2015

All-Pay 2 X 2 Hex: A Multibattle Contest With Complementarities

Dan Kovenock
Chapman University, kovenock@chapman.edu

Sudipta Sarangi
Louisiana State University

Matt Wiser
University of South Alabama

Recommended Citation

Kovenock, D., Sarangi S., and Wiser, M. (2015). "All-Pay 2 x 2 Hex: A Multibattle Contest with Complementarities," *International Journal of Game Theory*, 44, pp. 571-597. DOI: 10.1007/s00182-014-0445-7

All-Pay 2 X 2 Hex: A Multibattle Contest With Complementarities

Comments
This is a pre-copy-editing, author-produced PDF of an article accepted for publication in *International Journal of Game Theory*, volume 44, in 2015 following peer review. The final publication is available at Springer at
DOI: 10.1007/s00182-014-0445-7

All-Pay 2×2 Hex: A Multibattle Contest With Complementarities*

Dan Kovenock †   Sudipta Sarangi ‡   Matt Wiser §

May 10, 2014

Abstract

We examine a modified 2×2 game of Hex in which the winner of each cell is determined by a Tullock contest. The player establishing a winning path of cells in the game wins a fixed prize. Examining the polar cases of all cells being contested simultaneously versus all four cells being contested sequentially, we show that there is an increase in the total expected payoff for the players in the sequential case. We identify conditions under which players have identical and non-identical expected payoffs when the contest order is pre-specified. We also examine dissipation for random order contests. We thus provide a canonical model of a multibattle contest in which complementarities between battlefields are heterogeneous across both battlefields and players.

Keywords: Contests, All-Pay Auctions, Multibattle, Complementarity, Hex

JEL Classification: C72, C73, D72, D74

*We would like to thank Kalyan Chatterjee, Cary Deck, and Brian Roberson for their suggestions, as well as participants at the *Tournaments, Contests and Relative Performance Evaluation* (2011) conference at North Carolina State University and the *Public Economic Theory* (2011) conference at Indiana University for their comments. We would also like to thank two anonymous referees, an Associate Editor, and the Editor Bernhard von Stengel for useful comments. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

†Economics Science Institute, Chapman University. Email: kovenock@chapman.edu

‡Department of Economics, Louisiana State University; National Science Foundation, Arlington VA; and DIW Berlin. Email: sarangi@lsu.edu

§Department of Economics, Louisiana State University Email: mwiser1@lsu.edu

1 Introduction

We examine a competition comprised of multiple contests, combinations of which exhibit complementarity. In our game a benefit accrues to a player only by having won one of several winning combinations of contests. Players have different winning combinations, reflecting differing goals, but the combinations are such that there will always be exactly one winning player. The complementarity between contests arises because success in a single contest or set of contests may yield the same payoff as losing every contest but combined with one more contest win, may yield overall victory. These factors result in variations in the valuation of individual contests based on the identity and outcomes of contests already decided and the order of contests yet to be played.

The basic structure for this competition is given by the board game Hex.[1] In the canonical form, Hex is played by two players on a 11×11 grid of hexagonal cells. The players are conventionally labeled Black and White. Each player alternates claiming an unclaimed cell on the board. Black's objective is to connect the two black sides of the board with a path of his pieces, while White's is to connect the two white sides of the board with her pieces. As the game continues, the player who first is able to connect her two assigned sides is declared the winner. Draws are impossible, as the spaces are hexagons, i.e., we never see two pieces that join only at a corner.

Network security provides a good practical example of this type of structure, because data can be routed around compromised servers as long as a connection exists between two nodes, thus avoiding servers which have been hacked or damaged. Note that, intermediate relay nodes hold no intrinsic value since payoffs are entirely due to the final connection. Thus we do not need to worry about the value of individual cells; only the completed path. These networks are often geographically dependent, such as wireless relay towers and fibre optic switching stations. Another example of a similar network is a cellular phone system, where towers relay signals, and interference limits the number of towers in an area.

Related work can be found in the literature on Colonel Blotto games (Borel 1921, Borel and Ville 1938, Gross 1950, Gross and Wagner 1950, Friedman 1958).[2] These games feature two players who simultaneously allocate their respective fixed budgets of a resource across n different contests, with the player having a higher allocation in a given contest winning that contest. Players choose their allocations to maximize the expected number of battlefields won. In these early papers linkages between contests arise from the budget constraints; allocation of a unit of the resource

[1] Hex was independently invented by John Nash and the Danish mathematician Piet Hein in the 1940's.

[2] See Kovenock and Roberson (2012) for a survey of these and related games.

to one contest reduces the availability of the resource for other contests. Recently there has been a resurgence of interest in these games with extensions to the cases of asymmetric budgets and a positive opportunity cost of the resource in games with both continuous and discrete strategy spaces (Hart 2008, Kvasov 2007, Laslier 2002, Laslier and Picard 2002, Roberson 2006, Roberson and Kvasov 2012, and Weinstein 2012). Colonel Blotto games have also been examined under the assumption that the winner of each contest is determined probabilistically by the players' respective allocations according to a Tullock contest success function (Tullock, 1980), with the success function itself having been introduced previously (Tullock, 1975). Contributions employing the Tullock contest success function include Friedman (1958) and Robson (2005).

In addition to linkages between contests that arise through the cost of resource allocation, such as the individual budget constraints of the Colonel Blotto game, there are also linkages that arise through the way in which individual battlefield outcomes are aggregated in determining the players' payoffs. Szentes and Rosenthal (2003 a,b) examine a game in which players simultaneously allocate a resource at constant unit cost to n different battlefields. A player earns a prize of common and known value if he is the higher bidder in m of those battlefields. The special case where n is odd and $m = \frac{n+1}{2}$ is the game in which the player who wins a majority of the contests is the winner. Szentes and Rosenthal solved this game for $n = 3$, though for $n > 3$ this remains an open problem. The corresponding n battlefield majoritarian problem with a generalization of the Tullock contest success function with exponent $\alpha \leq 1$ was examined by Snyder (1989) who obtained some partial results. Klumpp and Polborn (2006) solved the n battlefield game for a Tullock contest success function with exponent $\alpha \leq 1$ more generally, characterizing the nature of the nondegenerate mixed strategy equilibria that arise when there are sufficiently many battlefields that no pure strategy equilibrium exists.

More complex linkages that arise from the way in which battlefield outcomes are aggregated have also been examined. Clark and Konrad (2007) examine a game with n battlefields, a constant unit cost of expenditure, and a Tullock contest success function with exponent $\alpha = 1$, in which one player must win all of the contests in order to win a prize, while the other only needs to win at most one contest. Kovenock and Roberson (2010) examine the corresponding game under the assumption that the high bidder in each contest wins the contest. Golman and Page (2009) examine a modified Colonel Blotto game, which they term "General Blotto," that takes the original budget-constrained Colonel Blotto game in which the high bidder wins each contest and adds compound contests formed by taking subsets of the sets of battlefields. In each of these added compound contests, a player's allocation is taken to be the product of the allocations in the battlefields defining the compound contest and the player with the higher such product wins. These contest wins are then added to those of the single battlefields to determine the number of contests won.

Our model is similar in spirit to the models with payoffs determined by a nonlinear aggregation of battlefield outcomes. Our examination of equilibrium under different assumptions governing the simultaneity and sequencing of contests aims to shed light on the impact of potential timing options for strategic behavior and payoffs. In our model players allocate resources at a constant and identical unit cost to four battlefields. In the main text we analyze the polar cases with all cells being contested simultaneously and sequentially, with the sequential case done both when both the order is known a priori and when it is random. Intermediate cases for sequential contests are also solved for in the Appendices. Each battlefield outcome is determined by a Tullock contest success function with exponent $\alpha = 1$ (the lottery contest success function). The battlefields are spatially distributed to correspond with cells in a 2×2 game of Hex and the overall contest winner is the player who wins a configuration of contests that would win the game of Hex. Consequently, multiple combinations of individual contest wins may earn a player the overall prize, but exactly one player will be the winner. Players are risk neutral so that they maximize the expected prize winnings minus the cost of their allocation to the four battlefields.

Like the literature on the Colonel Blotto game, the 2×2 game of All-pay Hex that we examine is meant as a type of "toy model" designed to shed some insight into the strategic considerations that might arise when rivals compete in multiple contests exhibiting linkages. In All-pay Hex these linkages arise because the "prizes" contested exhibit complementarities. Moreover, players have asymmetric preferences over combinations of contest victories.

The paper is laid out as follows. First, an example is described in Section 2, showing the general method of play. The model is then formally defined in Section 3, and results are obtained in Section 4. Finally, some robust conclusions about the game in general are presented in Section 5, with the mathematical details of the cases in the Appendices.

2 All Pay Hex: An Illustrative Example

The structure of Hex provides a useful starting point for examining complementarities. Given each player's goal of connecting the opposite sides, winning combinations will vary, and the value of a cell will depend on the use of this cell in creating a winning path. Since the canonical version of

Hex is computationally difficult, we will focus on a tractable 2 $\times$ 2 version with four cells being contested.

Our game differs from canonical Hex in that the players simultaneously place bids on a cell and the winner is probabilistically determined. Each player simultaneously commits a resource to the cells currently being contested. The winner of each cell j is determined by a Tullock contest success function, where $P(X_j, Y_j) = \frac{X_j}{X_j+Y_j}$ is the probability that player X wins, where X_j and Y_j are the amounts committed to cell j by players X and Y respectively. After a set of cells has been contested, players observe which player has won the game, thereby receiving a prize V. If neither player has won the game, the outcomes are observed and a new subset of cells is contested, repeating this process until a winner of the overall contest is found. These subsets of cells come from taking an ordered partition of the set of cells, with the subset being contested in round n being the elements in the nth subset of the partition.

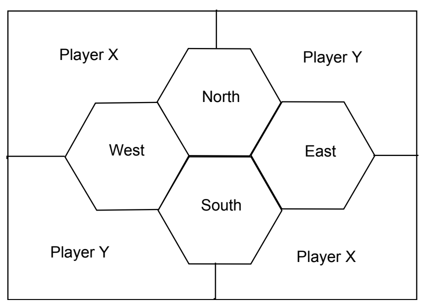

Figure 1: **The Cells Being Contested**

For the sake of illustration, let the prize for completing a connection or path be 100, with players simultaneously competing for all four cells shown in Figure 1. Suppose player X invests 20 in both the North and East cells, and only 1 in West and South, in an effort to have overwhelming force in 2 cells. Meanwhile, Player Y invests 10 in both the North and South, and 5 in the East and West.

We can now calculate the probabilities of X winning each cell. Based on the contest success function, in the North, she has a $\frac{2}{3}$ chance of victory, in the East $\frac{4}{5}$, in the West $\frac{1}{6}$, and in the South $\frac{1}{11}$. Thus player X has a chance of victory given by the sum of his probabilities of winning both the North and East $\frac{2}{3} \cdot \frac{4}{5}$, both the North and South but not the East $\frac{2}{3} \cdot \frac{1}{11} \cdot \frac{1}{5}$, and both the West and South but not the North $\frac{1}{6} \cdot \frac{1}{11} \cdot \frac{1}{3}$. This covers all winning sets for player X without double counting any sets, as all eight winning sets fall into exactly one of these three possibilities.

This gives a total chance of victory for player X of approximately 0.5505, i.e. his expected earnings are 55.05 at a cost of 42, for a net gain of 13.05. Player Y will have a 0.4495 probability of winning, and thus expected winnings of 44.95. However, player Y will have a net gain of 14.95, as she spent only 30.

3 The Model

We will now develop our formal model of the 2 × 2 case of All-Pay Hex. The game is played over the set of cells $A = \{N, S, E, W\}$.

Players: We denote the two risk neutral players in the game by X and Y. It is assumed that the players do not have a budget constraint.

Strategies: Since there are four cells, we will allow for the possibility that they can be contested at different points in time. Before the contest begins, the contest structure $\Re$ is announced.[3] Let $\Re = \{R_1, R_2, R_3, R_4\}$ be an ordered partition of A, where R_r is the set of cells being put up for contest in round r and let $|R_r| = C_r$. Since there are only four cells, we assume that the number of rounds cannot exceed this number.

In each round r, each player chooses a C_r-vector of bids with the bid for cell i by player $T \in \{X, Y\}$ being labeled T_i. We will require $T_i \geq 0$ and define $Z_i = \sum_{T=X,Y} T_i$.[4] The winner of each cell will be determined by the Tullock contest function, so as long as $Z_i > 0$ player T will win cell i with probability $\frac{T_i}{Z_i}$.

Payoffs: The payoff function of each player takes into account the expected benefits minus the costs. Each player obtains a identical benefit V from winning the game. We will require some additional notation for our calculations. Let $X^\star$ be the collection of winning sets of cells for player X, and $Y^\star$ for player Y. Specifically, from Figure 1 it can be seen that

$X^\star = \{\{N, E\}, \{N, S\}, \{S, W\}, \{N, S, E\},$ $\{N, S, W\}, \{N, E, W\}, \{S, E, W\}, \{N, S, E, W\}\}$	$Y^\star = \{\{N, W\}, \{N, S\}, \{S, E\}, \{N, S, E\},$ $\{N, S, W\}, \{N, E, W\}, \{S, E, W\}, \{N, S, E, W\}\}$

Note that each player has three minimal winning sets consisting of two cells, along with the supersets of these. The set $\{N, S\}$ is the common minimal winning set for either player. Thus the

[3]We also explore the implications of relaxing this assumption and having a random sequence later in the paper. We thank an anonymous referee for suggesting this.

[4]$T_i = 0$ will be limited to cases in which cell i has been made irrelevant. This is a technical condition imposed by the Tullock contest success function.

probability that player X wins the prize is

$$\sum_{\alpha \in X^\star} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{X_i}{Z_i} \frac{Y_j}{Z_j} \right) \tag{1}$$

Hence the payoff of player X can be written as

$$U_X \left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| \Re \right) = \sum_{\alpha \in X^\star} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{X_i}{Z_i} \frac{Y_j}{Z_j} \right) V - \sum_{i \in A} X_i \tag{2}$$

where the second term is the amount of money that X spends on bids. Note that the utility function for player X is a function of the amount invested in each cell, which will be influenced by the round structure $\Re$. Thus we take the amount invested by each player in the first round, the amount invested in the second round, and so on until all cells have been accounted for. Similarly, Player Y's payoff function is given by:

$$U_Y \left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| \Re \right) = \sum_{\alpha \in Y^\star} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{Y_i}{Z_i} \frac{X_j}{Z_j} \right) V - \sum_{i \in A} Y_i \tag{3}$$

4 Solving the Game

Before we analyze the game, we briefly discuss what happens when there are no complementarities between the cells. If there are no complementarities, the values of the cells are independent of one another. Then, if V_N is the common value to the players of winning the North cell, Player X's expected payoff for the contest occurring in the North will be $V_N \left(\frac{X_N}{Z_N} \right) - X_N$, and Player Y's expected payoff will be $V_N \left(\frac{Y_N}{Z_N} \right) - Y_N$. From the first order conditions we have $V_N \left(\frac{Y_N}{Z_N^2} \right) = V_N \left(\frac{X_N}{Z_N^2} \right) = 0$, which gives us $X_N = Y_N = \frac{V_N}{4}$. Corresponding calculations apply to the East, South, and West cells. Thus each player will spend a quarter of the value of each cell, and have an expected payoff of a quarter of the value of the cell.[5]

From the previous section it should be clear that a complete analysis of the problem requires examining several different cases to identify the effects of the complementarity involved. The role of complementarity also varies depending on whether the winning sets are contested separately or simultaneously. Hence we will focus on the two polar cases: all cells being contested simultaneously

[5]Note that this differs from the typical Colonel Blotto problem in the sense that there are no linkages across the contests, either through a critical value objective or a budget constraint that limits the resources to be allocated across the cells (see Kovenock and Roberson 2012).

and the four cells being contested sequentially. The intermediate cases, where cells are contested over 2 or 3 rounds, are discussed briefly, but full solutions are provided in Appendix B and summarized in Table 2. The main distinction between the simultaneous and sequential cases is that the sequencing of cells allows for the introduction of asymmetries that are not otherwise possible. Although there is only one case when all four cells are contested simultaneously, there are multiple distinct subcases for sequential contests, depending on the order of the cells. In the remainder of the paper we normalize V to 1.

4.1 Simultaneous Contests

To solve this game, we will simultaneously maximize the expected payoff of each player given by equations 2 and 3. We first introduce some additional notation. Let $\Theta = \frac{1}{Z_N Z_S Z_E Z_W}$. Let also $(X_E X_W) = \beta$, $(X_E Y_W) = \gamma$, $(Y_E X_W) = \delta$, and $(Y_E Y_W) = \zeta$. Player X has an expected utility function given by

$$U_X \left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| R_1 \right) = (X_N X_S)\Theta[\beta + \gamma + \delta + \zeta] + (X_N Y_S)\Theta[\beta + \gamma] + (Y_N X_S)\Theta[\beta + \delta] - (X_N + X_S + X_E + X_W) \quad (4)$$

where the first 8 terms come from the different winning combinations for player X. Using the fact that $\left(\frac{X_j}{Z_j}\right) = (1 - \left(\frac{Y_j}{Z_j}\right))$ we can combine terms and simplify to obtain

$$U_X(\cdot) = \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{X_E}{Z_E}\right) + \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_W}{Z_W}\right) - X_N - X_S - X_E - X_W \quad (5)$$

Similarly, player Y has an expected utility function given by

$$U_Y \left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| R_1 \right) = (Y_N Y_S)\Theta[\beta + \gamma + \delta + \zeta] + (Y_N X_S)\Theta[\gamma + \zeta] + (X_N Y_S)\Theta[\delta + \zeta] - (Y_N + Y_S + Y_E + Y_W) \quad (6)$$

where the first 8 terms come from the different winning combinations for player Y. Combining and simplifying as before gives us

$$U_Y(\cdot) = \left(\frac{Y_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right) + \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_W}{Z_W}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right) - Y_N - Y_S - Y_E - Y_W \quad (7)$$

We derive the unique simultaneous move pure strategy equilibrium in Appendix A. Following an argument similar to that employed by Klumpp and Polborn (2006), we show that since the probability that a cell is pivotal for player T (i.e., whether player T wins the contest is determined by the outcome in that cell) is the same for both players, each player will allocate the same amount to a given cell in equilibrium. That is, players play identical pure strategies in equilibrium. This significantly simplifies the derivation of equilibrium strategies. The remark below summarizes the main findings for the simultaneous case.

Remark 1: When all four cells are contested simultaneously, in the unique pure strategy Nash equilibrium both players will employ a symmetric strategy of spending $\frac{1}{8}$ on each of the North and South cells, and spending $\frac{1}{16}$ on each of the East and West cells, giving each player a probability of victory of $\frac{1}{2}$ and an expected payoff of $\frac{1}{8}$.

4.2 Sequential Contests with Pre-Specified Order

We now analyze sequential contests in which the order of play is known to both players before the contest begins. In order to solve for subgame perfect equilibrium strategies, we need to backward induct in the extensive form of the game, and obtain expected payoffs for all possible sequences in the final stage. These are then used in determining expected payoffs in the previous round, and so on to obtain the subgame perfect equilibrium. Given that the two players have different winning sets, the ordering of cells can bias the game in favor of one player, as winning sets can become available to the players at different times. For instance, in all cases where either North or South is the first cell contested, the game will still be symmetric between players X and Y. This is not necessarily true if East or West is the first cell contested. If East and West are the first two rounds contested, in either order, there will be no bias in favor of any player.

We now compare the two polar cases in terms of underdissipation. Underdissipation is the situation where the total aggregate expenditure of the players is less than the value of the prize being contested.[6]

Proposition 1. *All sequential structures have lower expected dissipation than the simultaneous case.*

Proof: See Appendix A.

[6]Similarly, overdissipation occurs when players spend more in the aggregate than the common value of the prize. The possibility of overdissipation in a game with Tullock contest success functions has been explored, for instance in (Baye et al., 1999).

The table below summarizes the payoffs of both players for the two polar cases. Detailed calculations of these results can be found in Appendix A.

Type	**Order**	$E[U_X]$	$E[U_Y]$
4	Simultaneous (NESW)	.125	.125
1-1-1-1	N or S as first round	.1797	.1797
1-1-1-1	E-W or W-E as first and second rounds	.1406	.1406
1-1-1-1	E-N or W-S as first and second rounds	.0731	.2315
1-1-1-1	W-N or E-S as first and second rounds	.2315	.0731

Table 1: Expected Payoffs under Simultaneous and Sequential Structures

In the table, "Type" refers to the number of cells contested in each round, and "Order" denotes which cells are contested in each round. In each case, the rounds are separated by dashes. The second row consists of those sequential structures where North or South is the first cell, and the third row has those with East and West as the first two rounds, with North and South in either order in the third and fourth rounds. The last two rows can be understood in a similar manner. $E[U_X]$ and $E[U_Y]$ denote the expected payoffs of players X and Y respectively.

Remark 2: In a four round contest, the ordering of the *two* cells to be contested in the third and fourth rounds is irrelevant. There are three possible cases after the first two rounds: (i) either one player has won, or (ii) only one of the remaining cells is relevant, in which case the other cell can be ignored, or (iii) one player must win both remaining cells. In the first two cases, since we can ignore irrelevant cells, the joint ordering for the last two cell does not matter. In the third case, since one player needs to win both remaining cells, the order in which the cells are contested is irrelevant. Thus, the ordering of cells in the third and fourth rounds does not matter in any situation.[7]

Intuitively, underdissipation occurs in the sequential case due to asymmetries between players in the number of winning subsets that have been covered by the contested cells at the end of each round. Using the knowledge of which cells have been won by which player, only a limited number of possibilities involving the remaining cells are feasible, allowing for better decisions. Sequencing creates these asymmetries, resulting in underdissipation. Our next two results identify conditions under which expected payoffs in the sequential case are asymmetric and symmetric respectively.

[7]Note that an intermediate case with only 3 rounds, such as E-W-NS, where East is the first round, West the second round and North and South are to be contested simultaneously in the third round, is not equivalent. This is shown in Appendix B, and is due to the possibility that both North and South are still relevant after the second round.

Proposition 2. *In the sequential case, asymmetric expected payoffs for players X and Y in equilibrium require that the cells in* $\{R_1, R_2\}$ *form an element of exactly one of* $X^\bigstar$ *or* $Y^\bigstar$*. If* $\{R_1, R_2\}$ *is an element of* $X^\bigstar$*, then* $E[U_X] \leq E[U_Y]$ *for the entire game, while if* $\{R_1, R_2\}$ *is an element of* $Y^\bigstar$*,* $E[U_Y] \leq E[U_X]$ *for the entire game.*

Proof: See Appendix A.

The proof of this proposition proceeds by contradiction and here we provide a quick sketch. If North and South constitute the first two rounds, they form a winning set for *both* players, and thus no asymmetry exists. Similarly, if East and West are the first two rounds, they form a winning set for neither player, and again, no asymmetry exists. In the remaining cases, one of either North or South, and one of either East or West constitute the first two rounds. Each of these possible combinations forms a minimal winning set for exactly one player, regardless of the order of these two rounds.

We will now consider two distinct examples of sequential structures. This will illustrate the intuition behind this result, as well as demonstrate the fact that this Proposition provides a necessary but not a sufficient condition for payoffs to be asymmetric.

Since the players have different winning sets, it may be possible for one player to obtain a winning set in a round in which the opposing player could not have done so. For example, consider the structure E-N-W-S. After the second round, the set of contested cells $\{E, N\}$ is an element of $X^\bigstar$. In this case player X has an expected payoff of approximately 0.0731 before the first round, while player Y has an expected payoff of approximately 0.2315, as shown in Appendix A.[8] Observe that if the player who could have completed a winning set has failed to do so, this player has fewer possible winning sets available to complete in subsequent rounds.

Consequently, if player X has not won after the second round, he could not have won *both* the North and East, eliminating his minimal winning set $\{N, E\}$ from consideration. It is still possible for player X to have won one of either North or East in the first two rounds; only winning both has been ruled out. This only leaves the minimal winning sets $\{\{N, S\}, \{S, W\}\}$ as potential sets to be contested for Player X. No such eliminations are possible for player Y, as winning both North and East does not comprise a winning set for Y.

[8]This expected payoff for X is the lowest value that exists for a single player in any structure $\Re$. The sequence W-N-E-S provides the lowest expected payoff for Y and is of identical value.

After a given round has been contested, if neither player has won, both players will be able to use the results of the contests up to that point for determining optimal strategies for the remaining rounds. The knowledge of what winning sets are still possible allows for recalculation of the value of each remaining cell by each player. If there is an asymmetry in the winning combinations that can still be possibly completed by the two players, they will have different opportunities to make use of this information. Thus, the knowledge of previous results may be more useful to one player, allowing her to obtain a higher expected payoff. Since such asymmetries cannot occur in the simultaneous case, the expected payoffs are identical.

Intuitively, two different factors are involved in the sequential case. Being able to form a winning set earlier provides a benefit from the greater chance of an early victory: early conclusion of the contest requires spending in fewer rounds. On the other hand, having a greater number of winning sets still available after a round allows a player to take advantage of the knowledge of the asymmetry in player strengths to increase her payoff.

In some situations these factors can balance each other out, so that the existence of a single player who is able to form a winning set after a particular round does not yield different expected payoffs. For example, the structure N-W-S-E results in identical expected payoffs of approximately 0.1797 for each player, as shown in Appendix A. In this case, if after the second round, player Y has not yet won, then the minimal winning set $\{N, W\}$ is no longer available to player Y. Because player X would not win by winning both North and West, not having won after two rounds have been contested does not mean player X did not win both North and West. Thus, player X may still have all minimal winning sets available. In spite of this difference in when the earliest minimal winning set can be completed, the expected payoffs for the entire game are the same for each player and we find that the two factors cancel each other out.

The fact that these two forces balance each other perfectly in this case results from the fact that although only Player Y has the possibility of victory after the second round, if Player X wins the North, then the second round of West becomes irrelevant, and thus Player X can win after the third round (which is only the second round in which players actually expend effort). Hence both players have the ability to win after incurring only two rounds of actual expenditures.

This implies that Proposition 2 only states a necessary condition, and not a sufficient one. In the sequential cases, the impact of this asymmetry favors the player that does not have a winning set available in the first two rounds. Note that under the intermediate cases (those lasting two or three rounds), the asymmetry can favor either player. In some of these cases, the player with the advantage is the one who has a greater number of winning sets that could be completed at the end of round r. For example, NE-S-W favors player X. On the contrary, this is not true for the sequence N-ES-W, which favors player Y. This can occur because the advantage of winning early may be greater than the advantage of having more winning sets available after the winners of some subset of cells have been determined.

Proposition 3. *Let $\Re$ contain a round R_i, such that either $\{N, S\} \subseteq R_i$ or $\{E, W\} \subseteq R_i$. Then both players have the same expected payoffs in equilibrium.*

Proof: See Table 2 and Appendices A and B.

We can see this result from the expected payoffs of the players given in Table 2. Appendix B illustrates how these payoffs are computed. The forces that give us Proposition 3 are similar to those responsible for Proposition 2. As seen in the computations, the players will have identical expected payoffs regardless of the round in which the $\{N, S\}$ or $\{E, W\}$ subset appears. Moreover, having NS or EW in the same round prevents the existence of a round which completes a minimal winning set for only one player. Also note that this is a sufficient condition for identical expected payoffs, not a necessary one. For example, the order N-E-S-W results in identical expected payoffs for the players, despite lacking such a round.

4.3 Contests with Random Order

We will now consider a sequential contest in which the order of rounds is determined randomly. First the cell to be contested in Round One is announced. After the players make their Round One decisions and the outcome is realized, the cell to be contested in Round Two is announced. Then, the players make their Round Two decisions and only after the realization occurs, is the Round Three cell announced. This process continues until all the cells are announced. Of course, given that there are only 4 rounds, once Round Three is over both players will know which cell remains to be contested, so the uncertainty really is over the first 3 rounds (Remark 2). Observe that there are 24 possible random permutations of North, South, East, and West. We begin by separating these into cases based on the first round.

Case 1: North or South as R_1

In 12 of the 24 possible orders, North or South will be the randomly drawn in the first round. Without loss of generality let North be randomly chosen first. If player X wins North, player X needs to win one of South and East to form a winning set, while if Player Y wins North, player Y must win one of South and West to form a winning set. This means that the winner of North must win one of two cells to be contested sequentially, while the loser of North must win two cells contested sequentially.[9]

The subgame consisting of East, West, and South when the player has won the North gives a player an expected payoff of $\frac{43}{64}$, while the subgame consisting of East, West, and South when the player lost the North has an expected payoff of $\frac{1}{64}$ (as obtained by backwards induction in Appendix A). Moving backwards from these subgames, in Round One we have the expected payoff conditions $U_X = \left(\frac{X_N}{Z_N}\right)\frac{43}{64} + \left(\frac{Y_N}{Z_N}\right)\frac{1}{64} - X_N$ and $U_Y = \left(\frac{Y_N}{Z_N}\right)\frac{43}{64} + \left(\frac{X_N}{Z_N}\right)\frac{1}{64} - Y_N$. This gives us the first order equations $\left(\frac{Y_N}{Z_N^2}\right)\frac{42}{64} = \left(\frac{X_N}{Z_N^2}\right)\frac{42}{64} = 1$, which yield $X_N = Y_N = \frac{21}{128}$, and thus $U_X = U_Y = \frac{23}{128} \approx .1797$.

Note that this is the same as the expected payoff in the sequential case where North or South is the first cell contested. Once the first round is resolved, one remaining cell becomes irrelevant, effectively leaving a two round game. The player who lost the first round must win both of the remaining relevant rounds, so the order of these two rounds does not matter. This holds true both when the contest sequence is known in advance, and when it is randomly drawn in each round. Thus the identical expected payoffs follow directly.

Case 2: East or West as R_1

In 12 of the 24 possible orders, East or West will be the cell randomly drawn first. Without loss of generality let East be randomly drawn in Round One. In these cases the ordering of the subsequent rounds matters. Thus, we must find expected payoffs for each player for all three (West, North, or South) possible second round contests. We then take the average across these three (expected payoffs) to find a total expected payoff. We will then use these average expected payoffs to find the optimal strategy and the resulting expected payoff for the first round.

Fourth Round: For the final round, either the cell will determine the overall winner, in which case each player will expend $\frac{1}{4}$, and thus have an expected payoff of $\frac{1}{4}$, or the cell is irrelevant and will not be contested.

Third Round: Moving backwards to the third round, once the cell being contested in the third round is announced, the cell to be contested in the fourth round is known by process of elimination. Thus the optimal strategies and expected payoffs for each player for the third round

[9] The third remaining cell has no impact on the overall winner, and thus can be ignored.

can be taken from the optimal strategies and expected payoffs found while obtaining the solutions for the sequential case (see Appendix A, case 2).[10]

Second Round: Here were must consider subcases, with the subcases differentiated by which player won the East in the first round, as well as which cell is contested in the second round. Consider the subgame where player X won East.[11] In that case the expected payoffs are the same regardless of whether the second round is West or South. This is because even if X wins the second round in either case, she will still require one of the remaining two cells. As we saw in the sequential cases, at the start of this subgame, X has an expected payoff of approximately 0.4245, and Y has an expected payoff of approximately 0.0455.

If the second round is North, in the subgame after player X won the East, X is victorious if she wins North. If X loses the North, X must win both the South and West. Again, taking the payoff from the fixed sequential case already solved, we see that in this subgame, X has an expected payoff of 0.2373, and Y of 0.1106.

First Round: Taking the average expected payoff, conditional on X having won the East, across all three possible second (and subsequent) rounds gives a total expected payoff for X of $\frac{1}{3}(0.4245 + 0.4245 + 0.2373) = 0.3612$ and for Y of $\frac{1}{3}(0.0455 + 0.0455 + 0.1106) = 0.0672$. Similarly, in the case where Y wins the East, X has an expected value for the subgame consisting of North, West, and South of 0.0672, while player Y has an expected value for the subgame of 0.3612. Thus player X obtains an expected payoff of $\left(\frac{X_E}{Z_E}\right)(0.3612) + \left(\frac{Y_E}{Z_E}\right)(0.0672) - X_E$, generating a first order condition $\left(\frac{Y_E}{Z_E^2}\right)(0.3612) - \left(\frac{Y_E}{Z_E^2}\right)(0.0672) = 1$. Because $X_E = Y_E$, due to winning East being equally valuable to each player as shown in Lemma 1 (see Appendix A), solving these yields $X_E = Y_E = 0.0735$, and thus expected payoffs of 0.1407.

Combining the Two R_1 Cases: Averaging the expected payoff of 0.1407 from the 12 cases where East or West is the first round with the 0.1797 from the 12 cases where North or South is the first round yields a total expected payoff to each player of $\frac{12}{24}0.1407 + \frac{12}{24}0.1797 = 0.1602$. This is lower than the overall average of all 24 sequential cases where the order is known, which is 0.1641. Since the total benefit to the players is the same, the lower expected payoff means the players must have greater expected expenditures. We summarize this as the following proposition.

Proposition 4. *The three contests can be arranged in order of increasing dissipation, where sequential contests with specified order have the lowest and simultaneous contests have the highest*

[10] All expected payoffs used here are taken from the calculations contained in Appendix A.

[11] The analogous results hold true if player Y won the East.

expected dissipation in equilibrium.

Since players have more difficulty determining which cells are likely to be important in the random case than they do in the pre-specified order case, resource expenditure becomes less efficient. This is especially true in the cases where East or West is drawn as the first cell to be contested, because future asymmetries remain unknown. Since effort reducing asymmetries arising in the random order case are probabilistic, and are revealed only as the game progresses, the degree of dissipation lies between sequential contests without random order and simultaneous contests. The small differences in expected values between the random and fixed sequential cases are driven by the large number of sequences in which the strategies in the random draws end up being the same as in the sequential draw. If North or South is drawn in the first round, the random order contest plays out the same as the known sequential cases where North or South are the first round. In both cases, after the first round is completed, one of the three remaining cells has been rendered irrelevant, and the other two cells are both required for the player who lost the first round to obtain victory. Thus the players' behavior will be the same in these cases. Unlike the simultaneous case, in the random order case, there is still some information available, allowing for more informed decision making and the possibility of an early victory.

The random sequence contest helps explain why the sequential contests have lower resource expenditure in general. As the contest progresses, the relative importance of each cell becomes clearer, with some cells becoming irrelevant due to the knowledge of the winners of previously contested cells. Thus the players know what can be safely ignored, and only expend effort in the cells that still matter. In the simultaneous case, the players lack this ability, and thus expend effort on cells that turn out to be irrelevant to the formation of the winning path, such as cases where the winner wins 3 or even all 4 cells. This is also why North and South are more valued, as these appear in more winning sets for each player. At least one of them appears in all winning sets, and North and South form a winning set by themselves, a fact that is not true for the East and West cells. Thus, lacking other information, North and South are likely to be more relevant than East and West.

5 Discussion

In this section we discuss a number of possible extensions and identify some future research questions. We see that complementarity plays a major role in determining expected payoffs. The fact that different sets of cells constitute winning sets for different players means that an asymmetry in the order in which winning sets for each player are contested. This creates an asymmetry in the expected payoffs of the players. Although omitted here in the interest of space, we also see similar results in the intermediate cases, shown in Appendix B, where the structure consists of 2 or 3 rounds. In these cases different expected payoffs for players can occur due to different winning sets. We now discuss some other possible variations of this game.

The most obvious variation would be to increase the size of the grid over which the competition takes place. Although tractability requires the number of cells to be limited, the logic of our results may be of use in structures with more cells. For example, in determining a structure for selling bandwidth on network routers, the owner should take into account the structure of connections the bidders wish to obtain. By identifying the routers that exhibit complementarity, the important routers for the structure may be found, thus allowing a reasonably good, though possibly suboptimal, solution to be found for the seller. We leave this as an open question for future research.

Throughout this paper we have assumed that the contest structure is externally imposed. However, the cells being contested could be owned by one or many individuals who would be interested in maximizing their own revenues. If the single owner of the cells is simply allowed to choose a contest structure, she will obviously choose one of the structures resulting in a total expected expenditure by the players of 0.75. However, if the contest structure is the result of decisions made by multiple owners, this may not hold see Kovenock et al. (2013).

If we use the generalized Tullock success function (Tullock, 1980), in which player X has a probability of winning cell i equal to $\frac{X_i^\alpha}{X_i^\alpha+Y_i^\alpha}$, where $0 < \alpha \leq 1$, we can obtain a pure strategy equilibrium for the simultaneous case, in which each player spends $\frac{V\alpha}{8}$ on the North and South, and $\frac{V\alpha}{16}$ on each of the East and West. However, our preliminary investigations show that the sequential cases are quite problematic, as α terms appear as both exponents and as coefficients, giving non-linear behavior with respect to α.

If instead of a costly, unlimited resource, we instead have a costless limited budget, in which unspent resources provide no benefit (Brams and Davis, 1974), we have a slightly different problem. In this case player X attempts to maximize the probability of victory given by

$$P_X\left(\{X_k, Y_k\}_{k\in\{N,S,E,W\}}\middle|\,\Re\right) = \sum_{\alpha\in X^\star}\left(\prod_{i\in\alpha, j\in A\setminus\alpha}\frac{X_i}{Z_i}\frac{Y_j}{Z_j}\right)V \qquad (8)$$

subject to the budget constraint $X_N + X_S + X_E + X_W = B$.

Although the value of the prize does not matter, in the simultaneous case each player will spend $\frac{1}{3}$ of their budget on the North and South cells, and $\frac{1}{6}$ on the East and West. Therefore, if given a budget $B = \frac{3}{8}V$, the decisions made will be identical to those in the simultaneous case. Calculations are omitted in the interest of space, but proceed in a manner similar to the main case.

However, the sequential cases lead to different solutions. For example, consider the E-N-W-S, E-N-S-W, and E-N-SW cases. There is no advantage to not spending the entire budget, and thus if all four cells are contested, the entire budget will be spent. Players would only reserve resources for future rounds of competition if future rounds are possible. Obviously, once spending decisions for three cells are made, it also fixes how much can be spent on the fourth cell. Thus all three of these structures are equivalent with budget constraints, because after the first two rounds, the only decision left is how to divide the remaining budget between the West and South cells. In contrast, with the costly resource the expected payoffs and expected expenditures vary between these cases. Furthermore, for the costly resource case, the identity of the player with the greater expected payoff differs depending on whether there is one of the two four-round structures (E-N-W-S or E-N-S-W) listed versus the three-round structure (E-N-SW). Obviously this is not true in the constrained budget case. This difference between the costly resource and costless limited resource cases is likely due to there being no advantage to winning the contest quickly compared to winning it in the final round for the case of budget constraints. Thus although the costless resource case is an interesting problem, it lies outside the scope of the current paper.

References

[1] Michael R. Baye, Dan Kovenock, and Casper G de Vries: The incidence of overdissipation in rent-seeking contests. Public Choice *99*, 439-454 (1999)

[2] Emile Borel: La theorie du jeu les equations integrales a noyau symetrique. Comptes Rendus del Academie *173*, pp. 13041308 (1921)

[3] Emile Borel, and J. Ville: Application de la theorie des probabilits aux jeux de hasard. GautheirVillars, Paris. In: Borel, E., Cheron, A. (Eds.), Theorie mathematique du bridge a la porte de tous. Editions Jacques Gabay, Paris, 1991 (1938)

[4] Steven J. Brams and Morton D. Davis: The 3/2's rule in presidential campaigning. The American Political Science Rev. *68*, 113-134 (1974)

[5] Derek J. Clark and Kai A. Konrad: Asymmetric conflict: weakest link against best shot. J. of Conflict Resolution *51*, 457-469 (2007)

[6] Lawrence Friedman: Game theory models in the allocation of advertising expenditures. Operations Res. *6*, 699-709 (1958)

[7] Russell Golman and Scott E. Page: General Blotto: games of allocative strategic mismatch. Public Choice *138*, 279-299 (2009)

[8] Oliver Alfred Gross: The symmetric Blotto game. Rand Corporation Memorandum, RM-424 (1950)

[9] Oliver Alfred Gross and R. A. Wagner: A continuous Colonel Blotto game. Rand Corporation Memorandum, RM-408 (1950)

[10] Sergiu Hart: Discrete Colonel Blotto and General Lotto games. International J. of Game Theory *36*, pp. 441-460 (2008)

[11] Tilman Klumpp and Mattias K. Polborn: Primaries and the New Hampshire effect. Rev. of Economic Stud., *90*, 1073-1114 (2006)

[12] Dan Kovenock and Brian Roberson: The optimal defense of networks of targets. Purdue University Working Paper 1251 (2010)

[13] Dan Kovenock and Brian Roberson: Conflicts with multiple battlefields. Oxford Handbook of the Economics of Peace and Conflict, ed. Michelle R. Garfinkel and Stergios Skaperdos, Oxford University Press, 503-531 (2012)

[14] Dan Kovenock, Sudipta Sarangi, and Matt Wiser: All-Pay 2 $\times$ 2 Hex: A multibattle contest with complementarities. Working Paper, http://xythos.lsu.edu/users/mwiser1/Hex (2013)

[15] Dmitriy Kvasov: Contests with limited resources. J. of Economic Theory *136*, 738-748 (2007)

[16] Jean-Francois Laslier: How two-party competition treats minorities. Rev. of Economic Des. *7*, 297-307 (2002)

[17] Jean-Francois Laslier and Nathalie Picard: Distributive politics and electoral competition. J. of Economic Theory *103*, 106-130 (2002)

[18] Brian Roberson: The Colonel Blotto game. Economic Theory *29*, 1-24 (2006)

[19] Brian Roberson and Dmitriy Kvasov: The non-constant-sum Colonel Blotto game. Economic Theory *51*, 397-433 (2012)

[20] Alexander R.W. Robson: Multi-item contests. Australian National University Working Paper, 446 (2005)

[21] Balazs Szentes and Robert W. Rosenthal: Three-object two-bidder simultaneous auctions. Chopsticks and tetrahedra. Games and Economic Behav. *44*, pp. 114-133 (2003a)

[22] Balazs Szentes and Robert W. Rosenthal: Beyond chopsticks. Symmetric equilibria in majority auction games. Games and Economic Behav. *45*, pp. 278-295 (2003b)

[23] James M Snyder: Election goals and the allocation of campaign resources. Econometrica *57*, 637-660 (1989)

[24] Gordon Tullock: On the efficient organization of trials. Kyklos *28*, 745-762 (1975)

[25] Gordon Tullock: Efficient rent seeking. Towards a Theory of the Rent Seeking Society, edited by Gordon Tullock, Robert D. Tollison, James M. Buchanan, Texas A & M University Press, 3-15 (1980)

[26] Jonathan Weinstein: Two notes on the Blotto game. The B. E. J. of Theoretical Economics *12*, 1-13 (2012)

6 Appendix

6.1 Appendix A: Polar Cases: Simultaneous versus Sequential Cases

For Proposition 1 we will start by computing the expected payoffs from the simultaneous and sequential cases. For the simultaneous case, we first prove a lemma which shows that in any pure strategy Nash equilibrium, the players will play identical strategies.

Case 1: All Four Cells Simultaneously

Lemma 1 *In any pure strategy Nash equilibrium of the simultaneous contest game, both players will play identical strategies.*

Proof: Following Klumpp and Polborn (2006; p 1103), given a set of strategies for players X and Y across all cells other than j, there is some probability $p_j \in (0,1)$ that cell j is pivotal for both players. A pivotal cell is a cell such that winning that cell wins the overall contest, while losing the cell loses the overall contest. Recall that $X_j > 0$ and $Y_j > 0$ for each cell $j \in A$ (see footnote 4). From this it follows that the probability of winning any given cell j by either player is $\frac{T_j}{Z_j} > 0$. Hence the probability of any particular division of winning cells between players is non-zero implying the probability that any given cell is pivotal is non-zero. This is true in spite of the complementarities involved in the game of Hex. The fact that the North and South cells are pivotal for each player is easy to see. East is a pivotal cell for both players when X wins North and Y wins South. West is a pivotal cell if X wins South and Y wins North. As only one player can win the contest, any pivotal cell must be completing a winning path for both players. Clearly, the player who wins the pivotal cell will win the contest.

Given a set of strategies for the other cells generating a $p_j > 0$, player X's conditional payoff function can be written as $U_X(\cdot) = \left(\frac{X_j}{Z_j}\right) p_j - X_j$, while player Y's conditional payoff function can be written as $U_Y(\cdot) = \left(\frac{Y_j}{Z_j}\right) p_j - Y_j$. This gives the following first order conditions

$$\frac{\partial U_X}{\partial X_j} = \left(\frac{Y_j}{Z_j^2}\right) p_j - 1 = 0$$
$$\frac{\partial U_Y}{\partial Y_j} = \left(\frac{X_j}{Z_j^2}\right) p_j - 1 = 0$$

From this we obtain $\left(\frac{X_j}{Z_j^2}\right) = \left(\frac{Y_j}{Z_j^2}\right)$, and hence $X_j = Y_j$. Since the cell j had been chosen arbitrarily, it follows that in equilibrium players X and Y will play the same strategies, i.e. $X_N = Y_N$; $X_S = Y_S$; $X_E = Y_E$ and $X_W = Y_W$. This completes the proof. $\square$

Computing payoffs for the simultaneous case:

Recall that we can write player X's utility function as

$$U_X(\cdot) = \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{X_E}{Z_E}\right) + \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_W}{Z_W}\right) - X_N - X_S - X_E - X_W$$

Taking the first derivatives gives us

$$\begin{aligned}
\frac{\partial U_X}{\partial X_N} &= \left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_S}{Z_S}\right) + \left(\frac{Y_N}{Z_N^2}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{X_E}{Z_E}\right) - \left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_W}{Z_W}\right) - 1 \\
\frac{\partial U_X}{\partial X_E} &= \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{Y_E}{Z_E^2}\right) - 1 \\
\frac{\partial U_X}{\partial X_W} &= \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_W}{Z_W^2}\right) - 1 \\
\frac{\partial U_X}{\partial X_S} &= \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S^2}\right) - \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S^2}\right)\left(\frac{X_E}{Z_E}\right) + \left(\frac{Y_N}{Z_N}\right)\left(\frac{Y_S}{Z_S^2}\right)\left(\frac{X_W}{Z_W}\right) - 1
\end{aligned}$$

Recall that we can write player Y's expected value function as

$$U_Y(\cdot) = \left(\frac{Y_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right) + \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_W}{Z_W}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right) - Y_N - Y_S - Y_E - Y_W$$

whose first derivatives are

$$\begin{aligned}
\frac{\partial U_Y}{\partial Y_N} &= \left(\frac{X_N}{Z_N^2}\right)\left(\frac{Y_S}{Z_S}\right) + \left(\frac{X_N}{Z_N^2}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_W}{Z_W}\right) - \left(\frac{X_N}{Z_N^2}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right) - 1 \\
\frac{\partial U_Y}{\partial Y_E} &= \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{X_E}{Z_E^2}\right) - 1 \\
\frac{\partial U_Y}{\partial Y_W} &= \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_W}{Z_W^2}\right) - 1 \\
\frac{\partial U_Y}{\partial Y_S} &= \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S^2}\right) - \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S^2}\right)\left(\frac{Y_W}{Z_W}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S^2}\right)\left(\frac{Y_E}{Z_E}\right) - 1
\end{aligned}$$

Following Lemma 1, we can substitute $X_j = Y_j$, which makes these two sets of derivatives identical. Hence writing in terms of X_j, our first derivatives are

$$\begin{aligned}
\frac{\partial U_X}{\partial X_N} &= \frac{X_N}{4X_N^2}\frac{X_S}{2X_S} + \frac{X_N}{4X_N^2}\frac{X_S}{2X_S}\frac{X_E}{2X_E} - \frac{X_N}{4X_N^2}\frac{X_S}{2X_S}\frac{X_W}{2X_W} - 1 \\
\frac{\partial U_X}{\partial X_E} &= \frac{X_N}{2X_N}\frac{X_S}{2X_S}\frac{X_E}{4X_E^2} - 1 \\
\frac{\partial U_X}{\partial X_W} &= \frac{X_N}{2X_N}\frac{X_S}{2X_S}\frac{X_W}{4X_W^2} - 1 \\
\frac{\partial U_X}{\partial X_S} &= \frac{X_N}{2X_N}\frac{X_S}{4X_S^2} - \frac{X_N}{2X_N}\frac{X_S}{4X_S^2}\frac{X_E}{2X_E} + \frac{X_N}{2X_N}\frac{X_S}{4X_S^2}\frac{X_W}{2X_W} - 1
\end{aligned}$$

This simplifies to

$$\begin{aligned}
\frac{1}{4X_N}\frac{1}{2} + \frac{1}{4X_N}\frac{1}{4} - \frac{1}{4X_N}\frac{1}{2}\frac{1}{2} - 1 &= 0 \\
\frac{1}{2}\frac{1}{2}\frac{1}{4X_E} - 1 &= 0 \\
\frac{1}{2}\frac{1}{2}\frac{1}{4X_W} - 1 &= 0 \\
\frac{1}{2}\frac{1}{4X_S} - \frac{1}{4}\frac{1}{4X_S} + \frac{1}{4}\frac{1}{4X_S} - 1 &= 0
\end{aligned}$$

Solving this system of equations gives us $X_N = X_S = \frac{1}{8}$, and $X_E = X_W = \frac{1}{16}$ in equilibrium. By Lemma 1, this also gives us $Y_N = Y_S = \frac{1}{8}$, and $Y_E = Y_W = \frac{1}{16}$ in equilibrium.

To determine the nature of this equilibrium, we require a full set of second derivatives, evaluated at $X_N = X_S = Y_N = Y_S = \frac{1}{8}, X_E = X_W = Y_E = Y_W = \frac{1}{16}$. This gives us the following matrix of second derivatives

$$\begin{bmatrix} \frac{\partial^2 U_X}{\partial X_N^2} & \frac{\partial^2 U_X}{\partial X_N \partial X_E} & \frac{\partial^2 U_X}{\partial X_N \partial X_W} & \frac{\partial^2 U_X}{\partial X_N \partial X_S} \\ \frac{\partial^2 U_X}{\partial X_E \partial X_N} & \frac{\partial^2 U_X}{\partial X_E^2} & \frac{\partial^2 U_X}{\partial X_E \partial X_W} & \frac{\partial^2 U_X}{\partial X_E \partial X_S} \\ \frac{\partial^2 U_X}{\partial X_W \partial X_N} & \frac{\partial^2 U_X}{\partial X_W \partial X_E} & \frac{\partial^2 U_X}{\partial X_W^2} & \frac{\partial^2 U_X}{\partial X_W \partial X_S} \\ \frac{\partial^2 U_X}{\partial X_S \partial X_N} & \frac{\partial^2 U_X}{\partial X_S \partial X_E} & \frac{\partial^2 U_X}{\partial X_S \partial X_W} & \frac{\partial^2 U_X}{\partial X_S^2} \end{bmatrix} = \begin{bmatrix} -8 & 4 & -4 & 0 \\ 4 & -16 & 0 & -4 \\ -4 & 0 & -16 & 4 \\ 0 & -4 & 4 & -8 \end{bmatrix}$$

We need to find the characteristic polynomial of the determinant

$$\begin{vmatrix} -8-\lambda & 4 & -4 & 0 \\ 4 & -16-\lambda & 0 & -4 \\ -4 & 0 & -16-\lambda & 4 \\ 0 & -4 & 4 & -8-\lambda \end{vmatrix}$$

This matrix has a characteristic polynomial of $\lambda^4 + 48\lambda^3 + 784\lambda^2 + 4864\lambda + 7936$, which has roots of approximately $(-2.4805, -10.8345, -15.0041, -19.6809)$, and thus is negative definite. As the Hessian matrix is negative definite, the solution to first order equations must be a local maximum, and as it is the only equilibrium, this local maximum is the only possible interior solution.

In order to find a global maximum, we now have to consider the boundary cases. Negative bids are not allowed to begin with, and clearly placing bids greater than the value of the prize cannot result in positive values, so we will restrict all possible solutions to the $[0,1]^4$ hypercube. We will now consider boundary conditions.

First note that a bid of 1 placed on more than one cell will yield negative payoffs. Hence a bid of 1 must be restricted to only one cell and on this boundary point all other bids must be zero for non-negative profits. However this will never lead to a path and hence can also be ruled out. Thus we will look at boundary conditions involving some number of zero bids. Cases with three or four zero bids can again be ruled out, as they cannot result in a winning path, and thus cannot have positive profits.

One Zero Bid

If there is a maximum with exactly one zero bid, the zero must be in the East or West. If the zero bid is on the North or South, the player would have no incentive to have a positive bid on one of East or West, as losing one of North and South makes one of East or West irrelevant. To illustrate, we will take the case of player X leaving a zero bid in the West without loss of generality. Thus, we have an expected utility function of

$$U_X(\cdot) = \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right) - X_N - X_S - X_E$$

This has partial derivatives of

$$\begin{aligned}
\frac{\partial U_X}{\partial X_N} &= \left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_E}{Z_E}\right) + \left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right) - 1 \\
\frac{\partial U_X}{\partial X_E} &= \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_E}{Z_E^2}\right) - \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_E}{Z_E^2}\right) - 1 \\
\frac{\partial U_X}{\partial X_S} &= \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S^2}\right)\left(\frac{Y_E}{Z_E}\right) - 1
\end{aligned}$$

Again, by Lemma 1, $X_N = Y_N$, $X_S = Y_S$, and $X_E = Y_E$, giving us a solution of $X_N = \frac{3}{16}$, $X_E = X_S = \frac{1}{16}$. This yields an expected value for player X of $\frac{1}{16}$, which is worse than the interior solution. The Hessian matrix of partial second derivatives at this point is

$$\begin{bmatrix} -\frac{16}{3} & \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3} & -16 & -8 \\ \frac{8}{3} & -8 & -16 \end{bmatrix}$$

This is again negative definite, with eigenvalues of $-8, -\frac{4}{3}(11+\sqrt{57}), -\frac{4}{3}(11-\sqrt{57})$, thus making this a local maximum. Again, we must check the boundary for solutions. Those cases with a value of 1 for a bid cannot have positive expected value, while those with a zero bid for one of the three values take us to the case of having exactly two zero bids.

Two Zero Bids

If a player places zero bids on two cells, the remaining two cells must form a winning set. For illustration, we will take the case where East and West have zero bids without loss of generality, leaving player X's expected utility as

$$U_X(\cdot) = \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right) - X_N - X_S$$

which has first partial derivatives of

$$\begin{aligned}
\frac{\partial U_X}{\partial X_N} &= \left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_S}{Z_S}\right) - 1 \\
\frac{\partial U_X}{\partial X_S} &= \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S^2}\right) - 1
\end{aligned}$$

From Lemma 1, we know $X_N = Y_N$, $X_S = Y_S$, and thus we have $X_N = X_S = \frac{1}{8}$, giving as expected value of zero. The Hessian at this point is

$$\begin{bmatrix} -8 & 4 \\ 4 & -8 \end{bmatrix}$$

which has eigenvalues of $-4, -12$, and thus is negative definite, making this a local maximum. As for boundary conditions, we can still rule out bids of 1, and a 0 bid would leave three zero bids, and thus no way to complete a winning path.

Case 2 : Four cells sequentially

We now will solve the sequential cases. Combining these with the simultaneous cases gives us Proposition 1, while comparing the sequential cases proves Proposition 2. In order to obtain these results, we must work backwards from the terminal nodes. If only one cell remains to be contested, either the cell is irrelevant as we already have a winner, or the winner of this one cell will win the contest. Let the remaining cell be East without loss of generality, we have $U_X(\cdot|R_1, R_2, R_3, R_4) = \left(\frac{X_E}{Z_E}\right) - X_E$, and $U_Y(\cdot|R_1, R_2, R_3, R_4) = \left(\frac{Y_E}{Z_E}\right) - Y_E$. Taking derivatives gives us $\left(\frac{X_E}{Z_E^2}\right) = 1$, $\left(\frac{Y_E}{Z_E^2}\right) = 1$, so $X_E = Y_E = \frac{1}{4}$, giving $U_X(\cdot|\cdot, R_4) = U_Y(\cdot|\cdot, R_4) = \frac{1}{4}$.

Now we can work backwards to the previous stage. If there are two cells remaining, there are three possibilities. If the contest has already been won, they are both irrelevant, and we are done. However, if only one of the two remaining cells is relevant, we ignore the irrelevant cell. For simplicity assume that that irrelevant cell appears in Round 3. Then the fourth round will play out as described above for East and this reduces to the above. Finally, there is the possibility that both are relevant, with one player needing to win both and the other needing to win one of the two. We will assume without loss of generality that North and East remain to be contested sequentially, with Player X requiring both to win. Then $U_X(\cdot|\cdot, R_3, R_4) = \left(\frac{X_N}{Z_N}\right)\frac{1}{4} - X_N$, $U_Y(\cdot|\cdot, R_3, R_4) = \left(\frac{Y_N}{Z_N}\right) + \left(\frac{X_N}{Z_N}\right)\frac{1}{4} - Y_N$ gives the expected utility for each player in the North round. Taking derivatives gives us

$$\left(\frac{Y_N}{Z_N^2}\right)\frac{1}{4} = 1$$
$$\left(\frac{X_N}{Z_N^2}\right) - \left(\frac{X_N}{Z_N^2}\right)\frac{1}{4} = \left(\frac{X_N}{Z_N^2}\right)\frac{3}{4} = 1$$

Solving this gives $X_N = \frac{3}{64}$, while $Y_N = \frac{9}{64}$, and thus expected payoffs $U_X(\cdot|\cdot, R_3, R_4) = \frac{1}{64}$, $U_Y(\cdot|\cdot, R_3, R_4) = \frac{43}{64}$. These two solutions will be used extensively in the subcases below. This is as far backwards as we can work without having the specifics of the complementarity, so we now must break into subcases.

Subcase 2a: *North or South in R_1*

After the first round, one of the remaining cells will become irrelevant. The winner of the first round will need to win one of the remaining two relevant cells, which is a subgame with an expected payoff of $\frac{43}{64}$, and the loser of the first round will need to win both remaining relevant cells, a subgame with an expected value of $\frac{1}{64}$, as shown above. Thus, if North is the first cell, we have $U_X = \left(\frac{X_N}{Z_N}\right)\frac{43}{64} + \left(\frac{Y_N}{Z_N}\right)\frac{1}{64} - X_N$. Taking the derivative with respect to X_N gives $\left(\frac{X_N}{Z_N^2}\right)\frac{43}{64} - \left(\frac{X_N}{Z_N^2}\right)\frac{1}{64} = 1$. From Lemma 1, $X_N = Y_N$, so solving gives us $X_N = Y_N = \frac{21}{128}$ and $U_X = U_Y = \frac{23}{128}$.

Subcase 2b: *East and West in R_1 and R_2*

Without loss of generality, we consider the cases where East is the first round, results for West as opening round are symmetric. If player X wins the East, winning the West means he will need either the North or South, while losing the West means that the South is irrelevant, and the North will determine the overall winner. These have expected payoffs of $\frac{43}{64}$ and $\frac{1}{4}$ respectively for player X and $\frac{1}{64}$ and $\frac{1}{4}$ for player Y. Thus, if we consider the subgames in which player X wins the East, the expected payoff functions for the second round are

$$U_X(\cdot|\cdot, R_2, R_3, R_4) = \left(\frac{X_W}{Z_W}\right)\frac{43}{64} + \left(\frac{Y_W}{Z_W}\right)\frac{1}{4} - X_W$$
$$U_Y(\cdot|\cdot, R_2, R_3, R_4) = \left(\frac{Y_W}{Z_W}\right)\frac{1}{4} + \left(\frac{X_W}{Z_W}\right)\frac{1}{64} - Y_W$$

This gives us first order conditions of

$$\left(\frac{Y_W}{Z_W^2}\right)\frac{27}{64} = \left(\frac{X_W}{Z_W^2}\right)\frac{15}{64} = 1$$

and so $15X_W = 27Y_W$. Thus $X_W = \frac{27}{15}Y_W$, which when substituted into the first order conditions gives us $X_W = \frac{(15)(27^2)}{(42^2)(64)}$, $Y_W = \frac{(15^2)(27)}{(42^2)(64)}$. Using these payoffs in the expected payoff functions gives $U_X = \frac{47907}{112896}$ and $U_Y = \frac{5139}{112896}$. These payoffs will be reversed if player Y wins the East.

Thus, in the initial round, the expected payoff function for player X is $\left(\frac{X_E}{Z_E}\right)\frac{47907}{112896} + \left(\frac{Y_E}{Z_E}\right)\frac{5139}{112896} - X_E$, giving a first order condition of $\left(\frac{Y_E}{Z_E^2}\right)\frac{42768}{112896} = 1$, with $X_E = Y_E$ due to symmetry. Thus, the optimal strategy is $X_E = Y_E = \frac{42768}{451584}$, which yields expected payoffs of $\frac{63486}{451584}$.

Subcase 2c: *East or West in R_1 and North or South in R_2*

Without loss of generality, we consider the subcases where East is contested in the first round. If player X wins the East, winning the North means victory, while losing North means she must win both West and South, for an expected payoff of $\frac{1}{64}$ for X and $\frac{43}{64}$ for Y. Thus the expected payoff functions in the second round in the subgame where player X won the East are

$$U_X(\cdot|\cdot, R_2, R_3, R_4) = \left(\frac{X_N}{Z_N}\right) + \left(\frac{Y_N}{Z_N}\right)\frac{1}{64} - X_N$$
$$U_Y(\cdot|\cdot, R_2, R_3, R_4) = \left(\frac{Y_N}{Z_N}\right)\frac{43}{64} - Y_N$$

This gives the first order conditions of

$$\left(\frac{Y_N}{Z_N^2}\right)\frac{63}{64} = \left(\frac{X_N}{Z_N^2}\right)\frac{43}{64} = 1$$

Thus we have $43X_N = 63Y_N$, which with our first order condition gives $X_N = \frac{(43)(63^2)}{(64)(106^2)}$ and $Y_N = \frac{(43^2)(63)}{(64)(106^2)}$. Plugging these into the expected payoff equations give approximations of $U_X \approx 0.2373$, $U_Y \approx 0.1106$.

If player Y wins the East, winning either the South or North and West wins. This is the same as in the E-W-N-S case, which gives decimal approximations of $U_X \approx 0.0455$ and $U_Y \approx 0.4245$.

Thus for the first round, we have expected payoff functions of

$$U_X = \left(\frac{X_E}{Z_E}\right).2373 + \left(\frac{Y_E}{Z_E}\right).0456 - X_E$$
$$U_Y = \left(\frac{Y_E}{Z_E}\right).4245 + \left(\frac{X_E}{Z_E}\right).1106 - Y_E$$

These yield first order conditions of

$$\left(\frac{Y_E}{Z_E^2}\right)(.2373 - .0455) = \left(\frac{X_E}{Z_E^2}\right)(.4245 - .1106) = 1$$
$$\left(\frac{Y_E}{Z_E^2}\right).1918 = \left(\frac{X_E}{Z_E^2}\right).3139 = 1$$
$$Y_E.1918 = X_E.3139$$
$$Y_E \approx X_E 1.6365$$

Thus we find $X_E \approx 0.0452$, $Y_E \approx 0.0739$. Placing these in our expected payoff function gives $U_X \approx 0.0731$, $U_Y \approx 0.2315$.

Combining cases 1 and 2 gives us Proposition 1, while comparing the expected payoffs for each player across case 2 gives us Proposition 2. □

6.2 Appendix B: The Other Sequential Contests

To explain the other types of sequential contests here we will go through the details of one case. The other cases are similar and details of these cases can be found in the working paper version.[12]

We will provide an analysis of the Hex contest involving two rounds where two cells are contested in each round. This case will require a solution to the expected payoffs of each player if only 2

[12] http://xythos.lsu.edu/users/mwiser1/Hex

cells remain, they are being contested simultaneously, and neither cell has been rendered irrelevant by the first round. In such a case, one player requires winning both cells for victory, while the other player requires winning either cell for victory. To illustrate, we will take take the subgame where player X needs both the North and East cells. In this subgame, player X's expected payoff is $U_X(\cdot|\cdot, R_2) = \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right) - X_N - X_E$. As player Y wins this subgame as long as player X does not win both the North and East, which occurs with probability $1 - \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right)$, player Y has an expected payoff of $U_Y(\cdot|\cdot, R_2) = (1 - \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right)) - Y_N - Y_E$. Taking derivatives, we obtain the following set of first order conditions

$$\begin{aligned}
\frac{\partial U_X}{\partial X_N} &= \left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_E}{Z_E}\right) - 1 &= 0\\
\frac{\partial U_X}{\partial X_E} &= \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_E}{Z_E^2}\right) - 1 &= 0\\
\frac{\partial U_Y}{\partial Y_N} &= \left(\frac{X_N}{Z_N^2}\right)\left(\frac{X_E}{Z_E}\right) - 1 &= 0\\
\frac{\partial U_Y}{\partial Y_E} &= \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E^2}\right) - 1 &= 0
\end{aligned}$$

These may be solved simultaneously to obtain $X_N = Y_N = X_E = Y_E = \frac{1}{8}$, so player X has an expected payoff of 0, while player Y has an expected payoff of $\frac{1}{2}$. Thus, if both cells in the two cell subgame are relevant, the player who needs to win both has an expected payoff of 0, while the player that needs to win at least one of two cells has an expected payoff of $\frac{1}{2}$. If only one of the two cells is relevant, both players have an expected value in the subgame of $\frac{1}{4}$, as seen in Appendix A.

As we now have the expected values for the players in the final round, we can now solve for the first round. However, we will need to break this case into three subcases, those where East and West comprise the first round, where North and South comprise the first round, and all other combinations of two cells comprising the first round. The results for the other cases are summarized in the final table.

Case 3: East and West as the First Round If player X wins both the East and West in the initial round, she will complete a winning set with either North or South, and thus X will have an expected payoff of $\frac{1}{2}$, while Y will have an expected payoff of 0 in the ensuing second round. If players X and Y split East and West, only one of North and South will matter, and thus each player has an expected payoff of $\frac{1}{4}$ in the second round. Thus, player X has an overall expected payoff of:

$$\left(\frac{X_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{2} + \left(\frac{X_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{4} + \left(\frac{Y_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{4} - X_E - X_W$$

As East and West are symmetric, as well as players X and Y, by logic similar to Lemma 1, we know that $X_E = X_W = Y_E = Y_W$. Thus, taking the derivative with respect to X_E gives us the following

$$\left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{2} + \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{4} - \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{4} - 1$$
$$= \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{Y_E}{Z_E}\right)\frac{1}{2} - 1 = \left(\frac{Y_E}{Z_E^2}\right)\frac{1}{4} - 1 = \frac{1}{4X_E}\frac{1}{4} - 1$$

Setting this equal to zero gives us a solution of $X_E = X_W = Y_E = Y_W = \frac{1}{16}$, and using this yields expected payoffs for each player of $\frac{1}{8}$.

Type	**Order**	$E[U_X]$	$E[U_Y]$	$E[U_A]$
4	NESW	.125	.125	.75
3-1	NSW-E, NSE-W	.13	.13	.74
3-1	EWN-S, EWS-N	.1563	.1563	.6875
2-2	NE-WS, WS-NE	.3396	.1793	.4811
2-2	SE-NW, NW-SE	.1793	.3396	.4811
2-2	NS-EW	.125	.125	.75
2-2	EW-NS	.125	.125	.75
1-3	N-ESW, S-ENW	.125	.125	.75
1-3	E-NSW, W-NSE	.2292	.2292	.5417
2-1-1	EW-N-S, EW-S-N	.1328	.1328	.7344
2-1-1	NE-S-W, SW-N-E, NE-W-S, SW-E-N	.1583	.1471	.6947
2-1-1	NW-S-E, SE-N-W, NW-E-S, SE-W-N	.1471	.1583	.6947
2-1-1	NS-E-W, NS-W-E	.125	.125	.75
1-2-1	N-EW-S, S-EW-N	.1797	.1797	.6719
1-2-1	N-ES-W, S-WN-E	.1006	.2269	.6725
1-2-1	S-EN-W, N-WS-E	.2269	.1006	.6725
1-2-1	E-NS-W, W-NS-E	.2368	.2368	.5265
1-2-1	E-NW-S, W-SE-N	.2040	.3372	.4588
1-2-1	W-NE-S, E-SW-N	.3372	.2040	.4588
1-1-2	N-S-EW, S-N-EW	.1797	.1797	.6719
1-1-2	N-E-SW, S-W-NE	.1006	.2269	.6725
1-1-2	N-W-SE, S-E-NW	.2269	.1006	.6725
1-1-2	E-S-NW, W-N-SE	.0933	.2003	.7064
1-1-2	E-N-SW, W-S-NE	.2003	.0933	.7064
1-1-2	E-W-NS, W-E-NS	.125	.125	.75
1-1-1-1	N-(E,W,S), S-(E,W,N)	.1797	.1797	.6719
1-1-1-1	E-W-N-S, E-W-S-N, W-E-N-S, W-E-S-N	.1406	.1406	.7188
1-1-1-1	E-N-W-S, W-S-E-N, E-N-S-W, W-S-N-E	.0731	.2315	.6954
1-1-1-1	W-N-E-S, E-S-W-N, W-N-S-E, E-S-N-W	.2315	.0731	.6954

Type indicates the number of cells contested in each round

Table 2: Expected Payoffs For Each Player and Seller Under All Structures