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The role of algebraic inferences in Na'īm ibn Mūsā's *Collection of geometrical propositions*

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Abstract

Na'īm ibn Mūsā's lived in Baghdad in the second half of the 9th century. He was probably not a major mathematician. Still his *Collection of geometrical propositions*—recently edited and translated in French by Roshdi Rashed and Christian Houzel—reflects quite well the mathematical practice that was common in Thābit ibn Qurra's school. A relevant characteristic of Na'īm's treatise is its large use of a form of inferences that can be said 'algebric' in a sense that will be explained. They occur both in proofs of theorems and in solutions of problems. In the latter case, they enter different sorts of problematic analyses that are mainly used to reduce the geometrical problems they are concerned with to al-Khwārizmī's equations.

Na'īm ibn Mūsā a vécu à Bagdad pendant la seconde moitié du IXème siècle. Il n'a pas été probablement un mathématicien majeur. Néanmoins, sa *Collection de proposition géométriques*—qui a été récemment éditée en traduite en franais par Roshdi Rashed and Christian Houzel—reflet assez bien la pratique mathématique qui était commune à l'école de Thābit ibn Qurra. Une caractéristique relevante du traité de Na'īm est son large usage d'une forme d'inférences qui peut être dite 'algébrique' en un sens qui sera expliqué. Celles-ci interviennent autant dans la preuve de théorèmes que dans la solution de problèmes. En ce dernier cas, elles participent de différentes sortes d'analyses qui sont surtout employées pour réduire les problèmes géométriques auxquelles elles s'appliquent à des équations au sens d'al-Khwārizmī.

Na'īm[1] ibn Mūsā lived in Bagdad in the second half of the 9th century. He was the son of the older of the three brothers Banū Mūsā, Muhammad ibn Mūsā, and the pupil of Thābit ibn Qurra. Still, he was probably not a major mathematician. He was the author of a *Collection of geometrical propositions*, only one copy of which is known. It fills fifteen folios of a manuscript preserved at the University Library of Istanbul (A 314, $122^{\text{v}} - 136^{\text{v}}$), probably dating back to 1510, on the basis of a transcription of Nasīr al-Dīn al-Tūsī. Based on this unique copy, Roshdi Rashed and Christian Houzel have recently edited Na'īm's *Collection* and translated it into French by adding a detailed commentary[2].

As a matter of fact, it contains no relevant new results and does not distinguish itself for its perspicuous logical structure. Nevertheless, as Rashed points out, it is one of the first examples of a particular kind of mathematical treatises that became quite usual during the 10th century, and it reflects quite well "the mathematical culture" of a 9th-century Baghdadi mathematician educated at Thābit ibn Qurra's school[3]. This culture includes some characteristics that I would like to draw attention to. My suggestion is that they reveal a crucial aspect in the evolution of geometry.

1 Two senses of 'algebra': some remarks on al-Khwārizmī's calculus of algebra and al-muqābala

One way to clarify this aspect is to distinguish two senses in which the term 'algebra' might be used by historians with respect to medieval and early modern mathematics.

As is well known, this term—or better, the term 'al-ğabr' from which it derives—first occurred in a mathematical context in al-Khwārizmī's *Book of Algebra and al-Muqābala*[4]. It is used there, together with the term 'al-muqābala', to designate two mathematical procedures that are taken to be so typical of a certain kind of "calculus [*hisāb*]"[5] that this calculus is named after them[6]. About two and a half centuries after this first occurrence, the same pair of terms was used in a similar way by al-Khayyām, who, instead of calculus, spoke of art[7]. The evolution of this calculus or art between al-Khwārizmī and al-Khayyām is well known. For my

[1]I thank Aline Auger, Heléne Bellosta, Charles Burnett, Annalisa Coliva, Massimo Galuzzi, Roshdi Rashed and two anonymous referees for valuable suggestions and comments on previous versions of my paper

[2]Cf. [9].

[3]Cf. [9], 6.

[4]Cf. [4] and, for an English translation, [3].

[5]Cf. [4], 94-97.

[6]On the nature of these procedures, cf. [1], 102-104 (note of R. Rashed).

[7]Cf. [10], 116-117 and 120-121.

purposes here, the relevant question is rather that of its status. According to Rashed[8], this should be understood as a "theoretic discipline" whose extensions apply both to numbers and to geometric magnitudes, but that is as such independent both of arithmetic and geometry. When referred to al-Khwārizmī, the term 'algebra'—taken as a shortcut for the expression 'calculus of algebra and al-muqābala'—should thus be understood as the name of a branch of mathematics that is common to arithmetic and geometry but does not reduce to them.

This discipline is concerned with a sort of combinatorial system including three basic elements, or modes: the numbers, the roots, and the squares[9]; let us denote them by 'N', 'R' and 'S', respectively. They can be combined either in pairs composed of two different modes or in pairs composed of a pair of two different modes and the remaining mode. The possible combinations are thus six: $< S, R >$, $< S, N >$, $< R, N >$, $<< S, R >, N >$, $<< S, N >, R >$, and $<< R, N >, S >$. These combinations correspond to as many equalities: $S = R$, $S = N$, $R = N$, $S + R = N$, $S + N = R$, and $R + N = S$. And each of these equalities corresponds to a problem.

These are essentially different from the problems that mathematicians had tackled before. One reason is that they exhaust a range of possibilities that is fixed in advance[10]. The subject matter of al-Khwārizmī's algebra is given by the elements of this range of possibilities that, according to Rashed, are nothing but (polynomial) equations[11]. This reason is linked to another one: al-Khwārizmī's problems are both canonical problems and forms of problems. When they are understood as canonical problems, they are about numbers, roots, and squares. When they are understood as forms of problems, they apply to problems that are, in turn, about numbers or geometrical magnitudes. It follows that to establish how to solve a canonical problem is the same as establishing how to solve infinite many problems about numbers or geometrical magnitudes.

Hence, al-Khwārizmī's algebra is both a discipline dealing with (polynomial) equations and a technique for solving arithmetical and geometrical problems, and, insofar as it is such a technique, it does not have any specific objects, since its objects are just those of arithmetic and geometry.

This is enough, I think, to fix a clear meaning for the composite term 'al-Khwārizmī's

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[8] Cf. [8], 20, and [4], vii and 12-13.

[9] Cf. [4], 95-96 and [3], 8 (in his French translation, Rashed uses the term 'modes' to refer to these basic elements; Rosen uses rather 'kinds'; in [8], 22, Rashed uses 'types' and adds in parenthesis the Arabic term 'al-durūb' and the Latin term 'modus'). The combinatorial nature of al-Khwārizmī's system is emphasized in [4], 18-24.

[10] Cf. [4], 22-24.

[11] Cf. [8], 22: "[...] la notion d'équation apparaît dès le départ, pour elle même, et, peut-on dire, de manière générique, dans la mesure où elle ne surgit pas simplement au cours de la solution d'un problème, mais est délibérement appelée à désigner une classe infinie de problèmes." Cf. also [4], 23.

algebra', or for the simple term 'algebra' when specifically referred to al-Khwārizmī. But what about the term 'algebra', in general? An obvious way to fix its meaning is to generalize the sense just ascribed to 'al-Khwārizmī's algebra'. 'Algebra' would then be understood as the general name of a changing subject-matter, that is, as a common name for the different forms taken by this subject-matter. In this sense, it would refer, in general, to the branch of mathematics dealing with (polynomial) equations, insofar as it provides a mathematical technique to tackle certain classes of arithmetical and geometrical problems. This last specification is in fact appropriate only if the term 'algebra' is used to speak of medieval and early modern mathematics up to the middle of the 17th-century, at most. Starting from this date, things changed quite radically. But, for the limited purposes of my paper, there is no need to consider what happened afterwards.

This is a first, general, sense that the term 'algebra' can reasonably have for historians of mathematics speaking about medieval and early modern mathematics. But this is not the only one. For the term 'algebra' could also be suitably used in order to account for some specific features of what this same term refers to, in its first sense. One way to fix this other sense is to focus on the nature of the arguments that are used in medieval and early modern mathematics to solve both the different sorts of (polynomial) equations or the particular arithmetical and geometrical problems associated with them.

Al-Khwārizmī's solution of his equations relies on both arithmetical procedures and geometrical arguments. This is possible because he interprets these equations either as arithmetical or as geometrical conditions. Consider $S + R = N$. The way al-Khwārizmī deals with this equation is well known.

He first presents a solution[12] that apparently depends on the interpretation of it as an arithmetical condition: supposing that two (positive) numbers μ and ν are given, the problem is that of finding another number $\varkappa$, such that

$$(\varkappa \times \varkappa) + (\mu \times \varkappa) = \nu, \tag{1}$$

where '$\times$' and '$+$' denote the operations of multiplication and addition of numbers, respectively. The solution goes than as follows:

$$\frac{\mu}{2} = \varkappa_1 \rightarrow \varkappa_1 \times \varkappa_1 = \varkappa_2 \rightarrow \varkappa_2 + \nu = \varkappa_3 \rightarrow \sqrt{\varkappa_3} = \varkappa_4 \rightarrow \varkappa_4 - \varkappa_1 = \varkappa \tag{2}$$

After having presented this solution, he shows its "cause"[13]. In order to do that, he interprets the equation as a geometrical condition: supposing that two segments a and b are

[12] Cf. [4], 100-101.
[13] Cf. [4], 108-113.

given, the problem is that of finding another segment x, such that

$$S(x) + R(a, x) = S(b), \tag{3}$$

where '$S(x)$' and '$R(a, x)$' denote the square with side x and the rectangle with sides a and x, respectively, and '+' designates the operation of addition of rectangles. This condition is certainly geometric, but it is non positional: it simply concerns the additive properties of two squares and a rectangle whose sides are given by three generic segments of which two are given and the third is unknown. The related problem is thus a geometric non-positional, additive problem.

In tackling this problem, al-Khwārizmī seems to mix two different forms of reasoning. On the one hand, he seems to understand the segments a, b and x as measured by numbers, that is, as resulting, respectively, from the multiplication of a unitary segment u for three numbers μ, λ and $\varkappa$, so as to have: $a = \mu u$, $b = \lambda u$, and $x = \varkappa u$, provided that $\lambda \times \lambda = \nu$. On the other hand, he constructs with these segments, and independently of their measurements, an appropriate configuration of rectangles that provides a positional interpretation of this problem[14]. The solution (2) is thus interpreted as an algorithm concerned with the numerical measurements of some appropriate segments and rectangles, and is associated with a model providing a geometrical positional interpretation of the given equation, that is independent of the consideration of the numerical measurements of these segments and rectangles. The cause al-Khwārizmī is concerned with pertains to this model and is thus independent of these arithmetical measurements, too. But, once it has been displayed, it is possible to come back to these measurements and prove that the arithmetical solution is correct since it can be interpreted according to this model.

Al-Khwārizmī's argument runs as follows. Suppose that x is the side of the square ABCD (fig. 1) that is: $\mathsf{AB} = x$, and $\mathsf{ABCD} = S(x)$. Suppose also that the equal rectangles p, q, r, and s have sides equal to x and one fourth of a, that is: $\mathsf{IE} = \frac{a}{4}$, and $\mathsf{p} = \mathsf{q} = \mathsf{r} = \mathsf{s} = R\left(\frac{a}{4}, x\right)$, and $\mathsf{j} = \mathsf{k} = \mathsf{m} = \mathsf{n} = S\left(\frac{a}{4}\right)$. As

$$\begin{array}{c} \mathsf{p} + \mathsf{q} + \mathsf{r} + \mathsf{s} = 4R\left(\frac{a}{4}, x\right) = R(a, x) \\ \text{and} \\ \mathsf{j} + \mathsf{k} + \mathsf{m} + \mathsf{n} = 4S\left(\frac{a}{4}\right) = S\left(\frac{a}{2}\right) \end{array}, \tag{4}$$

the condition (3) is thus equivalent to the condition

$$\mathsf{EFGH} = S\left(\frac{a}{2}\right) + S(b), \tag{5}$$

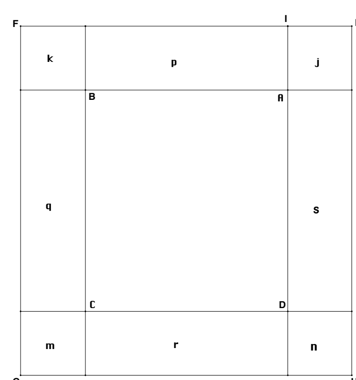

Figure 1.

[14]As a matter of fact, al-Khwārizmī constructs two distinct configurations of rectangles that provide two distinct (but of course equivalent) positional interpretations of the problem. For short, I consider here only the first of them [cf. [4], 108-111].

and to find x it is enough to find the side of a square equal to $S\left(\frac{a}{2}\right)+S(b)$, let us say z, and subtract $\frac{a}{2}$ from it.

To prove that the arithmetical solution is correct, it is then enough to admit that, if s is any segment and σ any number of the kind involved in the condition (1), then

$$(\sigma \times \sigma)\, S(s) = S(\sigma s) \qquad \text{and} \qquad \sigma S(s) = S\left(\sqrt{\sigma} s\right), \tag{6}$$

and to remark the correspondence displayed in the following table:

	Arithmetical solution		Geometrical interpretation	
	given,	you get:	given,	you get:
1.	μ	$\frac{\mu}{2} = \varkappa_1$	μ, u	$\frac{\mu u}{2} = \frac{a}{2} = 2\mathsf{EI} = \varkappa_1 u$
2.	$\varkappa_1$	$\varkappa_1 \times \varkappa_1 = \varkappa_2$	$\varkappa_1, u$	$(\varkappa_1 \times \varkappa_1)\, S(u) = S\left(\frac{a}{2}\right) = \mathsf{j}+\mathsf{k}+\mathsf{m}+\mathsf{n} = \varkappa_2 S(u)$
2′.			ν, u	$\nu S(u) = S(b)$
3.	$\varkappa_2, \nu$	$\varkappa_2 + \nu = \varkappa_3$	$\varkappa_2, \nu, u$	$\varkappa_2 S(u) + \nu S(u) = S\left(\frac{a}{2}\right) + S(b) = \mathsf{EFGH} = \varkappa_3 S(u)$
4.	$\varkappa_3$	$\sqrt{\varkappa_3} = \varkappa_4$	$\varkappa_2, u$	$\sqrt{\varkappa_3} u = z = \mathsf{EF} = \varkappa_4 u$
5.	$\varkappa_1, \varkappa_4$	$\varkappa_4 - \varkappa_1 = \varkappa$	$\varkappa_1, \varkappa_4, u$	$\varkappa_4 u - \varkappa_1 u = x = \mathsf{EF} - 2\mathsf{EI} = \mathsf{AB} = \varkappa u$

Al-Khwārizmī's argument is not only apt to provide a basis for proving that the solution (2) is correct (for the numbers complying with the equalities (6)). Insofar as it is independent of the arithmetical measurements of the segments it involves, it also provides a basis for a purely geometrical solution, provided that the equation is interpreted as the condition 3. From such an argument it follows, indeed, that, to get the segment x, it is enough to split up the segment a into four equal parts so as to get $\frac{a}{4}$, then construct a square equal to the sum of 4 times $S(\frac{a}{4})$ and $S(b)$, take the side of this square and subtract $\frac{a}{2}$ from it.

This is a quite simple construction that relies on *El.*, I.47, the Pythagorean theorem. This is not the only theorem of the Elements that enters implicitly in Al-Khwārizmī's argument. The equalities (4) follow from El. II.1, from which also the equalities (6) follow, if $\sigma \times \sigma$ and $\sqrt{\sigma}$ are (positive) rational numbers[15]. Both the theorems *El.*, I.47 and *El.*, II.1, however, have to be interpreted as purely additive conditions, here. This is a crucial point that deserves a clarification. These theorems, as well as any other geometrical theorem of the *Elements*, are concerned in such a treatise with particular geometrical configurations, and are proved by relying on them: *El.* II.1 concerns a number of rectangles constructed on contiguous portions of the same segment; *El.*, I.47 concerns the three squares constructed on a given right-angled triangle. They depend, thus, as such, on the mutual positions of the geometrical objects that enter these configurations. Still, the segment x that complies with the condition (3) can be constructed by relying on *El.*, I.47, and the equalities (4) and (6) can, be derived from

[15]On the relations between Al-Khwārizmī's algebra and the *Elements* cf. [4], 30-56.

El. II.1 only if these theorems are applied to generic squares and rectangles that are supposed to be appropriately added one to another. This is possible only if the configurations that these theorems are about are interpreted as particular models for some additive conditions, and these very theorems are understood as concerning these last conditions, rather than these particular models. So conceived—that is, interpreted as additive theorems—*El.*, I.1 and *El.* II.47 provide respectively a rule of inference relative to the addition of rectangles, and a rule for summing squares.

In another paper of mine[16], I have called the inferences depending on the application of a rule of inference like the one provided by *El.*, I.1 'geometrical non-positional inferences', and I have suggested that they are part of a system of techniques or an art, which I have called 'algebra'. This art included, together with geometrical non-positional inferences, also arithmetical inferences, and was used to perform a particular sort of analysis, which I have called 'trans-configurational' as opposed to Pappusian (or intra-configurational) analysis[17]. This art should not be identified with what 'algebra' refers to in its first sense, since it was, as such, independent of the specific aims of dealing with (polynomial) equations. When it is used to denote this art, the term 'algebra' is thus used in another, essentially different, sense.

My present purpose is to show how the geometric side of this art—namely, geometrical non-positional inferences—work in Naʿīm's *Collection*. These inferences occurred quite often in Greek geometry, and are an essential ingredient of al-Khayyām's solutions of cubic equations[18]. Naʿīm's *Collection* presents, in the second half of the ninth century, a use of them that it is so frequent to constitute a relevant phenomenon from an historical point of view. Insofar as Naʿīm's *Collection* is concerned only with geometry, the arithmetical side of algebra (in its second sense) has no relevant role in it. There is thus no need to emphasize the geometrical nature of the inferences that occur in it. For short, I shall, thus, refer, in what follows, to geometrical non-positional inferences through the general term 'algebraic inference'.

In most cases, these inferences occur in Naʿīm's *Collection* within analytic arguments. Still, no one of these arguments complies with a sufficient approximation with the two patterns of problematic analysis that I have described in my previous paper[19]. When considered with respect to the distinction between these two patterns, they are hybrid arguments including aspects of both of them. I shall distinguish their different functions and forms.

[16]Cf. [7].

[17]Cf. [7], sect. 2.

[18]Cf. [10] and [7].

[19]Cf. footnote (16) above.

2 Thābit ibn Qurra's double reductions

Naʿīm's *Collection* strongly manifests the influence of Naʿīm's master, Thābit ibn Qurra and, in particular, of his short treatise on the "restoring of the problems of algebra through geometrical demonstrations"[20]. Since this influence is particularly relevant to Naʿīm's use of algebraic inferences, some remarks on this treatise are useful before considering Naʿīm's.

What is relevant for my purposes in Thābit's treatise is the way in which algebraic inferences make it possible to transform the geometric non-positional conditions corresponding to al-Khwārizmī's equations into other conditions of the same sort. Similar transformations already occur in al-Khwārizmī's arguments. An example is given by the passage from the condition (3) to the condition (5). To make these transformations al-Khwārizmī relies on algebraic inferences, but uses them locally, within an argument that essentially depends on the consideration of a positional model for these conditions. What is new in Thābit's treatise is that these same transformations are obtained without relying on any positional model.

Let us consider again $S + R = N$. Thābit refers[21] to a diagram that displays a positional model of this equation: he identifies S with the square BACD (fig. 2), the number of R with the numerical measurement of the given segment EB, and R itself with the rectangle EBDG constructed on EB and a side of BACD. It follows that the rectangle EACG is identified with $S+R$ and is thus equal to a given square, identified with N. The problem is thus that of looking for a segment (BA) such that the rectangle whose sides are this very segment and the sum of this segment and another given segment (EB) is equal to a given square. The diagram is useful to fix the reference of the names of the relevant segments and rectangles. But it is not essential. *El.*, II.3, interpreted as an additive theorem, is sufficient for reducing the condition (3) to the equivalent condition:

$$R(x, x + a) = S(b), \qquad (7)$$

that corresponds to this last problem. Thābit possibly grasped the equivalence between his diagrammatic argument and such an application of *El.*, II.3. This is suggested by the way he continues. He relies on *El.*, II.6 for deducing that if F is the middle point of EB, the rectangle constructed on BA and EA plus the square constructed on FB is equal to the square constructed on FA. The fact that Thābit does not construct

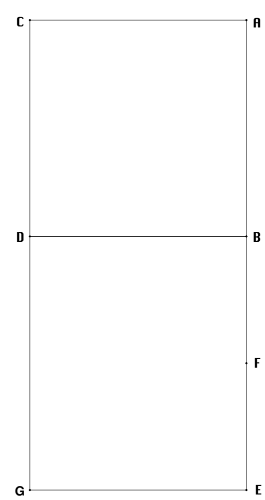

Figure 2

[20]Cf. [5]. A French translation of Thābit's treatise is provided by the conjunction of the three quotations inserted in [4], 33-34, 37-38 and 41. I base my reconstruction on this translation (I thank R. Rashed for having allowed me to consult it before publication).

[21]Cf. [4], 33-34.

this rectangle and these squares suggests that he interprets *El.*, II.6 as an additive theorem. So interpreted, this theorem states that

$$R(x, x+a) + S\left(\frac{a}{2}\right) = S\left(x + \frac{a}{2}\right). \tag{8}$$

And it is enough to compare this equality with the equality (7), to get:

$$S(b) + S\left(\frac{a}{2}\right) = S\left(x + \frac{a}{2}\right). \tag{9}$$

This is a new non-positional condition that immediately suggests how to construct x: by subtracting $\frac{a}{2}$ from the hypothenuse of the right-angled triangle whose other sides are b and $\frac{a}{2}$ itself.

Thābit offers a perfectly analogous argument also for $S+N = R$[22]. He identifies S with the square ABDC (fig. 3), the number of R with the numerical measurement of the given segment EB, and R with the rectangle constructed on BE and a side of ABDC. Hence, the rectangle EACG is identified with $R-S$ and is thus equal to a given square, identified with N. The problem is then that of looking for a segment (BA) such that the rectangle whose sides are this very segment and the difference of a given segment (EB) and this same segment is equal to a given square. Also in this case *El.*, II.3, interpreted as an additive theorem, is sufficient for reducing the condition

$$S(x) + S(b) = R(a, x) \tag{10}$$

to the condition

$$R(x, a-x) = S(b), \tag{11}$$

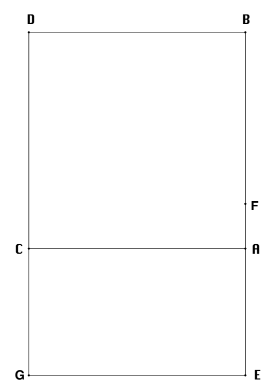

Figure 3

that corresponds to this last problem. Thābit then relies on *El* .II.5 for deducing that if F is the middle point of EB, the sum of the rectangle constructed on AB and EA and the square constructed on AF is equal to the square constructed on FB, that is:

$$R(x, a-x) + S\left(x - \frac{a}{2}\right) = S\left(\frac{a}{2}\right), \tag{12}$$

or, according to the equality (11),

$$S\left(\frac{a}{2}\right) - S(b) = S\left(x - \frac{a}{2}\right). \tag{13}$$

[22]Cf. [4], 37-38.

Again, this is a new non-positional condition that immediately suggests how to construct x: by adding $\frac{a}{2}$ to a side of the right-angled triangle whose hypothenuse is $\frac{a}{2}$ itself, and whose other side is b.

Thābit's arguments can be understood as double reductions based on algebraic inferences alone: the conditions (3) and (10) are first reduced to the conditions (7) and (11), and these are then reduced to the conditions (9) and (13). The conditions (3), (7), (9) and (10), (11), (13) are respectively equivalent, but also essentially different from each other: the rectangles that occur in the first members of (7) and (11) are such that *El.*, II.6 and *El.*, II.5 apply to them, when interpreted as additive theorems; in (9) and (13) the segment x occurs once, so that these conditions make manifest how to construct it.

If Thābit's arguments are understood as I have suggested, they can be seen as geometrical counterparts of arithmetical arguments that are not difficult to conceive. Hence, they show that, though geometrical non-positional inferences are far from constituting a formalism globally analogous to the arithmetical one, they can be used to transform certain geometric non-positional conditions in other equivalent, but essentially different ones, just as the arithmetical formalism does with arithmetical conditions. This should justify my use of the term 'algebra' (in its second sense) to designate the art that include these inferences together with arithmetical ones.

3 Na'īm's *Collection*

The vast majority of the propositions included in Na'īm's *Collection* are problems, and many of them are not properly solved, but rather reduced either to problems of application of areas with excess or defect of a square or of a rectangle whose sides are to each other in a certain numerical proportion, or to problems of partition or prolongation of a given segment according to a non-positional additive condition concerning rectangles. These problems are equivalent to al-Khwārizmī's equations understood as geometrical conditions.

The equivalence between the problem of finding a segment x which complies with the conditions (3) and (10) and a problem of application of an area with excess or defect of a square is proved in the first parts of Thābit's previous arguments.

The equivalence between the problem of solving an equation of al-Khwārizmī geometrically and a problem of application of areas with excess or defect of a rectangle whose sides are in a certain numerical proportion to each other can be easily proved by an argument analogous to Thābit's. Suppose, as before, that EB (fig. 2 and 3) is a given segment. This last problem requires one to construct a point A, respectively on the prolongation of EB and on EB itself, such that the rectangle EACG, constructed on EA and a segment AC which is to AB in the given ratio of n to m, is equal to a given square. Let a and x be the given and the required segments. The conditions of these problems are, respectively:

$$R\left(\frac{n}{m}x, a+x\right) = S(b) \qquad \text{and} \qquad R\left(\frac{n}{m}x, a-x\right) = S(b), \tag{14}$$

where $S(b)$ is a given square. According to *El.*, II.1, these conditions are equivalent to

$$R\left(x, a+x\right) = \frac{m}{n}S(b) \qquad \text{and} \qquad R\left(x, a-x\right) = \frac{m}{n}S(b), \tag{15}$$

that correspond, respectively, to problems of application of an area with excess and defect of a square.

Similar arguments can easily prove the equivalence between the problem of solving an equation of al-Khwārizmī and a problem of partition or prolongation of a given segment according to a non-positional additive condition concerning rectangles.

The second parts of Thābit's previous arguments prove, moreover, that a problem of application of an area with excess or defect of a square—and thus an equivalent problem of partition or prolongation of a given segment according to a non-positional additive condition concerning rectangles—can easily be solved through the construction of an appropriate right-angled triangle. It is thus clear why Naʿīm does not insist on the solution of problems like these and confines himself to reducing other problems to them. One of the aims of his treatise is to show the equivalence between a number of geometrical problems and al-Khwārizmī's equations understood as geometric conditions, so as to provide a repertory of geometrical problems to be solved through the solution of these equations. Algebraic inferences appear to be an essential tool for achieving this aim.

Rashed and Houzel's edition of Naʿīm's treatise contains forty-seven propositions: forty-six of them are enumerated from 1 to 42, with the addition of items $4'$, 39*bis*, 41*b* and 41*c*; in place of the proposition 41 appears a proposition 41*a* followed by an unfinished solution, which is then replicated (and solved) as proposition 41*b*; another proposition is added without any number after proposition 42. Since this last proposition and propositions 36 and 41*c* were probably added by al-Tūsi, and propositions 30 and 31 are mere replications of propositions 28 and 28, by excluding proposition 41*a*, we have forty-one distinct propositions to be assigned to Naʿīm: five theorems and thirty-six problems. Though the copy edited by Rashed and Houzel does not contain any explicit reference to the *Elements*, Naʿīm's proofs and solutions[23] depend on many results contained in them. Many of these proofs and solutions consist in usual arguments that make no relevant use of algebraic inferences and do not apply any sort of analysis. I shall leave them aside in order to concentrate on the propositions that Naʿīm proves or solves

[23]For short, I use the term 'solution' and its cognates to refer either to genuine solutions or to reductions to simpler problems.

by applying analytical arguments and/or by relying on algebraic inferences. I shall consider some examples of these propositions and reconstruct Na‘īm's arguments relative to them[24], by making explicit the propositions of the *Elements* they rely on.

3.1 Theorems: the example of proposition 35

The five theorems are propositions 5, 9, 12, 24, and 35. Propositions 5, 9 and 12 concern triangles; proposition 24 is a lemma for the solution of proposition 25 and concerns any convex quadrilateral; proposition 35 concerns the sum of the sides of the regular hexagon and the regular decagon inscribed in the same circle. None of their proofs relies on a previous analysis. The proofs of propositions 5 and 35 make use, however, of algebraic inferences. I take the latter as an example.

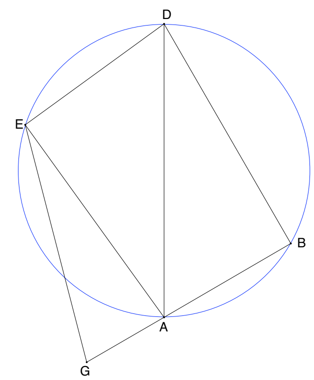

Figure 4

Na‘īm proves[25] that the sum of the sides of the regular hexagon and the regular decagon inscribed in the same circle is equal to the chord of three tenths of this circle. Supposing that AB, ED and GA (fig. 4) are respectively the sides of the hexagon, the pentagon and the decagon inscribed in the circle AEDB of diameter AD, this is the same as proving that $\mathsf{AE} = \mathsf{GB}$. To do this, Na‘īm begins by remarking that

$$\begin{array}{lll} a. & \mathsf{GB} : \mathsf{AB} = \mathsf{AB} : \mathsf{GA}, & (\text{E}\textit{l}., \text{XIII.9}), \\ b. & S(\mathsf{ED}) = S(\mathsf{GA}) + S(\mathsf{AB}), & (\textit{El}., \text{XIII.10}), \\ c. & S(\mathsf{BD}) = 3S(\mathsf{AB}), & (\textit{El}., \text{XIII.13 and VI.15}), \end{array} \qquad (16)$$

[24] Na‘īm's arguments are often inaccurate and obscure. I shall try to clarify them by helping myself to Rashed and Houzel's commentary [cf. [9], 11-68]. Still, I shall not supply these arguments with appropriate conditions specifying their exact domain of validly, since Na‘īm does not do it and my main point is independent of this matter.

[25] Cf. [9], 120-123.

and continues as follows:

$$
\begin{array}{llll}
a. & S(\mathsf{BD}) & = S(\mathsf{AB}) + 2R(\mathsf{GB}, \mathsf{GA}), & (16.a,\ 16.c, \text{ and } El., \text{VI.16}),\\
b. & & = S(\mathsf{AB}) + 2R(\mathsf{AB}, \mathsf{GA}) + 2S(\mathsf{GA}), & (a \text{ and } El., \text{II.3}),\\
c. & & = S(\mathsf{GB}) + S(\mathsf{GA}), & (b \text{ and } El., \text{II.4}),\\
d. & S(\mathsf{BD}) + S(\mathsf{AB}) & = S(\mathsf{GB}) + S(\mathsf{GA}) + S(\mathsf{AB}), & (\text{c}),\\
e. & S(\mathsf{AD}) & = S(\mathsf{GB}) + S(\mathsf{ED}), & (d,\ 16.b \text{ and } El., \text{I.47}),\\
f. & S(\mathsf{AE}) + S(\mathsf{ED}) & = S(\mathsf{GB}) + S(\mathsf{ED}), & (e \text{ and } El., \text{I.47}),\\
g. & S(\mathsf{AE}) & = S(\mathsf{GB}), & (f),\\
h. & \mathsf{AE} & = \mathsf{GB}. & (g).
\end{array}
\tag{17}
$$

The use of algebraic inferences is so obvious that no commentary is needed.

3.2 Problems

According to their nature, and independently of the way they are solved, the thirty-eight problems included in Naʿīm's *Collection* can be distributed into four groups.

The first group includes ten problems, that is, propositions 1, 2 , 7, 15, 26-29, 32, and 34. They require one to determine a triangle, a quadrilateral or a circle, supposing that some elements of them are given and/or they comply with some conditions.

The second group includes fifteen problems, that is, propositions 3, 6, 8, 13, 14, 16-23, 25, and 33. A triangle or a quadrilateral being given, they require one to draw an appropriate straight line that cuts some of its sides, or some of these sides and some chords so as to form some segments, triangles or quadrilaterals satisfying a certain condition.

The third group includes nine problems, that is, propositions 4, 4′, 37-39, 39*bis*, 40, 41*b*, and 42. Propositions 4 and 4′ consist of al-Khwārizmi's equations $S + R = N$ and $S + N = R$, directly understood as geometrical conditions. The other ones are problems of partition or prolongation of a given segment according to a non-positional additive condition concerning rectangles.

The fourth group includes only two problems, propositions 10 and 11. The former requires one to circumscribe a square about a scalene triangle, the latter requires one to inscribe a square in a scalene triangle.

Let us consider now Naʿīm's solutions.

Both problems of the fourth group are solved through a classical Euclidean construction, without relying on any sort of analysis and without making any relevant use of algebraic inferences.

Four problems of the second group—propositions 17, 18, 22 and 25—are reduced to other problems by relying on an appropriate analysis. The other eleven problems of this group are solved in the same way as those of the fourth group. Proposition 25 is reduced to proposition 23; propositions 17, 18 and 22 are reduced to problems of partition or prolongation of a given segment according to a non-positional additive condition concerning rectangles. In the case of proposition 17, the new problem is reduced, in turn, to a problem of application of an area with defect of a square through a new analysis that makes relevant use of algebraic inferences. Apart from that, Naʿīm's arguments concerning the problems of the second group make no relevant use of algebraic inferences.

Two problems of the first group—propositions 2 and 34—are solved like those of the fourth group. The other problems are solved by relying on some sort of analysis. In the case of three of them—propositions 15, 26 and 32—this is a classical, Pappusian analysis that makes no relevant use of algebraic inferences. Proposition 26 is solved only partially, and proposition 27 demands a complete solution, which Naʿīm does not provide. This could nevertheless have been easily obtained through an argument analogous to that which Naʿīm relies on for solving propositions 28-29. Like proposition 7, these last propositions are reduced to problems of partition or prolongation of a given segment according to a non-positional additive condition concerning rectangles, through an appropriate analysis that makes a relevant use of algebraic inferences. In the case of proposition 7, the new problem is reduced, in turn, to a problem of application of an area with excess of a rectangle whose sides are in a certain numerical proportion to each other, through a new analysis that makes a relevant use of algebraic inferences, too. This is also the case of proposition 1.

Two problems of the third group—propositions 4 and 4′—have a special status, since they consist, as has been said, of al-Khwārizmi's equations $S + R = N$ and $S + N = R$, understood as geometrical conditions. Naʿīm reduces them to the problems of constructing two appropriate right-angled triangles, repeating Thābit's arguments. He then continues by presenting two geometrical models for these problems including appropriate gnomons, as suggested by the second of al-Khwārizmi's positional interpretations of the geometrical problem related to the equation $S + R = N$[26]. Five other problems of the third

[26] Cf. [4], 110-113. Al-Khwārizmi identifies S or $S(x)$ with the square CBAD [fig. a], and the half of the number of roots with the numerical measurement of $\mathsf{HC} = \mathsf{C'C}$. It follows that R or $R(a, x)$ is identified with the sum

group—propositions 37-39, 41b and 42 —are reduced to problems of application of an area, through an appropriate analysis making a relevant use of algebraic inferences. Finally, propositions 39*bis* and 40 are proved to be equivalent to two problems requiring one to construct a triangle similar to another given one and equal to a given polygon; in both cases, the proof makes a relevant use of algebraic inferences.

Here I'm interested only in the problems solved through some sort of analysis. Among these arguments, some make relevant use of algebraic inferences, while others do not. Though I am mainly interested in the former, the latter deserve some consideration too. I will take them in inverse order.

3.2.1 Analysis without algebra: proposition 17, first part

The arguments that do not make relevant use of algebraic inferences consist of the solutions of propositions 15, 26 and 32 of the first group[27], and the solution of propositions 17 (first part), 18, 22 and 25 of the second group. The analyses used to solve propositions 15, 26 and 32 comply with the classical pattern of Pappusian analysis: supposing that a problem has been solved, one draws a diagram that represents its solution and reasons on this diagram so as to understand how the required elements can be constructed starting from the given ones. The analyses used to solve propositions 17 (first part), 18, 22 and 25 are slightly different: supposing that each of these problems is solved, Na'īm draws a diagram that represents the solution and reasons on this diagram so as to show that the required elements can be constructed if other unknown elements, satisfying some new conditions, are previously constructed. In the case of propositions 17 (first part), 18 and 22 these conditions are non-positional ones. Insofar as the analyses occurring in the solution of these propositions reduce the original problem to

of the equal rectangles HCDS and C′B′BC, N or $S(b)$ is identified with the gnomon C′B′ASHC, and $S\left(\frac{a}{2}\right)$ is identified with the square RC′CH. Thus the whole square RB′AS is known, and its side minus the segment HC is the segment required. Insofar as he can rely on Thābit's reductions, Na'īm can invert this argument and extend it also to the case of the equation $S+N=R$. He identifies half of the number of roots with the numerical measure of HC [figs. a and b] and constructs on it the square RC′CH that is thus identified with $S\left(\frac{a}{2}\right)$. He then constructs the square RB′AS equal to RC′CH plus (for proposition 4) or minus (for proposition 4′) the known number, that is, equal to $S\left(\frac{a}{2}\right)+S\left(b\right)$ or $S\left(\frac{a}{2}\right)-S\left(b\right)$. The side RB′ of this square is known and equal to $x+\frac{a}{2}$ or $\frac{a}{2}-x$. As $\mathrm{RC}'=\mathrm{HC}=\frac{a}{2}$ is know, this is also the case for $\mathrm{C'B'}=\mathrm{CB}=x$.

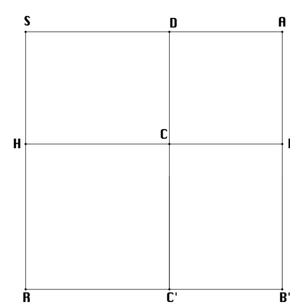

Figure a

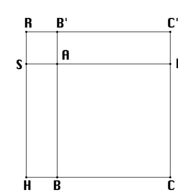

Figure b

[27] The synthesis of proposition 32 relies on the solution of proposition 15 .

a different one, they are trans-configurational (they reduce a certain configuration of known and unknown objects associated to a certain problem to another configuration of known and unknown objects associated to a distinct, though equivalent problem). But they do not comply with the pattern of trans-configurational analysis, as I have described it elsewhere[28]: the analyses that comply with this pattern reduce non-positional problems to other non-positional ones, do not rely on diagrams, and make use of algebraic inferences; the analyses applied to the solution of the previous propositions reduce positional problems to non-positional ones, rely on appropriate diagrams and do not make use of algebraic inferences.

A nice example is given by the first part of Naʿīm's solution of proposition 17[29].

Supposing that a triangle CBA (fig. 5) and a point D on the prolongation of its side CB are given, this proposition requires the construction a straight line DE that cuts CA and BA at two points H and E such that triangle HEA is λ times triangle DCH (Naʿīm first supposes that $\lambda = 2$, then claims that the same argument holds if HEA is any multiple of DCH[30]). Here is Naʿīm's argument.

Let us suppose that the problem is solved and let us draw from C the straight line CL parallel to DA, that cuts DE in G. According to *El.*, I.37, the triangles DGA and DCA are equal[31], and this is thus also the case for the triangles HGA and DCH. Hence, if $\mathsf{GE} = (\lambda - 1)\,\mathsf{HG}$, the problem is solved, since in this case, according to *El.*, VI.1, $\mathsf{GEA} = (\lambda - 1)\,\mathsf{HGA}$ and thus: $\mathsf{GEA} + \mathsf{HGA} = \mathsf{HEA} = \lambda\mathsf{HGA} = \lambda\mathsf{DCH}$.

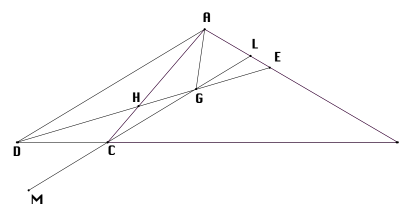

Figure 5

The original problem is thus reduced to a new one: to draw from D a straight line DE that cuts CA and CE in two points H and G such that $\mathsf{GE} = (\lambda - 1)\,\mathsf{HG}$. But, although this new problem concerns segments whereas the original one concerned triangles, it also requires the construction of point H. Hence, a new reduction is necessary. Let us suppose that the new problem is solved. The triangles DEA and DHA are respectively similar to the triangles GEL and CGH. As $\mathsf{DE} = \mathsf{DH} + \mathsf{HG} + \mathsf{GE}$ and $\mathsf{GE} = (\lambda - 1)\,\mathsf{HG}$, it follows that

$$\mathsf{DA} + \lambda\mathsf{CG} : (\lambda - 1)\,\mathsf{CG} = \ \mathsf{DA} : \mathsf{GL}. \tag{18}$$

[28] Cf. footnote (16) above.

[29] Cf. [9], 94-97.

[30] But notice that, in order to avoid the occurrence of negative numbers, one has to suppose that $\lambda > 1$. The case $\lambda = 1$ is considered in proposition 16.

[31] This is proved in the solution of the proposition 16: cf. [9], 94.

It is thus enough to prolong CL up to a point M such that $\mathsf{DA} = \lambda\mathsf{MC}$, to get

$$\begin{gathered}\mathsf{MG} : (\lambda - 1)\,\mathsf{CG} = \mathsf{MC} : \mathsf{GL}, \\ \text{or, according to } El., \text{VI.16}, \\ R\,(\mathsf{MG}, \mathsf{GL}) = R\,((\lambda - 1)\,\mathsf{CG}, \mathsf{MC})\,. \end{gathered} \tag{19}$$

Since MC and ML are given (as DA and CL are as well), the problem is thus reduced to a problem of partition of a given segment according to a non-positional additive condition concerning rectangles: cut the given segment ML, on which a point C is given, at another point G such that proportion (19) is satisfied. By making $\mathsf{MG} = a$, $\mathsf{GL} = b$ and $\mathsf{CG} = x$, this problem consists in finding a segment x such that

$$R\,((\lambda - 1)\,x, a - x) = R(a, b), \tag{20}$$

where the segments a and b and the number λ are given. But for *El.*, II.1 and *El.*, II.3, this condition is equivalent to

$$S\,(x) + S\,(\alpha) = R\,(a, x)\,, \tag{21}$$

where α is such that $S\,(\alpha) = R\left(\frac{1}{\lambda - 1}a, b\right)$. The solution of this new problem is thus reduced to the solution of al-Khwārizmī's equation $S + N = R$ understood as a geometrical condition.

3.2.2 Analysis with algebra applied to the solution of geometrical positional problems: proposition 1

Among Na'īm's analytic arguments that make a relevant use of algebraic inferences, some are used to reduce a positional problem to a non-positional one. This is the case of propositions 1, 7, and 28-31, of the first group. Although these analyses are trans-configurational and make a relevant use of algebraic inferences, none of them complies with the pattern of trans-configurational analysis, as I have described it elsewhere[32]. This is due to the fact that they apply to positional problems and rely on appropriate diagrams.

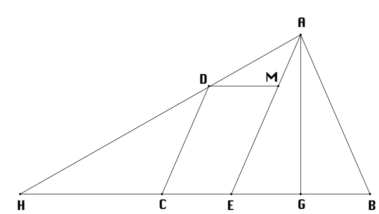

Figure 6

Let us consider the example of proposition 1. Supposing that the four sides of a quadrilateral are given and that two of its internal angles are equal to each other, this proposition require one to determine this quadrilateral[33]. Here is Na'īm's argument.

Suppose that CBAD (fig. 6) is the required quadrilateral and that $\mathsf{C\hat{B}A} = \mathsf{D\hat{C}B}$. Let EA be parallel to CD and DM be parallel to CB, so that $\mathsf{EA} = \mathsf{BA}$, $\mathsf{DM} = \mathsf{CE}$ and $\mathsf{EM} = \mathsf{CD}$. Prolong

[32]Cf. footnote (16) above.

[33]Cf. [9], 70-71. More specifically, Na'īm require one to find its diagonal.

DA to cut the straight line CB at H. Since the sides of the quadrilateral are given, also EA, EM, $\mathsf{MA} = \mathsf{EA} - \mathsf{EM}$, HD, and $\mathsf{HA} = \mathsf{HD} + \mathsf{DA}$ are given. Consider the last of these segments. One has:

$$\begin{array}{llll} a. & S(\mathsf{HA}) & = S(\mathsf{GA}) + S(\mathsf{HG}), & (\textit{El.}, \text{I.47}), \\ b. & & = S(\mathsf{EA}) - S(\mathsf{EG}) + S(\mathsf{HE} + \mathsf{EG}), & (a \text{ and } \textit{El.}, \text{I.47}), \\ c. & & = S(\mathsf{BA}) + S(\mathsf{HE}) + 2R(\mathsf{HE}, \mathsf{EG}) \ , & (b, \text{ and } \textit{El.}, \text{II.4}), \\ d. & & = S(\mathsf{BA}) + S(\mathsf{HE}) + R(\mathsf{HE}, \mathsf{EB}), & (c, \text{ and } \textit{El.}, \text{II.1}), \\ e. & & = S(\mathsf{BA}) + R(\mathsf{HE}, \mathsf{HB}), & (d, \text{ and } \textit{El.}, \text{II.3}), \\ f. & & = S(\mathsf{BA}) + R(\mathsf{HE}, \mathsf{CB}) + R(\mathsf{HE}, \mathsf{HC}), & (f, \text{ and } \textit{El} \ ., \text{II.1}). \end{array} \tag{22}$$

As $\mathsf{HC} : \mathsf{HE} = \mathsf{CD} : \mathsf{EA}$, and CD and EA are known, the ratio of HC to HE is known, too. Na‘īm takes this ratio to be that of 2 to 1, but his argument holds true in general, though it seems that he supposes that CD and EA are commensurable. Let this ratio be λ. From (22.f) it follows that:

$$R(\mathsf{HE}, \mathsf{CB}) + R(\mathsf{HE}, \lambda\mathsf{HE}) = S(\mathsf{HA}) - S(\mathsf{BA}) \tag{23}$$

The original problem is thus reduced to that of finding the segment HE that complies with this condition. Insofar as CB, HA and BA are given, this is a problem of prolongation of a given segment according to a non-positional additive condition concerning rectangles. If $\mathsf{CB} = a$, $\mathsf{BA} = b$, $\mathsf{HA} = h$ and $\lambda\mathsf{HE} = x$, according to *El.*, II.1, the condition (23) reduces to

$$R\left(\frac{1}{\lambda}x, a + x\right) = S(\alpha), \tag{24}$$

where α is such that $S(\alpha) = S(h) - S(b)$, and, as Na‘īm explicitly claims, this last problem is equivalent, in turn, to a problem of application of an area with excess of a rectangle whose sides are in a certain numerical proportion to each other.

3.2.3 Al-Khwārizmī's equations and problems of partition or prolongation of a given segment according to a non-positional additive condition concerning rectangles

In order to complete my account, I have to consider Na‘īm's treatment of al-Khwārizmī's equations and problems of partition or prolongation of a given segment according to a non-positional additive condition concerning rectangles. These are all the problems of the third group, together with the problems to which Na‘īm reduces propositions 7 , 17, 18, 22, and 28-29. In the case of propositions 18, 22 , and 28-29, Na‘īm confines himself to state these last problems, and, in the case of propositions 4 and 4′, he repeats, as has been said, Thābit's arguments, supplementing them with two additions suggested by an argument of al-Khwārizmi. In the other cases,he reduces the original problems to other non-positional ones, by relying on analyses that makes relevant use of algebraic inferences. Still, these analyses doesn't comply with the pattern of trans-configurational analysis, as I have described elsewhere[34], since they relies on diagrams, that is, they depend on a positional interpretation of the original, non-positional, problems. I will illustrate the issue with two examples.

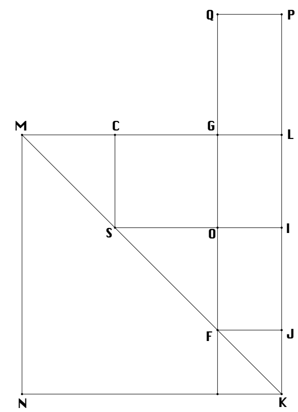

Figure 7

Proposition 17, second part In the first part of his argument concerning proposition 17[35], Naʿīm had reduced the original problem to the problem of cutting a given segment ML, on which a point C is given, at another point G so that condition (19) is satisfied. Let us see how he solves this problem.

Let NKLM (fig. 7) be the square constructed on the given segment ML, KM its diagonal, SC the perpendicular to ML drawn from C up to this diagonal, and SI the parallel to ML drawn from S up to KL. Since ML and C are given, the rectangle SILC is known. Let us prolong KL to P, so that $\mathsf{LP} = (\lambda - 1)\,\mathsf{MC}$. The required point G should be such that the rectangle FJPQ with sides $\mathsf{PQ} = \mathsf{GL}$ and $\mathsf{JP} = \mathsf{KP} - \mathsf{GL}$ is equal to $(\lambda - 1)\,\mathsf{SILC}$. This is easy to prove. Suppose that point G satisfies the condition $\mathsf{FJPQ} = (\lambda - 1)\,\mathsf{SILC}$. Then, from $\mathsf{LP} = (\lambda - 1)\,\mathsf{MC}$ it follows that $\mathsf{FJLG} = (\lambda - 1)\,\mathsf{SILC} - R\,(\mathsf{GL}, (\lambda - 1)\,\mathsf{MC})$, and thus:

$$\begin{array}{llll} a. & \mathsf{FJLG} & = (\lambda - 1)\,\mathsf{SILC} - (\lambda - 1)\,R\,(\mathsf{GL}, \mathsf{IL}), & (\textit{El.}, \text{I.1}), \\ b. & & = (\lambda - 1)\,\mathsf{SILC} - (\lambda - 1)\,\mathsf{OILG}, & (a), \\ c. & & = (\lambda - 1)\,[\mathsf{SILC} - \mathsf{OILG}], & (b \text{ and distributivity of external multiplication on subtraction}) \\ d. & & = (\lambda - 1)\,[\mathsf{SOGC}], & (c). \end{array} \quad (25)$$

Since $\mathsf{JL} = \mathsf{MG}$ and $\mathsf{OG} = \mathsf{MC}$, this last equality is the same as $R\,(\mathsf{MG}, \mathsf{GL}) = (\lambda - 1)\,R\,(\mathsf{CG}, \mathsf{MC})$. It is then enough to apply *El.*, I.1 to get condition (19).

The original problem of partition of a given segment according to a non-positional additive condition concerning rectangles is thus reduced to a problem of application of an area with defect of a square.

Proposition 42 My last example concerns proposition 42[36]. It requires dividing a given segment into two parts such that the rectangle constructed on the first of these parts and λ

[34] Cf. footnote (16) above.
[35] Cf. the previous footnote (29).
[36] Cf. [9], 136-141.

times the second one plus the square constructed on another given segment be equal to the square on the second part plus the square constructed on a third given segment. Suppose that a is the given segment to be divided, and that b and c are two other given segments. This amounts to searching for the segment x such that

$$R\,(x, \lambda(a-x)) + S(b) = S(x-a) + S(c), \tag{26}$$

Na'īm chooses a particular value of λ—that is, $\lambda = 3$ —but his argument holds true for any λ greater than 2. It runs as follows.

Let DB (fig. 8) be the segment a. Prolong it to P so that $2\mathsf{PD} = \lambda\mathsf{DB}$, and construct the square DBAC. Let HC and FC be the segments b and c. The squares HGZC and FEVC are thus given and are identified with $S(b)$ and $S(c)$, respectively. The problem is solved if DB is cut at a point K, such that

$$(i)\;\; (\lambda+1)\,\mathsf{KB} = 2\mathsf{SB} \qquad ; \qquad (ii)\;\; \mathsf{PSLJ} = \mathsf{DBGH} + \mathsf{FEC} \tag{27}$$

provided that PJ, SL and KI are perpendicular to PB, and JT is parallel to it (so that KBTI is a square). The proof is quite long, but not different in nature from the previous one.

Let O be the intersection point of PC and JT, and OQ be perpendicular to PB. Since the triangles PDC and PQO are similar and $\mathsf{QO} = \mathsf{SL}$ from $2\mathsf{PD} = \lambda\mathsf{DB}$ and $\mathsf{DB} = \mathsf{DC}$, it follows that $\mathsf{PQ} : \lambda\mathsf{DC} = \mathsf{SL} : 2\mathsf{DC}$, or $2\mathsf{PQ} = \lambda\mathsf{SL}$. Let also X be such that $4\mathsf{XK} = \lambda\mathsf{SL}$. Provided that XW is perpendicular to PB, one gets, according to *El.*, II,1,

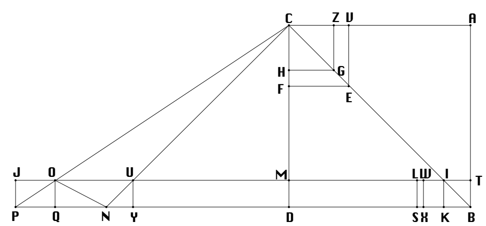

Figure 8

$$\begin{aligned} 2\mathsf{PQOJ} &= 4\mathsf{PQO} \\ &= 2R(\mathsf{PQ},\mathsf{SL}) = R(2\mathsf{PQ},\mathsf{SL}) = R(\lambda\mathsf{SL},\mathsf{SL}) \\ &\qquad\qquad = R(4\mathsf{XK},\mathsf{SL}) = 4R(\mathsf{XK},\mathsf{SL}) = 4\mathsf{XKIW}, \end{aligned}$$

and thus:

$$\mathsf{SKIL} = \mathsf{SXWL} + \mathsf{XKIW} = \mathsf{SXWL} + \mathsf{PQO}. \tag{28}$$

Now, from $\mathsf{KB} = \mathsf{SL}$ and $4\mathsf{XK} = \lambda\mathsf{SL}$, it follows

$$\begin{array}{lll} a. \;\; \mathsf{SXWL} & = R(\mathsf{SL}, \mathsf{SB} - \mathsf{XK} - \mathsf{SL}) & \\ b. & = R(\mathsf{SL}, \frac{\lambda+1}{2}\mathsf{SL} - \frac{\lambda}{4}\mathsf{SL} - \mathsf{SL}), & (a \text{ and } (27.i)), \\ c. & = R(\mathsf{SL}, \frac{\lambda-2}{4}\mathsf{SL}), & (b), \\ d. & = \frac{\lambda-2}{4}S(\mathsf{SL}), & (c \text{ and } El., \text{II},1) \\ e. & = \frac{\lambda-2}{4}\mathsf{KBTI} & (d). \end{array} \tag{29}$$

Moreover, if N is such that $\mathsf{ND} = \mathsf{DB}$ (so that the triangle NBC is isosceles), U is the intersection point of NC and JT, and UY is perpendicular to PB (so that $\mathsf{NY} = \mathsf{SL}$), from $2\mathsf{PD} = \lambda\mathsf{DB}$, $2\mathsf{PQ} = \lambda\mathsf{SL}$, and $\mathsf{QO} = \mathsf{SL}$ it follows:

$$\begin{array}{llll} a. & \mathsf{OU} & = \mathsf{PD} - \mathsf{PQ} - \mathsf{UM}, & \\ b. & & = \frac{\lambda}{2}\mathsf{ND} - \frac{\lambda}{2}\mathsf{SL} - (\mathsf{ND} - \mathsf{SL}), & (a), \\ c. & & = \frac{\lambda-2}{2}\mathsf{ND} - \frac{\lambda-2}{2}\mathsf{SL}, & (b), \\ d. & & = \frac{\lambda-2}{2}(\mathsf{ND} - \mathsf{SL}), & (c \text{ and distributivity of external multiplication on subtraction}), \end{array} \tag{30}$$

and,

$$\begin{array}{llll} a. & 2\mathsf{PNUO} & = 2\mathsf{PNO} + 2\mathsf{NUO}, & \\ b. & & = R(\mathsf{PD} - \mathsf{ND}, \mathsf{SL}) + R(\mathsf{OU}, \mathsf{SL}), & (a), \\ c. & & = R\left(\frac{\lambda-2}{2}\mathsf{ND}, \mathsf{SL}\right) + R\left(\frac{\lambda-2}{2}(\mathsf{ND} - \mathsf{SL}), \mathsf{SL}\right), & (a \text{ and } (30)), \\ d. & & = R\left(\frac{\lambda-2}{2}(2\mathsf{ND} - \mathsf{SL}), \mathsf{SL}\right), & (c \text{ and } \textit{El.}, \text{II.1}), \\ e. & & = \frac{\lambda-2}{2}R(2\mathsf{ND} - \mathsf{SL}, \mathsf{SL}), & (d \text{ and } \textit{El.}, \text{II.1}), \\ f. & & = (\lambda - 2)\,\mathsf{NDMU}, & (e), \\ g. & & = (\lambda - 2)\,\mathsf{DBIM}. & (d). \end{array} \tag{31}$$

Since $\mathsf{KBTI} = 2\mathsf{KBI} = 2(\mathsf{DBIM} - \mathsf{DKIM})$, from (29) and (31), one gets, according to the distributivity of external multiplication on subtraction,

$$\mathsf{SXWL} = \frac{\lambda-2}{4}\mathsf{KBTI} = \frac{\lambda-2}{2}(\mathsf{DBIM} - \mathsf{DKIM}) = \mathsf{PNUO} - \frac{\lambda-2}{2}\mathsf{DKIM}, \tag{32}$$

and, from this equality and $\mathsf{PQO} = \mathsf{POJ}$:

$$\begin{array}{llll} a. & \mathsf{PSLJ} & = \mathsf{PKIJ} - \mathsf{SKIL}, & \\ b. & & = \mathsf{PKIJ} - \mathsf{PQO} - \mathsf{SXWL}, & (a \text{ and } (28)), \\ c. & & = \mathsf{PKIO} - \mathsf{SXWL}, & (b), \\ d. & & = \mathsf{NKIU} + \mathsf{PNUO} - \mathsf{SXWL}, & (c), \\ e. & & = \mathsf{NKIU} + \frac{\lambda-2}{2}\mathsf{DKIM}, & (d). \end{array} \tag{33}$$

From this equality, (27.ii), and $\mathsf{NDMU} = \mathsf{DBIM}$, it follows that:

$$\mathsf{DBGH} + \mathsf{FEC} - \mathsf{DBIM} = \mathsf{NKIU} + \frac{\lambda-2}{2}\mathsf{DKIM} - \mathsf{NDMU}, \tag{34}$$

that is,

$$\mathsf{MIGH} + \mathsf{FEC} = \frac{\lambda}{2}\mathsf{DKIM} \qquad \text{or} \qquad 2\mathsf{MIC} + 2\mathsf{FEC} = \lambda\mathsf{DKIM} + 2\mathsf{HGC}. \tag{35}$$

As $\mathsf{DB} = a$, $\mathsf{HC} = b$ and $\mathsf{FC} = c$, if one supposes that $\mathsf{KB} = x$, this is the same as

$$S(a - x) + S(c) = \lambda R(x, a - x) + S(b),$$

which, according to *El.*, II.1, is equivalent to the condition (26), which was to be proved.

Since DBGH and FEC are known—and equal to $\frac{1}{2}R(\mathsf{DB} + \mathsf{HC}, \mathsf{DB} - \mathsf{HC})$ and $\frac{1}{2}S(\mathsf{FE})$, respectively—to find a segment KB that complies with condition (27) is the same as applying to the given segment PB a rectangle equal to a given square with defect of a rectangle whose base SB is to its altitude $\mathsf{SL} = \mathsf{KB}$ as $\lambda+1$ is to 2. Thus, Na'īm's argument reduces the original problem of partition of a given segment according to a non-positional additive condition concerning rectangles to a problem of application of an area with defect of a rectangle whose sides are in a certain numerical proportion to each other. Although it largely depends on the mutual position of the relevant segments, this argument is replete with algebraic inferences. Moreover, it is, so to say, organized according to an algebraic plan: its structure seems to have been conceived so as to apply these inferences.

4 Conclusion

My examples should have shown that in his treatise Na'īm makes a large and relevant use of algebraic inferences. They occur both in proofs of theorems and in solutions of problems. In the latter case, they enter different sorts of problematic analyses which comply neither with the pattern of Pappusian analysis, nor with that of trans-configurational analysis, as I have described it elsewhere[37]. The consideration of Na'īm's *Collection* thus suggests that, during the second half of the 9th century, algebra—in the second sense of this term—was commonly practiced by Baghdadi mathematicians and especially used as a crucial tool for solving geometrical problems through appropriate analyses. The example of Na'īm's treatise also suggests that these analyses were mainly used to reduce a large family of these problems to al-Khwārizmī's equations understood as geometrical conditions. It seems thus that algebra, in the second sense of this term, was employed to enlarge the domain of application of the algebra, in the first sense of this term.

Though these two senses are thus essentially distinct from each other, they refer, respectively, to two pieces of mathematical practice that, for a Baghdadi mathematician of the second half of the 9th century, were essentially connected. What was normal for such a mathematician was certainly not so for other mathematicians working at different times and in different contexts. Still, it seems to me that this connection between a systematic discipline dealing with (polynomial) equations and a local art to be used to transform positional

[37]Cf. footnote (16) above.

conditions into non-positional ones, and then these last conditions into other non-positional ones, was an invariant structural feature of geometry for quite a long period. Insofar as this local art was largely dependent on the second book of the *Elements*, a similar judgment goes together with a historical appreciation of this book: it cannot be understood as a treatise in geometrical algebra, in the sense that has been traditionally given to such an expression[38]; but it certainly played a special role in the evolution of geometry, since it provided it with appropriate tools for transforming positional problems into non-positional ones.

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[38] Cf. [12]. On this matter, cf. also [11], [2], and [6].

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