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Marco Panza
Chapman University, panza@chapman.edu

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Review of Paolo Mancosu (ed.), *The Philosophy of Mathematical Practice*, Oxford Univ. Press, Oxford, New York, etc., 2008 (XII + 448 pages)

by Marco Panza

The philosophy of mathematical practice is not only a research topic, it is overall a disciplinary field that is extending its importance and attracting the interest of an increasing number of scholars from different communities.

The book edited by Paolo Mancosu provides a comprehensive and vivid account of the philosophy of mathematical practice, by showing it at work on different and multifarious topics, and by suggesting a substantial agenda for its development. This is also a momentous program for philosophy of mathematics as a whole, aiming at "bringing some fresh air" into it (1). Mancosu's book also gives voice to a community of scholars working in this area that does not only include the authors of the essays contained in it, but also a large group of scholars sharing their approach and often collaborating with them.

The structure of the book is conceived for these purposes. In addition to a short Preface (V-VI) and an Introduction (1-21), both by Mancosu himself, it includes sixteen chapters devoted to eight different topics: two chapters for each topic. The first chapter of each pair provides "a general introduction" to the relevant subject-area; the second consists of "a research paper in that area" (14).

Chapters 1-2 ("Visualizing in Mathematics", 22-42; "Cognition of Structure", 43-64), by Marcus Giaquinto, concern visualisation in mathematics. Giaquinto argues that visualisation is an irreplaceable ingredient of some proofs, since eliminating it from them would amount to getting essentially distinct proofs: though possibly supporting the same theorems, these new proofs would come with a reformulation of the latter producing a significant change. The research essay discusses the way in which visual representation allows "our cognitive grasp of structures" (43). It mainly concerns small finite structures, but also explains how visualisation can help in knowing simple infinite structures.

Chapters 3-4 ("Diagram-Based Geometric Practice", 65-79; "The Euclidean Diagram (1995)", 80-133), by Kenneth Manders, address the role of diagrams in some forms of geometric argumentation, especially Euclid's. Chapter 4 reproduces a paper, written in 1995, which, though never published hitherto, largely circulated and had an important role in promoting much work on this topic. Chapter 3 provides a survey of this work and sketches some "tasks for the future" (75). According to Manders, diagrams are an indispensable part of Euclid's arguments (namely those of books I and III of the *Elements*): the "verbal part" of these arguments "consists of a reason-giving ordered progression of assertions, each with the surface form of an ascription of a feature to the diagram" and each step in this progression is "licensed by attributions either already in force in the discursive text or made directly based on the diagram" (86-87). Both the attributions and the corresponding attributes can be either "exact" or "co-exact": "exact attributes are those which, for at least some continuous variations of the diagram, obtain only in isolate cases"; "co-exact attributes are those conditions which are unaffected by some range of every continuous variation of a specified diagram" (92). Manders' basic point is that whereas exact attributions can only be licensed by appropriate previous entries in the discursive text, diagrams play an indispensable role in licensing many co-exact attributions.

Chapters 5-6 ("Mathematical explanation: Why it Matters", 134-150; ("Beyond Unification", 151-178), by Paolo Mancosu, and by Johannes Hafner and Paolo Mancosu, respectively, are about mathematical explanation. Chapter 5 offers a survey of the recent discussion on the topic, by distinguishing mathematical explanation of scientific facts from explanation within mathematics itself. It is especially concerned with Mark Steiner's and Philip Kitcher's models. Steiner's model specifically concerns mathematics. According to it, a mathematical explanation of a scientific fact obtains when the relevant argument is such that an explanation within mathematics is got if the specific scientific part of the argument is removed. Its crucial idea is that mathematical proofs are explanatory whenever they involve a property (termed "characteristic property") which is "unique " of the relevant entities or structures "within a family or domain" of entities or structures (143). The main idea of Kitcher's model (which is also intended to apply to scientific explanation, and develops and crucially transforms an idea of Michael Friedman) is that explanation is got through unification by generalisation: a scientific argument is all the more explanatory the smaller number of general argument patterns it applies. Mancosu's discussion is critical. His criticism is mainly supported by the argument offered in chapter 6, where a particular case study from real algebraic geometry is considered. His basic point is that mathematical explanation is a matter of pluralism: no unique and general model can hope to account for all its possible forms; only a detailed analysis of mathematical practice can reveal some of these forms, and possibly, as Hafner and Mancosu have argued in previous publications, classify them.

Chapters 7-8 ("Purity as an Ideal of Proof", 179-197; "Reflections on the Purity of Method in Hilbert's *Grundlagen der Geometrie*", 198-255) by Michael Detlefsen and Michael Hallet, respectively, are devoted to purity conceived as an ideal of proofs and methods. Their concern is mainly historical. Chapter 7 describes different forms taken by this ideal, by insisting on the topical conception of purity: proofs should not transcend the topic of the corresponding theorem. In the most extreme form, this ideal requires that proofs only rely on the conceptual resources needed for understanding the statements of the corresponding theorems. Detlefsen discusses the way this ideal was addressed by Aristotle, Leibniz, Bolzano, Gauss, Dedekind, Frege, and Hilbert, by also considering more modern conceptions. Hilbert's case provides the subject matter of chapter 8. According to Hallett, Hilbert's is mainly concerned with a critical discussion of purity: this depends on appropriateness, but appropriateness cannot depend on "intuitive or informal assessment" (248), to the effect that, typically, "higher mathematics" is appropriate "to instruct or adumbrate intuition, or, at the very least to instruct us about it and what it entails" (249).

Chapters 9-10 ("Mathematical Concepts and Definitions", 256-275; "Mathematical Concepts: Fruitfulness and Naturalness", 276-301), by Jamie Tappenden, concern fruitfulness and naturalness of concepts and definitions. The former propound a principle extracted by Arnauld and Nicole's *Logique*: "nothing is more important in science than classifying and defining well […][but] it depends much more on our knowledge of the subject matter being discussed than on the rules of logic" (256). This is not against logic. It is rather in favour of the necessity of detailed and context-driven analyses as appropriate supports for judging whether a definition is the "right" one. Two examples are considered: the case of Legendre symbol and that of the appropriate definition of prime numbers. Legendre symbol denotes a function defined on natural numbers whose consideration allows a compact statement of the law of quadratic reciprocity. The question is whether the introduction of this function only results in an advantage of simplicity or it has epistemic virtues. A natural number greater than 1 is prime if and only if, whenever it divides a product *ab*, it divides either *a* or *b*. The question is whether defining them this way is appropriate or the usual definition is better. In both cases the answers depend on subtle mathematical considerations. Chapter 10 deals with a single, but more general example: "the Riemann-Dedekind approach to 'essential characteristic properties'" (278). The crucial question here is whether functions are to be classified according to their global nature or to their singularities. The Riemann-Dedekind approach advocates the latter option, which, clearly, cannot be cashed out in terms of a general logical principle.

Chapters 11-12 ("Computers in Mathematical Inquiry", 302-316; "Understanding proofs", 317-353), by Jeremy Avigad, discuss the role of computers in mathematics. Avigad wonders whether their use results in some epistemic gain, and how this gain can be accounted for. In chapter 11, two sorts of such gains are considered: "the ability of computers to deliver appropriate 'evidence' for mathematical assertions […][and] appropriate mathematical 'understanding'", respectively (302). The emphasis is on the way evidence and understanding should be conceived in order to explain the way in which computers can help acquire them. Chapter 12 pursues the same aim, in more detail, with respect to understanding. Avigad suggests conceiving understanding in terms of the ability of doing something. In the second part of the paper, he tests this approach on four case studies concerning formal verification.

Chapters 13-14 ("What Structuralism Achieves", 354-369; "'There is no Ontology Here': Visual and Structural Geometry in Arithmetic", 370-406), by Colin MacLarty, deal with structuralism and category theory. The aim is less that of advocating structuralism than that of arguing that mathematical practice does not force the admission of a "classical ontology" (356), but can be accounted for by appealing to the sort of structuralism which is typical of category theory, or "working structuralism", as MacLarty calls it (360) by insisting that it is embodied in mathematical practice. Chapter 13 offers a comprehensive discussion of such a form of structuralism. Chapter 14 is devoted to a single case study relative to Alexander Grothendieck's theory of schemes, a theory pursuing André Weyl's idea of an "algebraic geometry over the integers" (385). This provides a philosophy-driven survey of a large and crucial fragment of 20th century mathematics.

Chapters 15-16 ("The Boundary Between Mathematics and Physics", 407-416; "Mathematics and Physics: Strategies of Assimilation", 417-440), by Alasdair Urquhart, are about the connections between mathematics and mathematical physics. According to Urquhart, modern mathematical physics deals with models having a so rough relation with physical reality that it is tempting to describe them as "mathematical objects in their own right" (409). However, these models do not appeal, typically, to "normal mathematical methods", and it is often doubtful that "the objects themselves are even mathematically well defined" (410). This difference of methods has increased in the mid-20th century, but the last years have seen a movement of reconvergence. This raises the issue of the way the methods of physics can be integrated within mathematics and become acceptable for its standards. In Chapter 16, four examples of "nonrigorous mathematics" derived from a beneficial interaction with the physical practice are considered: infinitesimal methods, the umbral calculus, the theory of distributions, the replica method. The chapter addresses a plea for extending philosophy of mathematics so as to include the consideration of similar forms of mathematics.

The advocacy of a philosophy of mathematics closer to history and practice is not new. In his Introduction, Mancosu discusses two recent traditions embodying this aim. The former is the "maverick tradition": the philosophical movement principally promoted by Imre Lakatos and Philip Kitcher, and more recently followed by David Corfield. The latter is rooted in Quine's empiricism and naturalism and has its major exponent, in more recent times, in Penelope Maddy.

As influential as these traditions might have been, they failed in "substantially redirect the course of philosophy of mathematics" (5). But then, what license the hope that the approach promoted by Mancosu could have a more successful outcome? Mancosu tackles this question in his Introduction.

He remarks that the maverick tradition was largely "concerned with metaphilosophical issues" (17), like mathematical progress, that were tackled by pointing to a small number of alleged paradigmatic case studies. The approach he favours is opposite: a much vaster spectrum of topics and studies are taken to serve more restricted aims. Moreover, by contrast to Lakatos's and Corfierld's attitudes, Mancosu is not intended to make a case against logical analysis, analytic philosophy, or the foundationalist tradition. Far from denying the interest and importance of these approaches, he merely calls for an extension of current philosophy of mathematics. Similar considerations also apply to the second tradition, which is overall concerned with set theory. Mancosu argues that, "while set theory is a very important subject of methodological investigation, there are central phenomena that will be missed unless we cast our net more broadly and extend our investigations to other areas of mathematics" (19).

These differences are not enough, however, to warrant that Mancosu's effort will result in a long-running success. The perspectives are excellent, overall because his effort involves a quite large community of scholars (as it is also attested by the recent foundation of an Association for the Philosophy of Mathematical Practice). But only future development will be able to establish the long-running success that these scholars (and myself) hope for. It is thus important to wonder which lines of research are the most appropriate for this purpose. Let me close with some remarks on it.

The introductory papers of Mancosu's book delineate a vast agenda that the research papers cannot but cover partially. This suggests many lines of research for future development. Still, barring some exceptions, the former papers do not fix a compact disciplinary content for the philosophy of mathematical practice, *i. e*. a systematic net of competences, pieces of information, and notions that a philosopher of mathematical practice is required to have and transmit. This is natural, considering the programmatic nature of the book. Nevertheless, this is an important lack to overcome in the next future. The prospects of the philosophy of mathematical practice largely depend on this.

The second remark is more specific. In his Introduction (1), Mancosu remarks that much of the work that is presently done in philosophy of mathematics "can be seen as an attempt to address a set of epistemological and ontological problems that were raised with great lucidity in two classic articles by Paul Benacerraf": basically they are the problems "of explaining how, if there are abstract objects, we could have access to them". As much beneficial as focusing on these problems might have been, this "has also had the unwelcome consequence of crowding other important topics off the table".

I totally agree. But I also maintain that the future success of the philosophy of mathematical practice will largely depend on its relations with Benacerraf's agenda (and I think Mancosu would agree with me on this). Though recently so efficaciously set forth by Benacerraf, this agenda is a quite classical one, and, *mutatis mutandis*, was already suggested by Plato. Moreover, recent discussions on the foundations of mathematics had largely developed with respect to it. Hence, though Benacerraf's agenda can be hopefully supplied by other topics, it is hard to think that philosophy of mathematics can leave it on a side, or can avoid any foundationalist concern. A crucial problem that the philosophy of mathematical practice should then face is whether there is a way, closer to mathematical practice, for tackling Benacerraf's questions and foundationalist concerns. I think, for example, there is room for hoping that the very notion of a mathematical object and our epistemic relation with these objects would be clarified if they were conceived as being historically constituted by the very mathematical activity. My guess is that the long-running success of philosophy of mathematical practice will crucially depend on its capacity to integrate these themes within its own approach.

Mancosu's book is not only highly welcome, since it addresses the invaluable program of philosophy of mathematical practice. It is also a significant contribution to philosophy of mathematics as such. Its structure, the vastness of the topic considered, the clarity of Mancosu's Introduction and of the different essays included in it, the significance of the topics dealt with and of the theses advanced, as well as the robustness of the arguments that support these theses make it an indispensable companion for contemporary philosophers of mathematics, whatever their approach might be.