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Buyer's Equilibrium with Capacity Constraints and Restricted Mobility: a Recursive Approach

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Abstract

We study a decentralized trading model as in Peters (1984), where heterogeneous market participants face a trade-off between price and trade probability. We present a novel proof of existence of a unique demand vector in Nash equilibrium, based on a recursive approach that exploits the monotonicity of matching functions.

Keywords: Nash equilibrium, recursive methods, existence, heterogeneity, matching

JEL: C70, D390, D490, E390

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1 Introduction

We study existence of equilibrium in the static market considered in [5], where an endogenous matching process partitions a countable population of heterogeneous sellers and homogeneous buyers in separate trading groups of different sizes. We offer a new proof of existence and uniqueness of an equilibrium demand vector.

The model is a sequential game of complete information. Each seller has a fixed amount of a good that buyers desire in fixed amounts. Buyers can compensate sellers by direct utility transfers. The interaction evolves across three stages. First, sellers simultaneously and independently advertise a price or, equivalently, a utility level promised to any buyer who trades with the seller. In the second stage, buyers see all promised utilities and then, simultaneously and independently, decide to visit one seller; in this sense, the matching process is endogenous. Frictions arise in the third stage, when matches are realized and sellers may trade only if they have been visited by at least one buyer. Because sellers may be visited by more than one buyer but are capacity constrained, they ration buyers at random. Because buyers visit exactly one seller and cannot coordinate their visits, some sellers may remain unmatched. Hence, market participants face a trade-off between price and probability of trade.¹

Existence of equilibrium is usually studied in two steps. First, the equilibrium demand vector is determined in the “buyers’ game,” i.e., the continuation game where buyers choose which seller to visit, given a vector of promised utilities. Second, the equilibrium distribution of promised utilities is determined by studying the “sellers’ game,” i.e., the initial stage where sellers announce

¹Models of this type have been used to study issues in IO, eg., [6, 8], and form the basis of the so-called “directed search” literature, which mostly studies labor issues, e.g., [1, 2, 3, 4].

utilities taking as given the equilibrium demand vector.

This paper is about existence of equilibrium in the buyers' game: it develops a new proof of existence and uniqueness of an equilibrium demand vector. The proof is based on a recursive (or iterative) approach, and is offered as an alternative to the one developed in [5], which is based on a fixed-point argument. The proof provides an algorithm helpful to compute the equilibrium. Such methodological contribution deepens our understanding of equilibrium and present a novel avenue of analysis to study heterogeneous markets with endogenous matching.

2 The model

Consider an economy with finitely many players. Following the notation in [5], let $\mathcal{J} = \{1, \dots, J\}$, with $J \geq 2$ be the set of heterogeneous sellers where heterogeneity can be multidimensional (in which case we would need an index to summarize it). Let $\mathcal{I} = \{1, \dots, I\}$ with $I \geq 2$ be the set of homogeneous buyers.

Each seller has one indivisible object, possibly differentiated, from which she derives no utility. Each buyer desires to consume one object; let $r_j > 0$ denote buyers' reservation value from consuming object j . Buyers, who have linear preferences² and sellers, who may be risk-averse, play a sequential game of complete information, over three stages. In the first stage, sellers simultaneously and independently announce a price for the object they have; full commitment to the announced price is assumed. Let $\mathbf{v} = (v_1, \dots, v_J) \subset \mathbb{R}_+^J$ denote a "promised utility," i.e., v_j is the indirect utility for a buyer who

²The work in [7] considers risk-averse buyers.

purchases the good offered by seller j at the price posted by that seller. Hence, \mathbf{v} is the strategy profile of all sellers. We think of v_j as a decreasing function of the price posted by seller j , hence we work with v_j instead of the price. We say that seller j “posts” $v_j \in [v_j, \bar{v}_j] \subset \mathbb{R}_+$.

In the second stage, buyers observe \mathbf{v} and then simultaneously and independently choose to visit a single seller. Let $\boldsymbol{\pi}(\mathbf{v}) = (\pi_1(\mathbf{v}), \dots, \pi_J(\mathbf{v})) \in \Delta^{J-1}$ denote the strategy profile of buyers when they act symmetrically, i.e., $\pi_j(\mathbf{v})$ denotes the probability that any buyer chooses to visit seller $j = 1, \dots, J$ after observing \mathbf{v} . We call a *match* an encounter between one seller and $i = 1, \dots, I$ buyers.

In the third stage, matches are realized and a trade process takes place in each match. Consider a match between a generic seller and i buyers; due to capacity constraints, at most one buyer trade with the seller. Let $\rho(i) \leq 1$ denote the probability that a generic buyer in the match trades with the seller. It is assumed that the assignment rule ρ is match- and buyer-independent. Conditional on being visited by $i \geq 1$ buyers, the seller trades with probability $i\rho(i) \leq 1$. That is, the seller trades with at most one buyer (due to capacity constraints) and may not trade at all.³

A payoff of zero is given to players who are unmatched or do not trade. If trade takes place, then the buyer receives the promised utility v_j and transfers utility $r_j - v_j$ to seller j . Thus, we can think of $r_j - v_j$ as a price. Players max-

³The work in [5] considers random rationing $\rho(i) = \frac{1}{i}$, so a buyer with no competitors trades with certainty. One can also think about cases in which buyers who are offered the good do not purchase it with some exogenous probability $\theta \in [0, 1)$. This can occur for instance if buyers’ valuations r_j for the good sold by seller j are i.i.d. and are realized after visiting the seller. Here, trade may fail to take place and we have $\rho(i) = \frac{1-\theta^i}{i}$, for all $i = 1, \dots, I$. Or there may be a seller-specific shock preventing the execution of a trade; if this occurs with probability a_j at seller j , then the analysis that follows is unchanged and we simply redefine the promised utility as $v_j = a_j v_j$.

imize their unconditional payoff (or expected utility) without the possibility to communicate with each other and to coordinate their actions.⁴

3 The main result

We study strongly symmetric equilibrium, i.e., an outcome in which all players of the same type adopt identical strategies in and out of equilibrium. This is the focus in the literature; see [5]. Given symmetry, we study the behavior of a generic buyer.

3.1 Preliminaries

Consider any seller. Let $q^i(I, \pi)$ denote the probability that $0 \leq i \leq I$ buyers visit this seller, when buyers identically choose to visit the seller with probability π . As a result of independent, simultaneous and symmetric choice of buyers, the distribution of buyers at any seller is binomial with parameters with π and I ; see [5]. Hence, $q^i(I, \pi)$ is a smooth of function of π , and it satisfies

$$q^i(I, \pi) := \frac{I!}{i!(I-i)!} \pi^i (1-\pi)^{I-i}, \quad \text{for all } i = 0, \dots, I. \quad (1)$$

Let $\mathcal{M}(\pi)$ denote the *unconditional* probability that a seller trades, given π . Let $\mathcal{H}(\pi)$ denote the *conditional* probability that a buyer trades conditional on visiting this seller, when every other buyer visits that same seller with probability π . Hence, $\mathcal{H}(\pi_j)v_j$ is the buyer's *payoff* conditional on visiting

⁴The seller's payoff function is not needed for the results, hence is not specified. Clearly, seller's ex-post profit should decrease in the utility promised to the buyer; see [5].

seller j . We have

$$\begin{aligned}\mathcal{H}(\pi) &:= \sum_{i=0}^{I-1} q^i(I-1, \pi)\rho(i+1) \\ \mathcal{M}(\pi) &:= \sum_{i=1}^I q^i(I, \pi)i\rho(i).\end{aligned}\tag{2}$$

The literature typically assumes that for each π , (i) \mathcal{M} is C^2 , $\mathcal{M}' > 0$ and $\mathcal{M}'' < 0$; (ii) \mathcal{H} is C^2 , $\mathcal{H}' < 0$ and $\mathcal{H}'' > 0$; (iii) $\mathcal{H}(\pi)^{-1}$ is convex; (iv) $\mathcal{H}(\pi)v$ is quasiconcave. To prove our main result we only need monotonicity of \mathcal{H} ; one can show that the above assumption emerge endogenously in a symmetric outcome.⁵

3.2 Recursive equilibrium in the buyers' game

Consider the subgame where buyers choose sellers based on the posted promised utility vector \mathbf{v} , i.e., the “buyers’ game.” In symmetric equilibrium all buyers visit seller j with identical probability $\pi_j(\mathbf{v})$ (other, non-symmetric equilibria are possible; see [2]).

Definition 1. Given \mathbf{v} , a symmetric equilibrium in the buyers’ game is a vector $\boldsymbol{\pi}(\mathbf{v})$ such that: $\sum_{j \in \mathcal{J}} \pi_j(\mathbf{v}) = 1$; if $\pi_j(\mathbf{v}) > 0$ for $j \in \mathcal{J}$, then $\mathcal{H}(\pi_j(\mathbf{v}))v_j = \max_{k \in \mathcal{J}} \mathcal{H}(\pi_k(\mathbf{v}))v_k$.

If $\pi_j(\mathbf{v}) > 0$, then the buyers’ payoff from visiting that seller is no less than the payoff from visiting any other seller. Hence, if buyers randomize visits across sellers, then their payoffs from visiting any of these sellers are

⁵For completeness, we do so in an online Appendix. The properties of \mathcal{M} are derived by direct differentiation, with some rearrangement. Mathematical induction is used to derive the properties of \mathcal{H} and $1/\mathcal{H}$.

equivalent, i.e.,

$$\mathcal{H}(\pi_j(\mathbf{v}))v_j = \mathcal{H}(\pi_k(\mathbf{v}))v_k \quad \text{for all } j, k \in \Gamma(\mathbf{v}) := \{i \in \mathcal{J} | \pi_i(\mathbf{v}) > 0\}.$$

Given \mathbf{v} , the set $\Gamma(\mathbf{v})$ is the collection of sellers who are “in the market,” i.e., the set of sellers that may be visited by a buyer. The complementary set $\mathcal{J} \setminus \Gamma(\mathbf{v})$ contains “idle” sellers, i.e., sellers that post a specific utility v but are *certain* not to be visited by any buyer. Given Definition 1

$$\begin{aligned} \bar{H}(\mathbf{v}) &:= \mathcal{H}(\pi_k(\mathbf{v}))v_k && \text{for } k \in \Gamma(\mathbf{v}) \\ V(v_k) &:= \mathcal{H}(0)v_k \leq \bar{H}(\mathbf{v}) && \text{for } k \in \mathcal{J} \setminus \Gamma(\mathbf{v}). \end{aligned} \tag{3}$$

$\bar{H}(\mathbf{v})$ is the expected utility derived from visiting any sellers who is in the market, in buyers’ equilibrium; this payoff is *independent* of k for $k \in \Gamma(\mathbf{v})$. Instead, $V(v_k)$ denotes the expected utility derived from visiting idle seller k ; this out-of-equilibrium payoff depends on k .

Proposition 1. Fix $\mathbf{v} \neq 0$. There exists a unique equilibrium vector $\boldsymbol{\pi}(\mathbf{v})$ for the buyers’ game.

Proof. Fix $\mathbf{v} \neq 0$, and consider the choice $\boldsymbol{\pi}(\mathbf{v})$ of a generic buyer. The proof is in two parts. First, construct a demand distribution $\boldsymbol{\pi}$ through an iterative process. Second, prove that this equilibrium $\boldsymbol{\pi}$ is unique by means of contradiction.

Without loss of generality, let $v_1 \geq v_2 \geq \dots \geq v_J$ with $v_1 > 0$. Suppose the buyer considers sellers one at a time, according to an iterative process. Let $\eta_j^s(\mathbf{v})$ denote the probability that this generic buyer visits seller j when the first $s = 1, \dots, J$ sellers have been considered for a possible visit. Hence,

$\sum_{j=1}^s \eta_j^s(\mathbf{v}) = 1$, so that $\eta_1^1(\mathbf{v}) = 1$. Since promised utilities \mathbf{v} are fixed, we omit \mathbf{v} as an argument when understood.

For the initial step of the iterative decision process, define η^1 as the J -dimensional vector

$$\boldsymbol{\eta}^1 := (\eta_1^1, 0, \dots, 0).$$

This means that when only seller one is being considered for a visit, then the buyer goes there with probability 1 because $v_1 > 0$. Since the sequence v_j is decreasing, for $2 \leq s \leq J$ recursively define

$$\boldsymbol{\eta}^s(\boldsymbol{\eta}^{s-1}) = \begin{cases} (\eta_1^s, \dots, \eta_s^s, 0, \dots, 0) \text{ s.t. } \mathcal{H}(\eta_j^s)v_j = \mathcal{H}(\eta_s^s)v_s, j \leq s, & \text{if } V(v_s) > \mathcal{H}(\eta_1^{s-1})v_1 \\ (\eta_1^{s-1}, \dots, \eta_{s-1}^{s-1}, 0, \dots, 0), & \text{otherwise} \end{cases}$$

Start by noting that $\boldsymbol{\eta} = \boldsymbol{\eta}^J$ satisfies Definition 1, therefore $\boldsymbol{\eta}$ is a candidate equilibrium vector of buyers' choices. It should be clear that continuity of \mathcal{H} , $\mathcal{H}' < 0$ and the recursive construction of $\boldsymbol{\eta}$ ensure that $\boldsymbol{\eta}$ exists. We also have that

$$\mathcal{H}(\eta_j^{s-1})v_j \leq \mathcal{H}(\eta_j^s)v_j \text{ for } j = 1, \dots, s-1.$$

This means that the vector $\boldsymbol{\eta}$ maximizes the payoff of the generic buyer.

We claim that if any other vector of buyers' choices $\boldsymbol{\pi}$ exists that satisfies Definition 1, then $\boldsymbol{\pi} = \boldsymbol{\eta}$. That is, a unique equilibrium demand distribution exists. To prove it, suppose by means of contradiction that $\boldsymbol{\pi} \neq \boldsymbol{\eta}$. Let $\Gamma_{\boldsymbol{\pi}}(\mathbf{v})$ and $\Gamma_{\boldsymbol{\eta}}(\mathbf{v})$ denote the sets of sellers who are in the market associated to $\boldsymbol{\pi}$ and $\boldsymbol{\eta}$. There are three cases to consider.

Case 1: $\Gamma_{\boldsymbol{\pi}} = \Gamma_{\boldsymbol{\eta}} = \Gamma$

If $\boldsymbol{\pi} \neq \boldsymbol{\eta}$ then $\pi_i \neq \eta_i$ for some $i \in \Gamma$. Without loss of generality, consider

$\pi_i < \eta_i$. If $\mathcal{H}(\pi_i)v_i > \mathcal{H}(\eta_i)v_i$, then

$$\mathcal{H}(\pi_j)v_j > \mathcal{H}(\eta_j)v_j \quad \text{for all } j \in \Gamma = \Gamma_\pi = \Gamma_\eta$$

because π and η are assumed to satisfy Definition 1 and, in particular the indifference condition $\mathcal{H}(\pi_i)v_i = \max_{j \in \mathcal{J}} \mathcal{H}(\pi_j)v_j$.

Now note that $\mathcal{H}' < 0$ implies $\pi_j < \eta_j$ for all $j \in \Gamma$. Hence, $\sum_{i \in \Gamma_\pi} \pi_i = 1 < \sum_{i \in \Gamma_\eta} \eta_i$.

This gives us the desired contradiction. Hence $\pi = \eta$.

Case 2: $\Gamma_\pi \subsetneq \Gamma_\eta$

We consider only this inclusion because the other way around simply means changing the role of η and π . This case is studied in two steps. First we claim that $\eta_i < \pi_i$ for all $i \in \Gamma_\pi$; intuitively, if there are less idle sellers under η than π , then the probability of visiting any seller cannot be greater under η than π . Suppose by means of contradiction that $\eta_i \geq \pi_i$ for some $i \in \Gamma_\pi$. Then, $\eta_k \geq \pi_k$ for all $k \in \Gamma_\pi$ by the same argument developed in case 1 above (indifference). This gives us a contradiction because $\sum_{i \in \Gamma_\eta} \eta_i > 1 = \sum_{i \in \Gamma_\pi} \pi_i$. Hence $\eta_i < \pi_i$ for all $i \in \Gamma_\pi$.

Now we show that $\pi \neq \eta$ is impossible. Fix $i, j \in \Gamma_\eta$, $i \in \Gamma_\pi$, $j \in \mathcal{J} \setminus \Gamma_\pi$. Notice that the last two relations imply $\mathcal{H}(\pi_i)v_i \geq \mathcal{H}(0)v_j$, hence $v_i > v_j$. Now recall that $\eta_i < \pi_i$ and $\mathcal{H}' < 0$. Therefore we have the contradiction

$$\overbrace{\mathcal{H}(0)v_j > \mathcal{H}(\eta_i)v_i}^{\text{seller } j \text{ is in the market under } \eta} \geq \overbrace{\mathcal{H}(\pi_i)v_i \geq \mathcal{H}(0)v_j}^{\text{seller } j \text{ is idle under } \pi}.$$

Therefore $\pi = \eta$.

Case 3: Neither $\Gamma_\pi \subset \Gamma_\eta$ nor $\Gamma_\pi \supset \Gamma_\eta$

Consider sellers i and j such that $i \in \Gamma_\eta \setminus \Gamma_\pi$ and some $j \in \Gamma_\pi \setminus \Gamma_\eta$. Without

loss of generality choose i and j so that $v_j \leq v_i$. Since $j \in \Gamma_\pi$ and $i \in \mathcal{J} \setminus \Gamma_\pi$, we have

$$\overbrace{\mathcal{H}(\pi_j)v_j \geq \mathcal{H}(0)v_i}^{\text{seller } i \text{ is idle under } \pi}, \text{ i.e. } v_j > v_i.$$

This gives us a desired contradiction which completes the proof that $\pi = \eta$. \square

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Supplementary Material: Endogenizing the regularity properties

Key properties of matching and payoff functions are usually assumed in the literature to prove existence of equilibrium in the entire game.¹

Here, we show that such properties emerge endogenously in a symmetric outcome. In doing so, we consider $\rho(i) = \frac{1-\theta^i}{i}$, for all $i = 1, \dots, I$, as it generalizes the typical case where $\theta = 0$.

Proposition. Let $q^i(I, \pi) := \frac{I!}{i!(I-i)!} \pi^i (1-\pi)^{I-i}$ for all $i = 0, \dots, I$ and $\rho(i) = \frac{1-\theta^i}{i}$, for all $i = 1, \dots, I$. For each $\pi \in [0, 1]$, we have

- $\mathcal{M}(\pi)$ is twice continuously differentiable, strictly increasing, and concave.
- $\mathcal{H}(\pi)$ is twice continuously differentiable, strictly decreasing, and convex.
- $\mathcal{H}(\pi)^{-1}$ is convex, i.e. $2(\mathcal{H}'(\pi))^2 - \mathcal{H}''(\pi)\mathcal{H}(\pi) \geq 0$
- $\mathcal{H}(\pi)v$ is quasiconcave.

Proof. Consider a generic seller.

The function $\mathcal{M}(\pi)$. Notice that

$$\begin{aligned} \frac{I\pi}{i+1} q^i(I-1, \pi) &= \frac{I\pi}{i+1} \cdot \frac{(I-1)!}{i!(I-1-i)!} \pi^i (1-\pi)^{I-1-i} \\ &= \frac{I!}{(i+1)!(I-(i+1))!} \pi^{i+1} (1-\pi)^{I-(i+1)} = q^{i+1}(I, \pi). \end{aligned}$$

From (2) in the paper, the functions $\mathcal{M}(\pi)$ and $\mathcal{H}(\pi)$ are twice continuously differentiable, because $q^i(I, \pi)$ is smooth in p for all i . Now notice

$$\begin{aligned} \mathcal{M}(\pi) &= \sum_{i=1}^I q^i(I, \pi) - \sum_{i=1}^I q^i(I, \pi)\theta^i = 1 - q^0(I, \pi) - \sum_{i=1}^I q^i(I, \pi)\theta^i \\ &= 1 - \sum_{i=0}^I q^i(I, \pi)\theta^i = 1 - \sum_{i=0}^I \binom{I}{i} (\pi\theta)^i (1-\pi)^{I-i} \\ &= 1 - (1 - (1-\theta)\pi)^I \end{aligned}$$

¹For a recent example consider properties (i)-(iii) in [1, Assumption 2].

where the last line follows from the binomial formula. It is clear that $\mathcal{M}'(\pi) > 0$ and $\mathcal{M}''(\pi) < 0$ by direct differentiation.

The functions $\mathcal{H}(\pi)$ and $1/\mathcal{H}(\pi)$. Without loss of generality, set $\theta = 0$. Thus

$$\begin{aligned}\mathcal{H}(\pi) &= \frac{\mathcal{M}(\pi)}{I\pi} = \frac{1 - (1 - \pi)^I}{I\pi} \\ \mathcal{H}'(\pi) &= \frac{1}{I\pi^2} [(1 - \pi)^{I-1}(1 - \pi + I\pi) - 1].\end{aligned}$$

- We show that $\mathcal{H}'(\pi) < 0$ for $I \geq 2$.
The proof is by induction on I . Define the term

$$A(\pi, I) := (1 - \pi)^{I-1}(1 - \pi + I\pi) - 1.$$

We have

$$\mathcal{H}'(\pi) = \frac{1}{I\pi^2} A(\pi, I)$$

Clearly $A(\pi, 1) \leq 0$. Consider the following induction hypothesis: $A(\pi, I) < 0$ for some $I \geq 2$. We need to prove that $A(\pi, I + 1) < 0$. We have

$$\begin{aligned}A(\pi, I + 1) &= (1 - \pi)^I(1 - \pi + I\pi + \pi) - 1 = (1 - \pi)^{I-1}(1 - \pi)(1 + I\pi) - 1 \\ &= (1 - \pi)^{I-1}(1 - \pi + I\pi - I\pi^2) - 1 < (1 - \pi)^{I-1}(1 - \pi + I\pi) - 1 \\ &= A(\pi, I)\end{aligned}$$

Hence $A(\pi, I) < 0$ for all $I \geq 1$. It follows that $\mathcal{H}'(\pi) < 0$.

- We show that $\mathcal{H}''(\pi) > 0$ for $I \geq 2$. By direct differentiation we have:

$$\mathcal{H}''(\pi) = \frac{\pi}{I\pi^4} \{A'(\pi, I)\pi - 2A(\pi, I)\}$$

where

$$A'(\pi, I) := -I(I - 1)(1 - \pi)^{I-2}\pi.$$

Hence, if $A'(\pi, I)\pi - 2A(\pi, I) > 0$, then $\mathcal{H}''(\pi) > 0$. The proof is by induction on I :

$$A'(\pi, I)\pi - 2A(\pi, I) = -I(I - 1)(1 - \pi)^{I-2}\pi^2 - 2(1 - \pi)^{I-1}(1 - \pi + I\pi) + 2$$

For $I = 2$ we have $A'\pi - 2A = 0$. For the induction hypothesis suppose $A'\pi - 2A > 0$ for some $I \geq 3$, that is

$$2[(1 - \pi)^{I-1}(1 - \pi + I\pi) - 1] < -I(I - 1)(1 - \pi)^{I-2}\pi^2$$

We need to show that this is true for $I + 1$, i.e.,

$$\begin{aligned} 2(1 - \pi)^I(1 - \pi + I\pi + \pi) - 2 &\leq -I(I + 1)(1 - \pi)^{I-1}\pi^2 \\ 2[(1 - \pi)^{I-1}(1 - \pi)(1 + I\pi) - 1] &\leq -I(I - 1)(1 - \pi)^{I-1}\pi^2 - 2I(1 - \pi)^{I-1}\pi^2 \\ 2[(1 - \pi)^{I-1}(1 - \pi + I\pi) - 1] &\leq -I(I - 1)(1 - \pi)^{I-1}\pi^2 \end{aligned}$$

By virtue of the induction hypothesis we have

$$2[(1 - \pi)^{I-1}(1 - \pi + I\pi) - 1] < -I(I - 1)(1 - \pi)^{I-2}\pi^2 < -I(I - 1)(1 - \pi)^{I-1}\pi^2$$

so the statement is true for $I + 1$.

- We prove $\mathcal{H}''(\pi)\mathcal{H}(\pi) - 2(\mathcal{H}'\pi)^2 < 0$.

Notice that

$$\begin{aligned} &\mathcal{H}''(\pi)\mathcal{H}(\pi) - 2(\mathcal{H}'\pi)^2 \\ &= \frac{\pi}{I\pi^4} \{A'(\pi, I)p - 2A(\pi, I)\} \frac{1 - (1 - \pi)^I}{I\pi} - 2 \left(\frac{1}{I\pi^2} \right)^2 [(1 - \pi)^{I-1}(1 - \pi + I\pi) - 1]^2 \\ &\propto [A'(\pi, I)\pi - 2A(\pi, I)] [1 - (1 - \pi)^I] - 2[A(\pi, I)]^2 \end{aligned}$$

We want to prove that

$$[A'(\pi, I)\pi - 2A(\pi, I)] [1 - (1 - \pi)^I] - 2[A(\pi, I)]^2 < 0.$$

Using the definition for A' and A this can be rearranged as

$$-(I - 1)\pi [1 - (1 - \pi)^I] < 2A(\pi, I)(1 - \pi).$$

Again, we use the proof by induction. Notice that for $I = 2$ the inequality above holds. For the induction hypothesis, suppose it also holds for some $I \geq 2$. Then we must show that it holds for $I + 1$, i.e.

$$-I\pi [1 - (1 - \pi)^{I+1}] < 2A(\pi, I + 1)(1 - \pi).$$

This inequality is rearranged as

$$\begin{aligned} -I\pi [1 - (1 - \pi)^I] + I\pi^2(1 - \pi)^I &< (1 - \pi)2[(1 - \pi)^{I-1}(1 - \pi + I\pi) - 1] \\ &= 2(1 - \pi)A(\pi, I) \end{aligned}$$

The left hand side can be rearranged as follows:

$$\begin{aligned}
& -(I-1)\pi [1 - (1-\pi)^I] - \pi [1 - (1-\pi)^I] + I\pi^2(1-\pi)^I \\
& \quad = -(I-1)\pi [1 - (1-\pi)^I] + \pi [-1 + (1-\pi)^{I-1}(1-\pi + I\pi) - I\pi^2(1-\pi)^{I-1}] \\
& \quad = -(I-1)\pi [1 - (1-\pi)^I] + \pi [A(I, \pi) - I\pi^2(1-\pi)^{I-1}],
\end{aligned}$$

therefore it is smaller than $2(1-\pi)A(\pi, I)$ because (i) $A(\pi, I) - I\pi^2(1-\pi)^{I-1} < 0$ due to $A(\pi, I) \leq 0$, and (ii) by induction hypothesis above. Hence $\mathcal{H}''(\pi)\mathcal{H}(\pi) - 2(\mathcal{H}'(\pi))^2 < 0$ for all $I \geq 2$.

Quasiconcavity of $\mathcal{H}(\pi)v$. If $\mathcal{H}(\pi)v = r \geq 0$, then the superior set is defined by $S_r = \{(\pi, v) \in [0, 1] \times [\underline{v}, \bar{v}] : \mathcal{H}(\pi)v \geq r\}$. The set S_r is convex because if $(\pi, v), (\pi', v') \in S_r$ then for $\lambda \in (0, 1)$, we have

$$\frac{r}{\mathcal{H}(\pi^\lambda)} \leq \lambda \frac{r}{\mathcal{H}(\pi')} + (1-\lambda) \frac{r}{\mathcal{H}(\pi)} \leq \lambda v' + (1-\lambda)v$$

where $\pi^\lambda = \lambda\pi' + (1-\lambda)\pi$. The first line follows from convexity of $\frac{1}{\mathcal{H}(\pi)}$; the second line follows from $(\pi, v), (\pi', v') \in S_r$. \square

References

- [1] Galenianos, M. and P. Kircher, (2012). On the Game-theoretic Foundations of Competitive Search Equilibrium. *International Economic Review*, 53 (1), 121.