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A Question of Fundamental Methodology: Reply to Mikhail Katz and His Coauthors

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Abstract

This paper is a response by several historians of mathematics to a series of papers published from 2012 onwards by Mikhail Katz and various co-authors, the latest of which was recently published in the *Mathematical Intelligencer*, "Two-Track Depictions of Leibniz's Fictions" (Katz, Kuhlemann, Sherry, Ugaglia, and van Atten, 2021). At issue is a question of fundamental methodology. These authors take for granted that non-standard analysis provides the correct framework for historical interpretation of the calculus, and castigate rival interpretations as having had a deleterious effect on the philosophy, practice, and applications of mathematics. Rather than make this case by reasoned historical argument, however, these papers proceed largely by piecemeal confutation of isolated quotations of their opponents taken out of their argumentative context, juxtaposed with extracts from the primary sources that are again decontextualized and presented as testifying to the unique viability of the nonstandard interpretation. Instead of attempting a full rebuttal, in this paper we limit ourselves to showing how a line of argument that has been a central theme of Katz and co-authors since 2012, concerning the so-called "A" and "B" tracks, has undergone a radical change of meaning between 2012 and 2021, finally attaining a full-blown inconsistency in the 2021 paper.

In the last ten years or so, Mikhail G. Katz has published with various co-authors more than 60 papers in the history and philosophy of infinitesimal mathematics.[1] These papers are part of a sustained programme to rewrite the history of the calculus from a point of view where non-standard analysis is assumed to give the correct framework for historical interpretation. As described in Bair et al. (2013a, p. 887) the aim is "a re-evaluation of the history and philosophy of mathematics, analysing the shortcomings of received views, and shedding new light on the deleterious effect of the latter on the philosophy, the practice, and the applications of mathematics". Here the "received view", which has had such a "deleterious" effect, is presented as that of most historians of mathematics in the domain under consideration. Scholars who have not adopted the new non-standard framework are vilified as being "against" infinitesimals.[2] Needless to say, this is a gross oversimplification. Whether infinitesimal methods are advocated by the historical actors has not been disputed by any of the historians of mathematics these authors criticize, whose concern is to articulate how these methods were understood by their proponents.

What is at stake here is a question of fundamental methodology in the history of mathematics. For the structure of the papers by Katz and co-authors is always the same: it consists mainly of intemperate criticism of the supposed "received views" based on chosen extracts from the said historians of mathematics and a list of chosen extracts from the primary sources, most of the time decontextualized and presented as testifying that the nonstandard interpretation is the only one able to provide a faithful account of these sources. Many examples could be given, involving different historians of mathematics, and different pieces of the history of analysis. But what is relevant is not who these scholars are or which aspects of this history are concerned, nor whether the theses criticized are in fact well argued for, plausible, or questionable. The delicate point is rather with the form of these criticisms. They do not consist in reasoned historical argument, but rather in piecemeal confutation of isolated quotations of their opponents taken out of context, whose sole aim is to show them as enemies of the group's understanding of infinitesimals. The aim is not to contribute to historical clarification, nor even to rectify possible shortcomings or errors, but to put various scholars in the pillory—with accompanying abusive epithets—for not enthusiastically recognising the pervasive presence of non-standard analysis in the history of mathematics.

For a long time, we have refrained from sending "replies" underlining the various shortcomings of these papers, because we thought that the debate should mainly consist in rational arguments and studies of implications of newly discovered documents. But, over the years, it became clearer and clearer that our interlocutors do not care much about rational discussion and scientific dialogue from different perspectives, but seek rather to disparage their alleged enemies, meanwhile changing their views in an inconsistent way in order to accommodate the various objections they receive. Our main concern here is to draw attention to this strategy, which we consider unacceptable in a properly rational debate. The latest example of that approach is provided by the paper recently published in the *Mathematical Intelligencer*, "Two-Track Depictions of Leibniz's Fictions" (Katz, Kuhlemann, Sherry, Ugaglia, and van Atten, 2021). Rather than attempting to counter it point by point, which

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[1] A full list of these papers is given by Katz on the website https://u.math.biu.ac.il/~katzmik/infinitesimals.html (accessed May 31, 2022).

[2] A long list of scholars who have conducted their research without having adopted the non-standard analytical standpoint is proudly exhibited by Katz on his webpage referenced above ("List of Critics"). As can be seen, all that is important is for these "critics" to be identified and rebutted.

would involve us in lengthy argument and detailed historical contextualization (and also probably encourage more of the same scatter-gun responses from Katz and his collaborators), we decided to focus instead only on its main line of argument concerning the so-called “A” and “B” tracks, a central theme of Katz and co-authors since 2012. We will limit ourselves to demonstrating that these designations of theirs, and related theses such as the contradictory nature of infinitesimals in Leibniz, have undergone a radical change of meaning between 2012 and 2021, finally attaining a full-blown inconsistency.

In the first series of papers published by M. Katz and al., “A” and “B” were meant to describe “separate methodologies in Leibniz”, one based on exhaustion arguments à la Archimedes, and the second relying “more directly on infinitesimals” à la Bernoulli (2013b, p. 575). The two methodologies were presented as corresponding to two pictures of the continuum:

> “• an A-continuum (for Archimedes), and
>
> • a B-continuum (for Johann Bernoulli, following Leibniz).
>
> The former is a ‘thin’ continuum, exemplified by what are called today the real numbers; the latter is a ‘thick’ continuum incorporating infinitesimals.” (2013b, p. 598)

As can be seen from this quotation, it was not a matter of conflicting interpretations between scholars, but of methodologies existing *in* Leibniz. The main claim was that Leibniz shared with Bernoulli a certain view of the continuum as consisting of infinitesimal numbers in addition to ordinary (or “assignable”) numbers. We may note in passing that this already involves anachronism at odds with a properly historical approach. For Leibniz did not conceive of numbers as constituting a continuum, nor did he allow infinite sets (infinite wholes, in his terminology). In the case he discussed with Bernoulli in 1698-1699, the question was rather about whether there exists an infinitieth term in an infinite series, which in the case of a decreasing series would stand for an infinitesimal quantity. Bernoulli insisted that there would indeed be such an infinitieth (although not necessarily last) term, thus entailing the existence of an infinitesimal. According to Katz et al. this entails a conception of infinite series as consisting of an infinite sequence of standard numbers followed by an infinitesimal part. Bernoulli was not very clear about what happens after the infinitieth term, but Katz et al. (2013b, p. 599) explained how this can be conceived by giving a construction using nonstandard analysis. In that paper, as can be seen in the summary above, Leibniz and Bernoulli were presented as being in strong agreement. Infinitesimals were conceived as numbers, and the view of numerical sequences described by nonstandard analysis was presented as compatible with the philosophers’ views. The assertion that infinitesimals were numbers was even presented as a feature of crucial importance:

> “the addition or concatenation of infinitesimals (of the same dimension) is no more difficult to conceive of than adding or concatenating finite magnitudes. This is especially important, because it allows one *to represent infinitesimals by means of numbers* and so apply arithmetic operations to them. This is the *fundamental difference* between the infinitary methods of Archimedes (and later Cavalieri) and the infinitary methods of Leibniz and his followers.” (2013b, p. 575; our emphasis).

Many objections have been raised to this general picture, the most obvious one being that it testifies to a non-historical and decontextualized methodology. Indeed, we possess the documents in which Leibniz discusses these questions with Bernoulli and everybody can see, by just reading them, that the two mathematicians strongly *disagreed* on all the issues presented above. There is no way that one can claim that Bernoulli defended a certain picture of the continuum "following Leibniz". Accordingly, there is no B-methodology *stricto sensu* in Leibniz. Leibniz's main argument is that it is not possible to treat infinitesimals as existing entities because this amounts to the introduction of an infinite number, which he takes to be a contradictory notion.[3]

After receiving this objection, Katz and al. completely changed their position, but without acknowledging this change, and as if it did not ruin their previous argument. In the paper published in this journal, which is supposed to give a survey of a long-lasting debate, "A" and "B" are no longer presented as a pair of methodologies in Leibniz, but as positions endorsed *by commentators* to understand the term "fiction" in Leibniz. "B" does not refer to Bernoulli any longer, but to the views endorsed by a generic interpreter named "Bob" (alias Katz and his co-authors). And here is the funny part: Bob now insists that Leibniz *disagreed* with Bernoulli and that this disagreement is based on the fact that an infinite number is a contradictory notion (2021, p. 1)! In order to "save" his position, Bob, who cannot be Bernoulli anymore, introduces a new distinction: one should not confuse, he says, infinite (and infinitesimal) number and infinite (and infinitesimal) magnitude – with infinite magnitude exemplified by the idea of a *linea infinita terminata*. A crucial point is that the second pair are supposed to be non-contradictory objects. According to Bob, this is now supposed to be the "proper" meaning of fiction in Leibniz.

We will not enter in the discussion of this new claim here,[4] but just content ourselves with pointing to a previous paper published by Bob, alias Sherry and Katz, in *Studia Leibnitiana* (2012). The aim of the paper was to explain the "proper meaning" of the term fiction in Leibniz in the direct line of the argument developed in the paper published in your journal. But here is how it goes:

> "Infinitesimals and imaginary roots are not, therefore, objects of mathematical science, and their fictional status rests ultimately upon this absence of objectivity. These concepts, *because they contain a contradiction*, prevent us from imagining objects in accordance with their definitions" (2012, p. 179; our emphasis)

At the time, this went along with the following claim: "because they involve contradictions, Leibniz rejects infinitesimals, along with negatives and imaginaries as objects of mathematical science" (2012, p. 190). Moreover, the conclusion of the paper made it clear that the contradiction related to infinitesimals was for Bob of the same type as the one based on infinite cardinalities, in concordance with the fact that at this time Bob thought that infinitesimals were truly represented by numbers. Nonstandard techniques of the kind

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[3] This had been a well-known fact amongst Leibniz scholars for some time, see for example Herbert Breger, "Das Kontinuum bei Leibniz", in A. Lamarra (ed.), *L'infinito in Leibniz : Problemi e terminologia*, Rome, Edizioni dell'Ataneo, 1990, pp 53–67, and in the same volume: George MacDonald Ross, "Are there real Infinitesimals in Leibniz's Metaphysics ?" (pp. 125-141); see also O. Bradley Bassler. "Leibniz on the Indefinite as Infinite", *The Review of Metaphysics*, vol. 51, no. 4, 1998, pp. 849–74.

[4] Arthur and Rabouin will dedicate a specific study to this, providing several sources in which Leibniz explicitly claimed that *lineae infinitae terminatae* are contradictory entities.

invented by Robinson were presented as a way to extend the realm of fictions related to infinite cardinalities by dealing with them in a consistent logical theory (how the two claims were compatible is not clear, but that is not the issue here).[5]

But if Leibnizian infinitesimals have nothing to do with infinite cardinalities, if they are non-contradictory objects, as claimed by Bob2021, what he is criticizing is not the A-interpretation, but the B-interpretation in the sense of Bob2012. We simply do not understand what the "B" position is here, as we do not understand whether or not Leibniz was supposed to have shared the views of Bernoulli (as claimed by Bob2013 but refuted by Bob2021). What we suspect is that "A" and "B" are mere artefacts invented to hide inconsistent positions and we recommend that Bob 2021 engage in a polemic with his own self from 2012-2013 directly, rather than confronting historians of mathematics with six papers of this sort a year.

Finally, we note that the interest of Katz et al. in infinitesimals is not confined to reaching an understanding of the ideas of Leibniz and Johann Bernoulli, but extends to suggesting a tradition that includes Euler and reaches to Cauchy, before being shut down by Weierstrass, upon which the "leading historians" are supposed to have anachronistically relied. This extension rests on the confused views about Leibniz and Bernoulli described above, but it aims at promoting an interpretation of Cauchy's analysis as relying on the kinds of intuitions that can be made rigorous in a modern non-standard infinitesimal-based theory of analysis. Here the A-track is defined as "the Weierstrassian approach (in the context of an Archimedean continuum)", and the B-track is "the approach with indivisibles and/or infinitesimals (in the context of what we will refer to as a Bernoullian continuum)". In order to protect themselves from what would be an absurd claim that Cauchy possessed the rigorous machinery of modern non-standard analysis, the authors supply the concept of hidden lemmas. These are provable results in non-standard analysis that have intuitive, if unrigorous, meanings; the suggestion is that Cauchy used these in his work in the B-track. This is to wrench Cauchy's ideas out of their historical context and misinterpret what he meant by infinitesimal in a way that is somewhere between tendentious and incorrect, as Gilain, Lützen, Schubring, and others have shown.[6]

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[5] "Infinite sets have the property that they can be shown to be equinumerous with a proper part of themselves. This contradicts the principle that the whole is greater than the part, a principle to which Leibniz was firmly committed. Infinite aggregates would, therefore, be fictions from a Leibnizian perspective. Robinson's creation of a consistent theory of infinitesimals consists, roughly, in defining the hyper-reals by set theoretic means and showing that any first order statement that holds for a real function holds for the hyper-real natural extension of that function. (…). To be sure, the demonstration of this result requires resources unavailable to Leibniz. But for Robinson those resources occupy the same role as infinitesimals and imaginary roots did for Leibniz." (2012, pp. 191-192).

[6] See, for example, Gert Schubring, "Review of Bair, J. et al., Cauchy's work on integral geometry, centers of curvature, and other applications of infinitesimals", (in *Real Anal. Exchange*, 45 (2020), no. 1, 127–149), Mathematical Reviews/MathSciNet, MR4196072.

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