x) \\ \hat{Y}_b & \Leftrightarrow H(\hat{y}_b) = 0 \Leftrightarrow h(1-x) < \sigma < h(0) \\ 0 & \Leftrightarrow H(0) \leq 0 \Leftrightarrow \sigma \geq h(0) \end{cases}$$

is the unique barter equilibrium in the presence of the hold-up problem. ■

Proof of Lemma 4. Let $u(q) = q^\sigma$, $c(q, y) = q(1+y)$, $M \in \{0, 1\}$. Suppose $\frac{\partial W_s(Y)}{\partial Y} < 0 \forall Y$. Then $Y_b^* = 0$ hence, $Y_b^* \leq Y_b$. Next, suppose $\frac{\partial W_s(Y)}{\partial Y} = 0$ for a unique $Y = \hat{Y}_b^*$. Then $Y_b^* = \hat{Y}_b^*$, such that $u(Q(\hat{Y}_b^*)) - c(Q(\hat{Y}_b^*), \hat{Y}_b^*) = \frac{p(\hat{Y}_b^*)c_Y(Q(\hat{Y}_b^*), \hat{Y}_b^*)}{2}$ (using (15) when $M = 0$). Define $T(y, Y) \equiv u'(Q)Q_y - c_q(q, y)q_y$, which, using (4)-(19) is:

$$T(y, Y) = \frac{1}{1+\sigma} \left[\frac{\sigma^{1+\sigma}}{(1+y)(1+Y)^\sigma} \right]^{\frac{1}{1-\sigma^2}} = \frac{q(y, Y)}{1+\sigma}$$

Using $H(y, Y)$ and $T(y, Y)$ it follows that:

$$W_y(y, Y)|_{y=Y=\hat{Y}_b^*} = p(\hat{Y}_b^*)Q(\hat{Y}_b^*) \left[-\frac{1}{2} + \frac{1}{1+\sigma} \right] > 0$$

and $W_y(y, Y)|_{y=Y} > 0 \forall Y < \hat{Y}_b^*$. Hence, from Lemma 2, $Y_b > Y_b^*$. Finally if $\frac{\partial W_s(Y)}{\partial Y} > 0 \forall Y$ then $Y_b^* = 1-x$ and $u(Q) - c(Q, Y) > \frac{p(Y)c_Y(Q, Y)}{2} \forall Y$ (from (16)). Hence, $W_y(y, Y)|_{y=Y} > 0 \forall Y$ and $Y_b = Y_b^* = 1-x$. ■

Proof of Lemma 5. Consider a symmetric monetary equilibrium $y = Y = Y_m$. Let $u(q) = q^\sigma$, $c(q, y) = q(1 + y)$, $M \in (0, 1)$ and $Y \in [0, 1 - x)$. Use (23) and elements of the proof of Lemma 2 to consider three different cases.

Case $\sigma \geq h(0)$. Since $H(Y) < 0 \forall Y$ then $W_y(Y) < 0 \forall Y$. Thus (14) implies existence of a unique symmetric equilibrium $Y_m = 0$.

Case $h(1 - x) < \sigma < h(0)$. Since $H(Y) > 0$ for $Y < \hat{Y}_b$, $H(\hat{Y}_b) = 0$, and $H(Y) < 0$ for $Y > \hat{Y}_b$, then $W_y(Y) < 0$ for $Y \geq \hat{Y}_b$, and so there are only two cases to consider. Either (i) $W_y(Y) < 0 \forall Y < \hat{Y}_b$ [i.e. $0 \leq H(Y) < M(1 - M)G(Y)$], or (ii) $W_y(Y) = 0$ for some $Y < \hat{Y}_b$ [i.e. $0 < H(Y) = M(1 - M)G(Y)$]:

- In case (i) $W_y(Y) < 0 \forall Y$, and once again there exists a unique symmetric equilibrium $Y_m = 0$.
- In case (ii) there can be an interior equilibrium, defined as any element of the set:

$$\hat{Y}_m \equiv \left\{ Y \in [0, \hat{Y}_b) \mid H(Y) = M(1 - M)G(Y) \right\}.$$

Note that $Y < \hat{Y}_b$ because $0 = H(\hat{Y}_b) < M(1 - M)G(\hat{Y}_b)$. The intermediate value theorem tells us that a *sufficient* condition for \hat{Y}_m to be non-empty is $H(0) > M(1 - M)G(0)$, i.e. $W_y(0) > 0$. A condition sufficient for \hat{Y}_m to be a singleton is $\frac{\partial W_y(Y)}{\partial Y} \leq 0 \forall Y \in \hat{Y}_m$ (i.e. $W_y(Y_m)$ is locally concave whenever it crosses zero, so $W(Y)$ has a unique maximum). Note that $\frac{\partial H(Y)}{\partial Y} < 0$ when $h(1 - x) < \sigma < h(0)$. Thus, using continuity arguments, $M > 0$ small is sufficient for existence of a unique \hat{Y}_m . It is easy to verify that $\frac{\partial \hat{Y}_m}{\partial x} < 0$. Furthermore, as $M \rightarrow 0$ then $\hat{Y}_m \rightarrow \hat{Y}_b^-$, thus by continuity $\frac{\partial \hat{Y}_m}{\partial M} < 0$ if M is sufficiently small.

Case $\sigma \leq h(1 - x)$. Since $H(Y) \geq 0 \forall Y$, then either (i) $W_y(Y) < 0 \forall Y < 1 - x$ [i.e. $0 < H(Y) < M(1 - M)G(Y)$], or (ii) $W_y(Y) = 0$ for some $Y < 1 - x$ [i.e. $0 < H(Y) = M(1 - M)G(Y)$]:

- In case (i) $W_y(Y) < 0 \forall Y$, and once again there exists a unique symmetric equilibrium $Y_m = 0$.
- In case (ii) there can be an interior equilibrium $Y_m = \hat{Y}_m$. Following the procedure outlined above, if $M > 0$ small then \exists a unique $\hat{Y}_m \in (0, 1 - x)$, such that $\frac{\partial \hat{Y}_m}{\partial x} < 0$, $\lim_{M \rightarrow 0} \hat{Y}_m = Y_b = 1 - x$, and $\frac{\partial \hat{Y}_m}{\partial M} < 0$ for M small.

Thus: (i) If $\sigma \geq h(0) = \sqrt{\frac{1}{1+x}}$ then $Y_m = Y_b = 0$; (ii) If $\sigma < h(0)$ then either $Y_m = 0 < Y_b$ or $Y_m = \hat{Y}_m < Y_b$.

To prove that $Y_m \rightarrow Y_b$ as $M \rightarrow 1$ and that $\frac{\partial Y_m}{\partial M} > 0$ for M close to 1, note that $Y_m \rightarrow Y_b$ as $M \rightarrow 1$ since $Q_m \rightarrow 0$ (hence $G(Y) \rightarrow 0$). Since $Y_m \leq Y_b \forall M$, then $\frac{\partial Y_m}{\partial M} > 0$ for M close to 1.

To prove \hat{Y}_m is convex in M notice that if in equilibrium Y_m is interior then $W_y(Y)|_{Y=Y_m} = 0$ which, using the implicit function theorem and (23), implies:

$$\frac{\partial \hat{Y}_m}{\partial M} = - \frac{(1-2M)G(Y) + M(1-M)\frac{\partial G(Y)}{\partial M}}{\frac{\partial H(Y)}{\partial Y} - M(1-M)\frac{\partial G(Y)}{\partial Y}} \Bigg|_{Y=\hat{Y}_m}$$

The numerator vanishes at most at one point since (i) $\frac{\partial G(Y)}{\partial M} < 0$, because

$$G(Y) = \left[\frac{r}{r + (1-M)p(Y)(1-p(Y))} \right]^2 \left(\frac{\mu(Y)}{1+y} \right)^{\frac{\sigma}{1-\sigma}}$$

and $\frac{\partial \mu(Y)}{\partial M} < 0$ (from Lemma 1), and (ii) $(1-2M)G(Y) < 0$ for all $M < \frac{1}{2}$. The numerator is always negative since \hat{Y}_m must be a maximum, hence $H(Y) - M(1-M)G(Y)$ must be concave around \hat{Y}_m . It follows that $\frac{\partial \hat{Y}_m}{\partial M} = 0$ at one point $\hat{M} \in (0, 1/2)$, $\frac{\partial \hat{Y}_m}{\partial M} > 0$ whenever $M > \hat{M}$, and $\frac{\partial \hat{Y}_m}{\partial M} < 0$ for $M < \hat{M}$. ■

Proof of Proposition 1. Let $u(q) = q^\sigma$, $c(q, y) = q(1+y)$. The proof follows from the proof of Lemma 5. Note that (i) if $Y_b = 0$ is an equilibrium then $W_y(Y) < 0 \forall Y$, so $Y_m = Y_b = 0$, (ii) if $Y_b = 1-x$ is an equilibrium then $Y_m < Y_b$ (by Lemma 1), and (iii) if $Y_b = \hat{Y}_b$ is an equilibrium, then either $Y_m = 0$ or $Y_m = \hat{Y}_m \in [0, \hat{Y}_b)$ where $W_y(\hat{Y}_m) = 0$ and $\hat{Y}_m < \hat{Y}_b$. ■

Proof of Proposition 2. Let $u(q) = q^\sigma$, $c(q, y) = q(1+y)$. To show that Y_m is generally inefficient we exploit Lemma 4, in which we have proved that $Y_b \geq Y_b^*$. Recall that $Y_m \rightarrow Y_b$ and $Y_m^* \rightarrow Y_b^*$ as $M \rightarrow 0^+$ and as $M \rightarrow 1^-$. Since (15) is continuous in M , it follows that $Y_m \geq Y_m^*$ for M close to zero and one.

To show that $\frac{\partial W(\hat{Y}_m)}{\partial M} > 0$ for M close to 0, note from the prior argument that $Y_m \geq Y_m^*$ for M around 0 and 1. Note that:

$$\begin{aligned} W(Y_m) &= rV_0(Y_m) + rM[V_m(Y_m) - V_0(Y_m)] \\ &= W_s(Y)|_{M=0, Y=Y_m} + rMc(Q_m(Y_m), Y_m) \end{aligned}$$

since $rV_0(Y_m) = W_s(Y)|_{M=0,1}$. We know that $Y_m = Y_b$ for $M = 0, 1$, and $Y_m \leq Y_b \forall M$, from Proposition 1. Consider the case where $Y_b = \hat{Y}_b$, and $Y_m = \hat{Y}_m$ for $M > 0$ small. In this case

$\frac{\partial Y_m}{\partial M} < 0$ for M small and $\frac{\partial Y_m}{\partial M} > 0$ for M large. From Lemma 4 note that $\frac{\partial W_s(Y)}{\partial Y} \Big|_{Y=Y_m, M=0} < 0$, since $W_s(Y)$ is concave in Y and $\hat{Y}_m|_{M=0} = \hat{Y}_b > \hat{Y}_b^*$, because \hat{Y}_b^* maximizes $W_s(Y)$. Therefore:

$$\frac{\partial W(\hat{Y}_m)}{\partial M} = \frac{\partial W_s(\hat{Y}_m)}{\partial \hat{Y}_m} \Big|_{M=0} \cdot \frac{\partial \hat{Y}_m}{\partial M} + r \left[c(Q_m(\hat{Y}_m), \hat{Y}_m) + M \frac{\partial c(Q_m(\hat{Y}_m), \hat{Y}_m)}{\partial M} \right]$$

By continuity, it follows that $\frac{\partial W(\hat{Y}_m)}{\partial M} > 0$ for M around 0: since $W_s(Y_b) = W(Y_m)$ from $M = 0$ it follows that, welfare in the monetary economy cannot be below welfare in the barter economy, for M close to 0. It also follows that $\frac{\partial W(\hat{Y}_m)}{\partial M} < 0$ for M around 1 since $Q_m \rightarrow 0$, $\frac{\partial \hat{Y}_m}{\partial M} > 0$, and as $M \rightarrow 1$ then $\hat{Y}_m \rightarrow \hat{Y}_b > \hat{Y}_b^*$ so that $\frac{\partial \hat{Y}_m}{\partial M} \rightarrow 0$. Thus welfare in the monetary economy is above welfare in the barter economy, for M close to 1, since $W(\hat{Y}_m)|_{M=1} = W(\hat{Y}_b)|_{M=1}$. ■

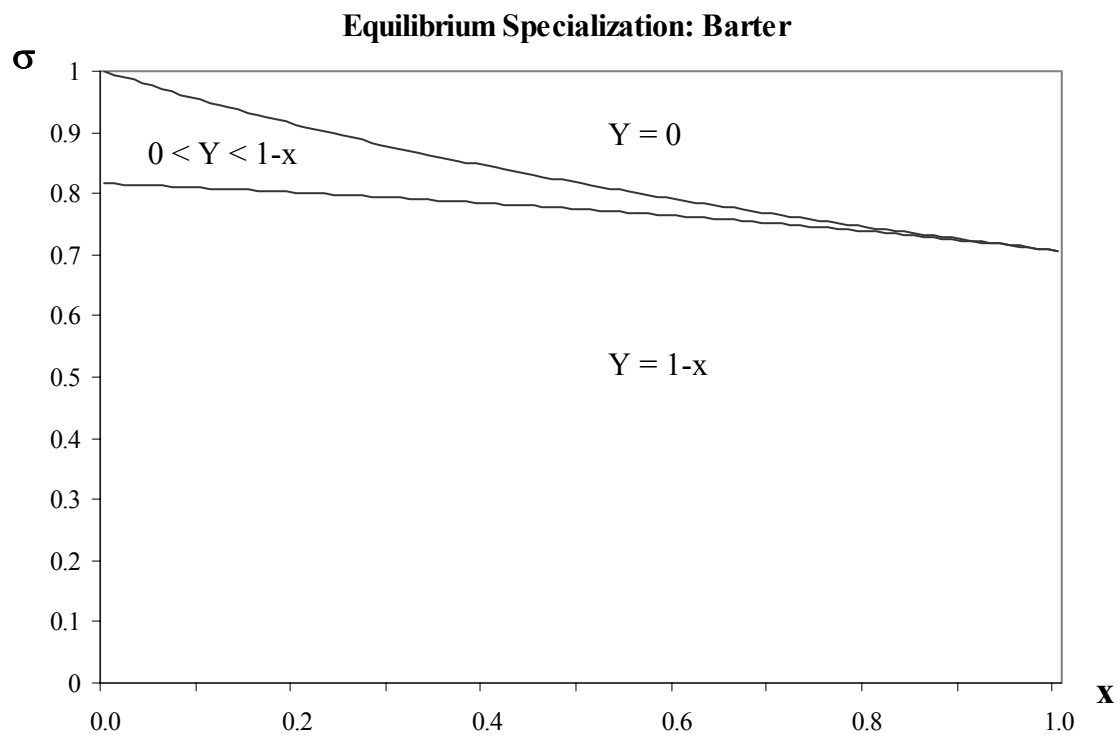


Figure 1

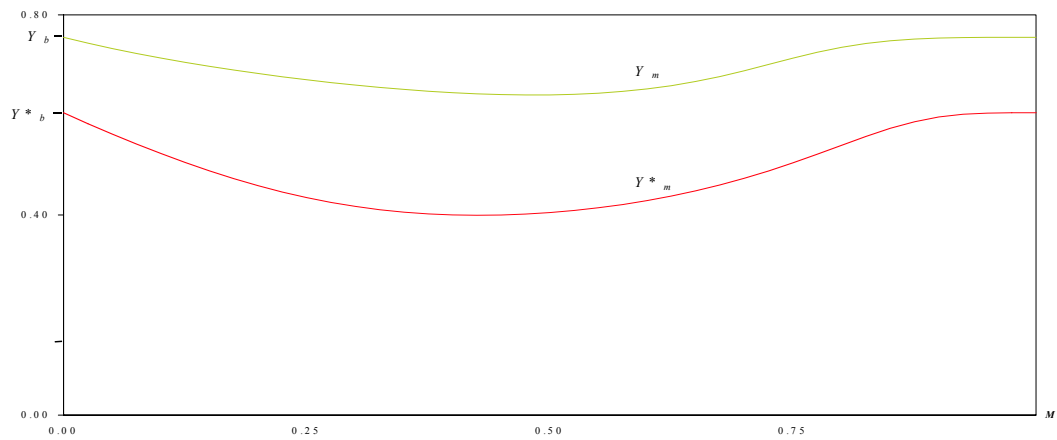


Figure 2

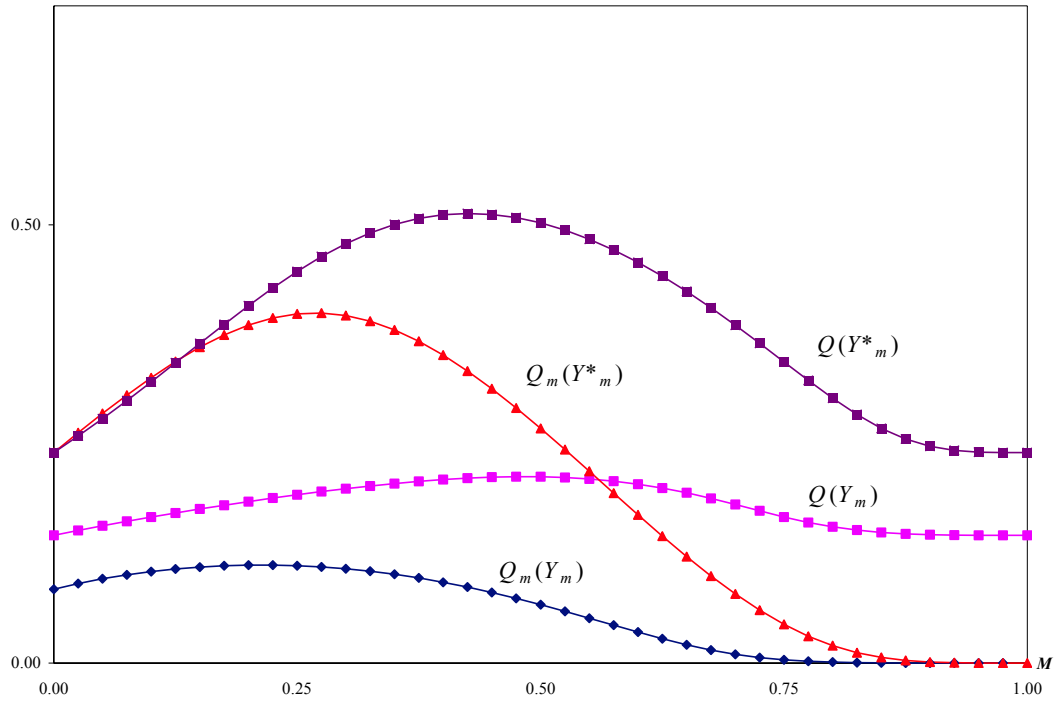


Figure 3

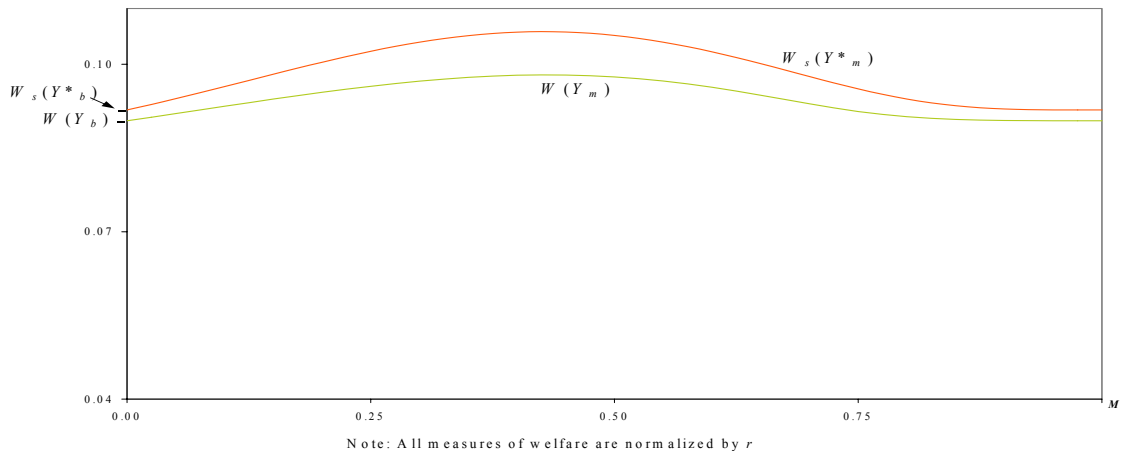


Figure 4