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WEAK FACTORIZATIONS, FRACTIONS AND HOMOTOPIES

A. KURZ AND J. ROSICKÝ*

ABSTRACT. We show that the homotopy category can be assigned to any category equipped with a weak factorization system. A classical example of this construction is the stable category of modules. We discuss a connection with the open map approach to bisimulations proposed by Joyal, Nielsen and Winskel.

1. INTRODUCTION

Weak factorization systems originated in homotopy theory (see [Q], [Bo], [Be] and [AHRT]). Having a weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$, we can formally invert the morphisms from $\mathcal{R}$ and form the category of fractions $\mathcal{K}[\mathcal{R}^{-1}]$. From the point of view of homotopy theory, we invert too few morphisms: only trivial fibrations and not all weak equivalences. Our aim is to show that this procedure is important in many situations.

For instance, the class *Mono* of all monomorphisms form a left part of the weak factorization system $(\textit{Mono}, \mathcal{R})$ in a category $R\text{-}\mathbf{Mod}$ of (left) modules over a ring R . Then $R\text{-}\mathbf{Mod}[\mathcal{R}^{-1}]$ is the usual stable category of modules. Or, in the open map approach to bisimulations suggested in [JNW], one considers a weak factorization system $(\mathcal{L}, \mathcal{O}_{\mathcal{P}})$, where $\mathcal{O}_{\mathcal{P}}$ is the class of $\mathcal{P}$ -open morphisms w.r.t. a given full subcategory $\mathcal{P}$ of path objects. Then two objects K and L are $\mathcal{P}$ -bisimilar iff there is a span

$$K \xleftarrow{f} M \xrightarrow{g} L$$

of $\mathcal{P}$ -open morphisms. Any two $\mathcal{P}$ -bisimilar objects are isomorphic in the fraction category $\mathcal{K}[\mathcal{O}_{\mathcal{P}}^{-1}]$ but, in general, the fraction category makes more objects isomorphic than just $\mathcal{P}$ -bisimilar ones.

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Any weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ with finite coproducts yields a cylinder object in $\mathcal{K}$ and thus a relation $\sim$ of homotopy between morphisms of $\mathcal{K}$. We will show that any two homotopic morphisms have the same image in the fraction category. Moreover, if $\mathcal{K}$ has finite coproducts and any morphism in $\mathcal{R}$ is a split epimorphism then the categories $\mathcal{K}[\mathcal{R}^{-1}]$ and $\mathcal{K}/\sim$ are equivalent.

2. WEAK FACTORIZATION SYSTEMS

Definition 2.1. Let $\mathcal{K}$ be a category and $f : A \rightarrow B$, $g : C \rightarrow D$ morphisms such that in each commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{v} & D \end{array}$$

there is a diagonal $d : B \rightarrow C$ with $d \cdot f = u$ and $g \cdot d = v$. Then we say that g has the *right lifting property* w.r.t. f and f has the *left lifting property* w.r.t. g .

For a class $\mathcal{H}$ of morphisms of $\mathcal{K}$ we put

$$\mathcal{H}^\square = \{g \mid g \text{ has the right lifting property w.r.t. each } f \in \mathcal{H}\} \text{ and}$$

$${}^\square\mathcal{H} = \{f \mid f \text{ has the left lifting property w.r.t. each } g \in \mathcal{H}\}.$$

Definition 2.2. ([Be]) A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{K}$ such that

- (1) $\mathcal{R} = \mathcal{L}^\square$, $\mathcal{L} = {}^\square\mathcal{R}$ and
- (2) any morphism h of $\mathcal{K}$ has a factorization $h = g \cdot f$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

The category of fractions $\mathcal{K}[\mathcal{S}^{-1}]$, where $\mathcal{K}$ is a category and $\mathcal{S}$ a class of morphisms in $\mathcal{K}$, was introduced in [GZ] (see [Bor], [KP]). This category has the same objects as $\mathcal{K}$ and is equipped with a functor $P : \mathcal{K} \rightarrow \mathcal{K}[\mathcal{S}^{-1}]$ sending any morphism from $\mathcal{S}$ to an isomorphism. Moreover, for every functor $F : \mathcal{K} \rightarrow \mathcal{X}$ sending any morphism from $\mathcal{S}$ to an isomorphism, there is a unique functor $\bar{F} : \mathcal{K}[\mathcal{S}^{-1}] \rightarrow \mathcal{X}$ such that $F = \bar{F} \cdot P$. It may happen that the category $\mathcal{K}[\mathcal{S}^{-1}]$ is not locally small, i.e., that one can have a proper class of morphisms between two given objects in $\mathcal{K}[\mathcal{S}^{-1}]$.

Observation 2.3. $\mathcal{K}[\mathcal{S}^{-1}]$ is a quotient of the category of zig-zags

$$K \xrightarrow{f_1} X_1 \xleftarrow{f_2} X_2 \longrightarrow \dots \quad L$$

where all morphisms going backwards are in $\mathcal{S}$ (see [KP], II.2). If $\mathcal{K}$ has finite limits and $\mathcal{S}$ contains all isomorphisms, is closed under compositions and stable under pullbacks then these zig-zags can be reduced to spans

$$K \xleftarrow{s} X \xrightarrow{f} L$$

with $s \in \mathcal{S}$. In fact, a zig-zag is reduced to a span by means of pullbacks as follows:

$$\begin{array}{ccccccc} K & \xrightarrow{f_1} & X_1 & \xleftarrow{f_2} & X_2 & \longrightarrow & \cdots & L \\ & \nwarrow \bar{f}_2 & & \nearrow \bar{f}_1 & & & & \\ & & \overline{X_1} & & & & & \end{array}$$

The equivalence relation on spans giving the category of fractions is easy to describe if $\mathcal{S}$ also has the property: if $t \cdot f = t \cdot g$ with $t \in \mathcal{S}$ then $f \cdot s = g \cdot s$ for some $s \in \mathcal{S}$. One then says that $\mathcal{S}$ admits a right calculus of fractions (see [Bor]).

Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$ having finite limits. Then $\mathcal{R}$ contains all isomorphisms, is closed under compositions and stable under pullbacks (see [AHRT]). Hence the fraction category $\mathcal{K}[\mathcal{R}^{-1}]$ is a quotient category of the category of spans. But $\mathcal{R}$ rarely admits a right calculus of fractions.

Example 2.4. Let $(\mathcal{L}, \mathcal{R})$ be a factorization system in a category $\mathcal{K}$ having finite limits. This means that $(\mathcal{L}, \mathcal{R})$ is defined by means of a unique diagonalization property, i.e., that d in 2.1 is unique. Then $(\mathcal{L}, \mathcal{R})$ is a weak factorization system (see 14.6 (3) in [AHS]). We show that $\mathcal{R}$ admits a right calculus of fractions. Consider

$$K \xrightarrow[h_2]{h_1} L \xrightarrow{t} M$$

such that $t \cdot h_1 = t \cdot h_2$ and $t \in \mathcal{R}$. It suffices to show that the equalizer e of h_1 and h_2 belongs to $\mathcal{R}$. This means that we have to show that e has the unique diagonalization property w.r.t. each morphism $f \in \mathcal{L}$. Consider a commutative square

$$\begin{array}{ccc} X & \xrightarrow{u} & N \\ f \downarrow & & \downarrow e \\ Y & \xrightarrow{v} & K \end{array}$$

with $f \in \mathcal{S}$. Since

$$\begin{array}{ccc}
 X & \xrightarrow{h_1 e u} & L \\
 f \downarrow & \nearrow h_1 v & \downarrow t \\
 Y & \xrightarrow{t h_1 v} & M
 \end{array}$$

commutes, the unique diagonalization property yields that $h_1 \cdot v = h_2 \cdot v$. Thus $v = w \cdot e$ for some $w : Y \rightarrow N$. Since e is a monomorphism, w is the unique diagonal in the starting square.

In the case of the factorization system $(\text{Iso}(\mathcal{K}), \text{Mor}(\mathcal{K}))$, the fraction category $\mathcal{K}[\text{Mor}(\mathcal{K})^{-1}]$ is the set of connected components of $\mathcal{K}$. Here, $\text{Iso}(\mathcal{K})$ consists of isomorphisms of $\mathcal{K}$ and $\text{Mor}(\mathcal{K})$ of all morphisms of $\mathcal{K}$.

Observation 2.5. The class

$$\overline{\mathcal{S}} = \{f \mid P(f) \text{ is an isomorphism}\}$$

is called the *saturation* of $\mathcal{S}$. It is easy to see that $\overline{\mathcal{S}}$ is closed under retracts in the arrow category $\mathcal{K}^{\rightarrow}$ and has the 2-out-of-3 property, i.e., with any two of f , g , $g \cdot f$ belonging to $\overline{\mathcal{S}}$ also the third morphism belongs to $\overline{\mathcal{S}}$.

The following definition is motivated by [JNW].

Definition 2.6. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$. Two objects K and L are called *bisimilar* if there is a span

$$K \xleftarrow{f} X \xrightarrow{g} L$$

with $f, g \in \mathcal{R}$.

Observation 2.7. Any two bisimilar objects are clearly isomorphic in the fraction category $\mathcal{K}[\mathcal{R}^{-1}]$. If $\mathcal{K}$ has finite limits then bisimilarity is an equivalence relation. We will see later that (see 3.6), even in this case, two objects K , L may be isomorphic in $\mathcal{K}[\mathcal{R}^{-1}]$ without being bisimilar.

Observation 2.8. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$ having an initial object 0 . Then the following two conditions are equivalent:

- (1) any morphism from $\mathcal{R}$ is a split epimorphism,
- (2) any morphism $0 \rightarrow K$ belongs to $\mathcal{L}$.

Indeed, (1) $\Rightarrow$ (2) because a diagonal in a square

$$\begin{array}{ccc} 0 & \longrightarrow & L \\ \downarrow & & \downarrow f \\ K & \xrightarrow{v} & M \end{array}$$

is $t \cdot v$ where t splits $f \in \mathcal{R}$. Conversely, (2) $\Rightarrow$ (1) because a diagonal in a square

$$\begin{array}{ccc} 0 & \longrightarrow & L \\ \downarrow & & \downarrow f \\ K & \xrightarrow{\text{id}_K} & K \end{array}$$

splits $f \in \mathcal{R}$.

Observation 2.9. Let $\mathcal{K}$ be a locally presentable category (cf. [AR]) and $\mathcal{C}$ a set of morphisms in $\mathcal{K}$. Then $({}^\square(\mathcal{C}^\square), \mathcal{C}^\square)$ is a weak factorization system and ${}^\square(\mathcal{C}^\square)$ is the smallest class $\mathcal{L}$ containing $\mathcal{C}$ which is

- (a) closed under retracts in $\mathcal{K}^\rightarrow$,
- (b) closed under compositions and contains all isomorphisms,
- (c) stable under pushouts,
- (d) closed under transfinite compositions, i.e., given a smooth chain of morphisms $(f_{ij} : K_i \rightarrow K_j)_{i < j < \lambda}$ from $\mathcal{L}$ (i.e., λ is a limit ordinal, $f_{jk} \cdot f_{ij} = f_{ik}$ for $i < j < k$ and $f_{ij} : K_i \rightarrow K_j$ is a colimit cocone for any limit ordinal $j < \lambda$), then a colimit cocone $f_i : K_i \rightarrow K$ has $f_0 \in \mathcal{L}$.

(see [Be] or [AHRT]). Even, ${}^\square(\mathcal{C}^\square)$ consists of retracts of transfinite compositions of pushouts of morphisms from $\mathcal{C}$.

3. HOMOTOPIES

Definition 3.1. Let $\mathcal{K}$ be a category with finite coproducts equipped with a weak factorization system $(\mathcal{L}, \mathcal{R})$. For an object K of $\mathcal{K}$, we get a *cylinder object* $\overline{K}$ by a factorization of the codiagonal

$$\nabla : K + K \xrightarrow{c_K} \overline{K} \xrightarrow{s_K} K$$

with $c_K \in \mathcal{L}$ and $s_K \in \mathcal{R}$. We denote by

$$c_K^1 = c_K \cdot i_1 \quad \text{and} \quad c_K^2 = c_K \cdot i_2$$

the compositions of c_K with the coproduct injections

$$K \xrightarrow{i_1} K + K \xleftarrow{i_2} K.$$

There is a well-known way of getting homotopy from cylinder objects (see [KP]). Having two morphisms $f, g : K \rightarrow L$, we say that f and g are homotopic and write $f \sim g$ if there is a morphism $h : \overline{K} \rightarrow L$ such that the following diagram commutes

$$\begin{array}{ccc} K + K & \xrightarrow{(f,g)} & L \\ & \searrow c_K \quad \nearrow h & \\ & \overline{K} & \end{array}$$

Here, $(f, g) \cdot i_1 = f$ and $(f, g) \cdot i_2 = g$. The homotopy relation $\sim$ is clearly reflexive and symmetric. But, in general, the homotopy relation is not transitive. On the other hand, $\sim$ is compatible with the composition.

Lemma 3.2. *Let $\mathcal{K}$ be a category with finite coproducts equipped with a weak factorization system $(\mathcal{L}, \mathcal{R})$. Let $f, g : K \rightarrow L$, $u : L \rightarrow M$ and $v : N \rightarrow K$ be in $\mathcal{K}$ and $f \sim g$. Then $u \cdot f \sim u \cdot g$ and $f \cdot v \sim g \cdot v$.*

Proof. Let $h : \overline{K} \rightarrow L$ make f and g homotopic. Then $u \cdot h$ makes $u \cdot f$ and $u \cdot g$ homotopic. Using a lifting property, there is a morphism t such that both squares in the following diagram are commutative

$$\begin{array}{ccc} N + N & \xrightarrow{v+v} & K + K \\ c_N \downarrow & & \downarrow c_K \\ \overline{N} & \xrightarrow{\quad t \quad} & \overline{K} \\ s_N \downarrow & & \downarrow s_K \\ N & \xrightarrow{\quad v \quad} & K \end{array}$$

Then $h \cdot t$ makes $f \cdot v$ and $g \cdot v$ homotopic. □

Observation 3.3. The homotopy relation does not depend on a choice of a cylinder object. The reason is that, for two cylinder objects

$$\nabla : K + K \xrightarrow{c_K} \overline{K} \xrightarrow{s_K} K$$

and

$$\nabla : K + K \xrightarrow{\bar{c}_K} \overline{\overline{K}} \xrightarrow{\bar{s}_K} K,$$

we always have a diagonal t in the square

$$\begin{array}{ccc}
 K + K & \xrightarrow{c_K} & \overline{K} \\
 \bar{c}_K \downarrow & \nearrow t & \downarrow s_K \\
 \underline{\overline{K}} & \xrightarrow{\bar{s}_K} & K
 \end{array}$$

Let $\mathcal{K}$ be a Quillen model category (see [H]) and let $\mathcal{L}$ consist of cofibrations and $\mathcal{R}$ of trivial fibrations. Then $(\mathcal{L}, \mathcal{R})$ is a weak factorization system and any f, g homotopic in our sense are left homotopic in the standard sense. But the converse does not hold.

Any weak factorization system $(\mathcal{L}, \mathcal{R})$ gives rise to a Quillen model category if we take all morphisms of $\mathcal{K}$ as weak equivalences ([AHRT] 3.7). Then any two morphisms $f, g : K \rightarrow L$ are left homotopic because we have a model category cylinder

$$\nabla : K + K \xrightarrow{\text{id}} K + K \xrightarrow{\nabla} K$$

(because ∇ is a weak equivalence).

Let $\mathcal{K}$ be a category with finite coproducts equipped with a weak factorization system $(\mathcal{L}, \mathcal{R})$. We get the quotient category

$$Q : \mathcal{K} \rightarrow \mathcal{K}/\sim.$$

Following 3.2, $Q(f) = Q(g)$ iff f and g are in the transitive closure of the homotopy relation $\sim$, i.e., iff there are $f_1, \dots, f_n$ such that

$$f \sim f_1 \sim \dots \sim f_n \sim g.$$

Lemma 3.4. *Let $\mathcal{K}$ be a category with finite coproducts equipped with a weak factorization system $(\mathcal{L}, \mathcal{R})$. Then $P(f) = P(g)$ for any morphisms $f \sim g$.*

Proof. We have

$$P(s_K \cdot c_K^1) = P(\nabla \cdot i_1) = P(\nabla \cdot i_2) = P(s_K \cdot c_K^2).$$

Since $P(s_K)$ is an isomorphism, we have $P(c_K^1) = P(c_K^2)$. Thus $P(f) = P(g)$ for $f \sim g$. $\square$

We therefore have a unique functor T such that

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{P} & \mathcal{K}[\mathcal{R}^{-1}] \\ & \searrow Q \quad \nearrow T & \\ & \mathcal{K}/\sim & \end{array}$$

commutes. In general, one cannot expect that T is an equivalence.

Example 3.5. Let $\mathcal{K}$ have finite coproducts and consider the factorization system $(\text{Iso}(\mathcal{K}), \text{Mor}(\mathcal{K}))$. Then a cylinder object for K is $K + K$ and thus any two parallel morphisms are homotopic. Hence Q is the posetal reflection of $\mathcal{K}$. On the other hand, P is (up to equivalence), the projection of $\mathcal{K}$ to the set of connected components of $\mathcal{K}$.

If $(\mathcal{L}, \mathcal{R})$ is a weak factorization system and $\sim$ the associated homotopy then a morphism $f : K \rightarrow L$ is called a *homotopy equivalence* if there is $g : L \rightarrow K$ with $g \cdot f \sim \text{id}_K$ and $f \cdot g \sim \text{id}_L$. Every homotopy equivalence is sent by Q and, following 3.4, by P as well, to an isomorphism. Since $\sim$ is not transitive, Q inverts more morphisms than just homotopy equivalences. In fact, $Q(f)$ is an isomorphism iff there is g such that both $(g \cdot f, \text{id}_K)$ and $(f \cdot g, \text{id}_L)$ are in the transitive closure of $\sim$.

The following gives an example of a homotopy relation that is not transitive and of homotopy equivalent objects that are not bisimilar.

Example 3.6. Let $\mathbf{Set}^{\mathcal{X}}$ be the category of multigraphs with loops, i.e., $\mathcal{X}$ is the category

$$\begin{array}{ccc} & \xrightarrow{r_1} & \\ E & \xleftrightarrow[v]{\quad} & V \\ & \xrightarrow{r_2} & \end{array}$$

where $r_1 \cdot v = r_2 \cdot v = \text{id}_V$. Here, E is the object of edges, V is the object of vertices, r_1 and r_2 yield the initial and the final vertex of an edge and v yields loops. Let $\mathcal{L} = \text{Mono}$ be the class of all monomorphisms and $\mathcal{R}$ consist of morphisms $g : K \rightarrow L$ such that

- (a) g is surjective on vertices and
- (b) if vertices $g(a)$ and $g(b)$ are joined by an edge in L then a and b are joined by an edge in K .

In fact, this is the weak factorization system $(\square(\mathcal{C}^\square), \mathcal{C}^\square)$ from 2.9 where $\mathcal{C}$ consists of the embedding of an empty multigraph into a vertex and of the embedding of two vertices not connected by an edge to the edge.

The cylinder object $c_K : K + K \rightarrow \overline{K}$ is obtained by joining the two copies $i_1(x)$ and $i_2(x)$ of a vertex x in K by two edges, one going from

$i_1(x)$ to $i_2(x)$ and the other going from $i_2(x)$ to $i_1(x)$. Moreover, having an edge from x to y , there is an edge going from $i_1(x)$ to $i_2(y)$ and an edge going from $i_2(x)$ to $i_1(y)$. This means that $\overline{K} = E' \times K$ where E' is the non-oriented edge, i.e., a complete graph on two vertices.

Morphisms $f, g : K \rightarrow L$ are homotopic iff for each vertex x of K , we have an edge from $f(x)$ to $g(x)$ and an edge from $g(x)$ to $f(x)$ in L . Moreover, having an edge from x to y , there is an edge going from $f(x)$ to $g(y)$ and an edge going from $g(x)$ to $f(y)$. Thus the homotopy relation is not transitive.

The multigraphs (loops are not depicted)

are homotopy equivalent. In fact, the first multigraph K is a retract of the second multigraph L via $u \cdot v = \text{id}_K$ and the other composition $v \cdot u$ is homotopic to id_L . Hence K and L are isomorphic in $\mathcal{K}[\mathcal{R}^{-1}]$. But K and L are not bisimilar. Indeed, assume that there exist

$$K \xleftarrow{f} M \xrightarrow{g} L$$

with $f, g \in \mathcal{R}$. There are $x, y \in M$ with $g(x) = b$ and $g(y) = c$. Since $f(x)$ and $f(y)$ are joined by an edge in K , x and y are joined by an edge in M and thus b and c are connected by an edge in L ; a contradiction.

Lemma 3.7. *Let $\mathcal{K}$ be a category having finite coproducts and $(\mathcal{L}, \mathcal{R})$ be a weak factorization system such that every morphism in $\mathcal{R}$ is a split epimorphism. Then every morphism in $\mathcal{R}$ is a homotopy equivalence.*

Proof. Let $r : K \rightarrow L$ be in $\mathcal{R}$ and consider f with $r \cdot f = \text{id}_L$. It suffices to show that $f \cdot r \sim \text{id}_K$. Consider the square

$$\begin{array}{ccc} K + K & \xrightarrow{(f \cdot r, \text{id}_K)} & K \\ c_K \downarrow & \nearrow t & \downarrow r \\ \overline{K} & \xrightarrow{r \cdot s_K} & L \end{array}$$

which commutes because $r \cdot f \cdot r = r$. Since $c_K \in \mathcal{L}$ and $r \in \mathcal{R}$, there is a diagonal t , which is a homotopy from $f \cdot r$ to id_K . $\square$

Remark 3.8. We have proved a stronger statement: if $r \in \mathcal{R}$ is a split epimorphism then r is a homotopy equivalence. Following 2.8, if $0 \rightarrow L$ is in $\mathcal{L}$ then every $r : K \rightarrow L$ from $\mathcal{R}$ is a homotopy equivalence.

Theorem 3.9. *Let $\mathcal{K}$ be a category having finite coproducts and $(\mathcal{L}, \mathcal{R})$ be a weak factorization system such that every morphism in $\mathcal{R}$ is a split epimorphism. Then*

- (1) *the categories $\mathcal{K}[\mathcal{R}^{-1}]$ and $\mathcal{K}/\sim$ are isomorphic,*
- (2) *$\overline{\mathcal{R}} = \{f|Q(f) \text{ is an isomorphism}\}$ and*
- (3) *$\mathcal{K}[\mathcal{R}^{-1}]$ is a locally small category.*

Proof. (1) Following 3.7, Q inverts all morphism from $\mathcal{R}$. Thus we get a unique functor U such that the triangle

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{P} & \mathcal{K}[\mathcal{R}^{-1}] \\ & \searrow Q & \swarrow U \\ & \mathcal{K}/\sim & \end{array}$$

commutes. Since both $U \cdot T$ and $T \cdot U$ are the identities, T is an isomorphism. It immediately yields (2) to (3). $\square$

Observation 3.10. Assume that $(\mathcal{L}, \mathcal{R})$ is a weak factorization system in a category $\mathcal{K}$ such that every morphism in $\mathcal{R}$ is a split epimorphism. Let $\mathcal{R}'$ consist of compositions $r \cdot f$ where $r \in \mathcal{R}$ and f splits some $s \in \mathcal{R}$, i.e., $s \cdot f = \text{id}$. Then two objects K and L are bisimilar iff there is $h : K \rightarrow L$ in $\mathcal{R}'$.

- (1) $\mathcal{R}'$ is closed under compositions.

Consider the composition

$$K_1 \xrightarrow{f_1} X_1 \xrightarrow{r_1} K_2 \xrightarrow{f_2} X_2 \xrightarrow{r_2} K_3$$

where $r_1, r_2 \in \mathcal{R}$ and $s_1 \cdot f_1 = \text{id}_{K_1}$, $s_2 \cdot f_2 = \text{id}_{K_2}$ for $s_1, s_2 \in \mathcal{R}$.

Let

$$\begin{array}{ccc} L & \xrightarrow{\bar{s}_2} & X_1 \\ \bar{r}_1 \downarrow & & \downarrow r_1 \\ X_2 & \xrightarrow{s_2} & K_2 \end{array}$$

be a pullback. Then $\bar{r}_1, \bar{s}_2 \in \mathcal{R}$ and, since

$$s_2 \cdot f_2 \cdot r_1 = r_1,$$

there is a unique $t : X_1 \rightarrow L$ with

$$\bar{r}_1 \cdot t = f_2 \cdot r_1 \quad \text{and} \quad \bar{s}_2 \cdot t = \text{id}_{X_1} .$$

Thus

$$r_2 \cdot f_2 \cdot r_1 \cdot f_1 = r_2 \cdot \bar{r}_1 \cdot t \cdot f_1$$

where $r_2 \cdot \bar{r}_1 \in \mathcal{R}$ and $s_1 \cdot \bar{s}_2 \cdot t \cdot f_1 = \text{id}_{K_1}$, $s_1 \cdot \bar{s}_2 \in \mathcal{R}$. Hence $r_2 \cdot f_2 \cdot r_1 \cdot f_1 \in \mathcal{R}'$.

(2) If $g \cdot h$, $g \in \mathcal{R}'$ then $h \in \mathcal{R}'$. Let $g = r_1 \cdot f_1$ and $g \cdot h = r_2 \cdot f_2$ where $r_1, r_2 \in \mathcal{R}$ and f_i splits some $s_i \in \mathcal{R}$ for $i = 1, 2$.

Consider

$$\begin{array}{ccccccc} K_1 & \xrightarrow{h} & K_2 & \xrightarrow{f_1} & X & \xrightarrow{r_1} & K_3 \\ & \searrow t & & & \uparrow \bar{r}_2 & & \nearrow r_2 \\ & & & & L & & \\ & \searrow f_2 & & & \downarrow \bar{r}_1 & & \\ & & & & Y & & \end{array}$$

where $r_1, r_2 \in \mathcal{R}$ and $s_1 \cdot f_1 = \text{id}_{K_2}$, $s_2 \cdot f_2 = \text{id}_{K_1}$ for $s_1, s_2 \in \mathcal{R}$. Take a pullback of r_1 and r_2 and consider the induced morphism t . Since $s_2 \cdot \bar{r}_1, s_1 \cdot \bar{r}_2 \in \mathcal{R}$,

$$(s_2 \cdot \bar{r}_1) \cdot t = s_2 \cdot f_2 = \text{id}_{K_1} ,$$

and

$$(s_1 \cdot \bar{r}_2) \cdot t = s_1 \cdot f_1 \cdot h = h ,$$

we get $h \in \mathcal{R}'$.

(3) Any $g \in \mathcal{R}' \cap \mathcal{L}$ is a split monomorphism.

Let $g = r \cdot f$ where $s \cdot f = \text{id}$ and $r, s \in \mathcal{R}$. Consider the square

$$\begin{array}{ccc} K & \xrightarrow{f} & X \\ \downarrow r \cdot f & \nearrow t & \downarrow r \\ L & \xrightarrow{\text{id}_L} & L \end{array}$$

If $g \in \mathcal{L}$ we get a diagonal t and we have

$$s \cdot t \cdot (r \cdot f) = s \cdot f = \text{id}_K .$$

(4) $\mathcal{R}'$ does not need to have the 2-out-of-3 property.

Let $\mathbf{Set}^{\mathcal{X}}$ be the category of multigraphs with loops from 3.6 and consider the following multigraphs K and L (loops are not depicted):

Let $f : K \rightarrow L$ be the embedding (i.e., $f(a) = a$ and $f(b) = b$) and $s, t : L \rightarrow K$ split f by means of $s(c) = a$ and $t(c) = b$. Then $s \in \mathcal{R}$ and thus $f \in \mathcal{R}'$. Assume that $t \in \mathcal{R}'$. Then $t = r \cdot g$ where $r \in \mathcal{R}$ and $s' \cdot g = \text{id}$ for some $s' \in \mathcal{R}$. Then there is an edge from $g(b)$ to $g(c)$ and thus an edge from b to c ; a contradiction.

4. STABLE EQUIVALENCES

A full subcategory $\mathcal{M}$ of a category $\mathcal{K}$ is *weakly reflective* if for every object K in $\mathcal{K}$ there is a morphism $r_K : K \rightarrow K^*$ with $K^* \in \mathcal{M}$ such that every morphism $f : K \rightarrow M$, $M \in \mathcal{M}$ factorizes through r_K , i.e., $f = g \cdot r_K$.

Observation 4.1. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$ with finite products and $\mathcal{L}^\Delta$ the full subcategory of $\mathcal{K}$ consisting of objects M injective w.r.t. any morphism from $\mathcal{L}$. This means that for every $f : X \rightarrow Y$ in $\mathcal{L}$ and every $h : X \rightarrow M$ there is $g : Y \rightarrow M$ with $g \cdot f = h$. Then $\mathcal{L}^\Delta$ is weakly reflective in $\mathcal{K}$ where a weak reflection is given by a factorization

$$K \xrightarrow{r_K} K^* \xrightarrow{t} 1$$

with $r_K \in \mathcal{L}$ and $t \in \mathcal{R}$ of the unique morphism into the terminal object 1. Let $(\mathcal{L}^\Delta)_{inj}$ consist of all morphism f such that each $M \in \mathcal{L}^\Delta$ is injective to f . Then

$$(\mathcal{L}^\Delta)_{inj} = \{f \mid \text{there exists } g \in \mathcal{L} \text{ with } g = h \cdot f \text{ for some } h\}$$

which means that $(\mathcal{L}^\Delta)_{inj}$ is the left cancellable closure of $\mathcal{L}$.

Indeed, it is easy to see that the left cancellable closure of $\mathcal{L}$ is contained in $\mathcal{L}^\Delta_{inj}$. Conversely, having $f : K \rightarrow L$ in $\mathcal{L}$, then a weak reflection $r_K : K \rightarrow K^*$ factorizes through f . Thus f belongs to the left cancellable closure of $\mathcal{L}$. Consequently, $\mathcal{L} = (\mathcal{L}^\Delta)_{inj}$ whenever $\mathcal{L}$ is left cancellable.

Conversely, consider a weakly reflective full subcategory $\mathcal{M}$ of $\mathcal{K}$. Then, following [AHRT] 1.5, $(\mathcal{M}_{inj}, (\mathcal{M}_{inj})^\square)$ is a weak factorization

system in $\mathcal{K}$. A weak factorization of $h : A \rightarrow B$ is

$$h : K \xrightarrow{\langle r_K, h \rangle} K^* \times L \xrightarrow{p_2} L.$$

Therefore, weak factorization systems $(\mathcal{L}, \mathcal{R})$ in $\mathcal{K}$ with $\mathcal{L}$ left cancellable are precisely $(\mathcal{M}_{inj}, (\mathcal{M}_{inj})^\square)$ for a weakly reflective full subcategory $\mathcal{M}$ of $\mathcal{K}$.

Let $\mathcal{M}$ be a weakly reflective full subcategory of an additive category $\mathcal{K}$. Morphisms $f, g : K \rightarrow L$ are called $\mathcal{M}$ -stably equivalent if $f - g$ factorizes through an object M from $\mathcal{M}$. It is easy to see that this is an equivalence relation compatible with the composition. One gets the stable category $\mathcal{K}/\mathcal{M}$ and the projection $S : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{M}$. We have the following results which are almost completely contained in [B] 4.5.

Lemma 4.2. *Let $\mathcal{M}$ be a weakly reflective full subcategory of an additive category $\mathcal{K}$. Then morphisms $f, g : K \rightarrow L$ are $\mathcal{M}$ -stably equivalent iff $f \sim g$ with respect to the weak factorization system $(\mathcal{M}_{inj}, (\mathcal{M}_{inj})^\square)$.*

Proof. Following 4.1, cylinder objects are

$$K \oplus K \xrightarrow{c_K} K^* \oplus K^* \oplus K \xrightarrow{p_3} K$$

where $p_i \cdot c_K^j = r_K$ for $i = 1, 2$ and $j = 1, 2$. Of course, $p_3 \cdot c_K^j = \text{id}_K$ for $j = 1, 2$. Hence $c_K^1 - c_K^2$ factorizes through $K^* \oplus K^*$ and thus c_K^1 and c_K^2 are $\mathcal{M}$ -stably equivalent. Since $\mathcal{M}$ -stable equivalence is compatible with the composition, $f \sim g$ implies that f and g are $\mathcal{M}$ -stably equivalent. Conversely, let f and g be $\mathcal{M}$ -stably equivalent, i.e., we have

$$f - g : K \xrightarrow{u} M \xrightarrow{v} L$$

with $M \in \mathcal{M}$. Then $f \sim g$ via

$$h : K^* \oplus K^* \oplus K \longrightarrow L$$

given by $h \cdot i_1 = 0$, $h \cdot i_2 = -v \cdot t$ and $h \cdot i_3 = f$ where

$$\begin{array}{ccc} K & \xrightarrow{u} & M \\ & \searrow r_K & \nearrow t \\ & K^* & \end{array}$$

□

The proof contains the description of a cylinder object using weak reflections to $\mathcal{M}$, which may be useful in homotopy theory of additive categories. We will need this in the following proof.

Consequently, homotopy equivalences coincide with $\mathcal{M}$ -stable equivalences, i.e., with morphisms f admitting g with $g \cdot f$ and $f \cdot g$ $\mathcal{M}$ -stably equivalent to the identities. A monomorphism f is called an $\mathcal{M}$ -monomorphism if its cokernel belongs to $\mathcal{M}$ and an epimorphism g is called an $\mathcal{M}$ -epimorphism if its kernel belongs to $\mathcal{M}$.

Theorem 4.3. *Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in an additive category $\mathcal{K}$ such that $\mathcal{L}$ is left cancellable. Then the following conditions are equivalent for a morphism h :*

- (i) $h \in \overline{\mathcal{R}}$,
- (ii) h is a homotopy equivalence,
- (iii) h is an $\mathcal{L}^\Delta$ -stable equivalence, and
- (iv) $h = g \cdot f$ with f a split $\mathcal{L}^\Delta$ -monomorphism and g a split $\mathcal{L}^\Delta$ -epimorphism.

Proof. (i) $\Rightarrow$ (iii). Consider $h \in \mathcal{R}$ and the factorization

$$h : K \xrightarrow{\langle r_K, h \rangle} K^* \oplus L \xrightarrow{p_2} L$$

from 4.1. We have

$$\begin{array}{ccc} K & \xrightarrow{\text{id}_K} & K \\ \langle r_K, h \rangle \downarrow & \nearrow t & \downarrow h \\ K^* \oplus L & \xrightarrow{p_2} & L \end{array}$$

and thus h is a retract of p_2 in $\mathcal{K}^\rightarrow$ via

$$\begin{array}{ccc} K & \xrightarrow{h} & L \\ \langle r_K, h \rangle \downarrow & & \downarrow \text{id}_L \\ K^* \oplus L & \xrightarrow{p_2} & L \end{array} \quad \text{and} \quad \begin{array}{ccc} K & \xrightarrow{h} & L \\ t \uparrow & & \uparrow \text{id}_L \\ K^* \oplus L & \xrightarrow{p_2} & L \end{array}$$

Since $p_2 \cdot i_2 = \text{id}_L$ and $i_2 \cdot p_2 = \text{id}_{K^* \oplus L} - i_1 \cdot p_1$, p_2 is an $\mathcal{L}^\Delta$ -stable equivalence with an $\mathcal{L}^\Delta$ -stable inverse i_2 . Hence $S : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{L}^\Delta$ factorizes through $P : \mathcal{K} \rightarrow \mathcal{K}[\mathcal{R}^{-1}]$ and thus each morphism from $\overline{\mathcal{R}}$ is an $\mathcal{L}^\Delta$ -stable equivalence.

(ii) $\Leftrightarrow$ (iii) following 4.2

(iii) $\Rightarrow$ (iv). Let $h = g \cdot f$ be a factorization of an $\mathcal{L}^\Delta$ -stable equivalence with $f \in \mathcal{L}$ and $g \in \mathcal{R}$. Following the proof of (i) $\Rightarrow$ (iii), g is a retract of a split $\mathcal{L}^\Delta$ -epimorphisms p_2 . Hence g is a split $\mathcal{L}^\Delta$ -epimorphism.

Since (i) $\Rightarrow$ (iii), g is an $\mathcal{L}^\Delta$ -stable equivalence and thus f is an $\mathcal{L}^\Delta$ -stable equivalence. Let t be an $\mathcal{L}^\Delta$ -stable inverse of f . Then $t \cdot f$ is $\mathcal{L}^\Delta$ -stably equivalent to the identity and thus we have a factorization

$$\text{id}_K - t \cdot f : K \xrightarrow{u} M \xrightarrow{v} K$$

through $M \in \mathcal{L}^\Delta$. Since $f \in \mathcal{L}$, we have a factorization

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ & \searrow u & \swarrow w \\ & M & \end{array}$$

We have

$$\text{id}_K - t \cdot f = v \cdot u = v \cdot w \cdot f$$

and thus

$$\text{id}_K = (v \cdot w + t) \cdot f.$$

Thus f is a split monomorphism. Consequently, f has a cokernel $g : L \rightarrow P$ which is a split epimorphism.

Consider a morphism h such that $S(h \cdot f) = 0$ in $\mathcal{K}/\mathcal{L}^\Delta$. Then $h \cdot f$ factorizes through $N \in \mathcal{L}^\Delta$

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \downarrow r & \searrow p & \downarrow h \\ N & \xrightarrow{s} & X \end{array}$$

Since $f \in \mathcal{L}$, there is $p : L \rightarrow N$ with $p \cdot f = r$. We have

$$(h - s \cdot p) \cdot f = h \cdot f - s \cdot p \cdot f = h \cdot f - s \cdot r = 0$$

and thus there is $q : P \rightarrow X$ such that

$$q \cdot g = h - s \cdot p.$$

Hence $S(q \cdot g) = S(h)$ and thus $S(g)$ is a weak cokernel of $S(f)$ in $\mathcal{K}/\mathcal{L}^\Delta$. Since $S(g)$ is a split epimorphism, $S(g)$ is a cokernel of $S(f)$ in $\mathcal{K}/\mathcal{L}^\Delta$. However, $S(f)$ is an isomorphism and therefore P is the null object in $\mathcal{K}/\mathcal{L}^\Delta$. Consequently, $P \in \mathcal{L}^\Delta$, which proves that f is a split $\mathcal{L}^\Delta$ -monomorphism.

(iv) $\Rightarrow$ (i) Let $f : K \rightarrow L$ be a split $\mathcal{L}^\Delta$ -monomorphism. Then f is an injection of a biproduct

$$f : K \longrightarrow K \oplus M$$

where $M \in \mathcal{L}^\Delta$. Since the corresponding projection belongs to $\mathcal{R}$ (see 4.1 or the proof of 1.6 in [AHRT]), we get $f \in \overline{\mathcal{R}}$. Any split $\mathcal{L}^\Delta$ -epimorphism $g : K \rightarrow L$ is a projection of a biproduct

$$L \oplus M \longrightarrow L$$

with $M \in \mathcal{L}^\Delta$ and thus belongs to $\mathcal{R}$. $\square$

Observation 4.4. Following the proof of (iii) $\Rightarrow$ (iv), any $g \in \mathcal{R}$ is a split $\mathcal{L}^\Delta$ -epimorphism. Conversely, following (iv) $\Rightarrow$ (i), any split $\mathcal{L}^\Delta$ -epimorphism belongs to $\mathcal{R}$. Thus compositions $g \cdot f$ from (iv) are precisely morphism from $\mathcal{R}'$ (cf. 3.10). Consequently, homotopy equivalent objects are precisely bisimilar objects.

Example 4.5. Consider the category $R\text{-Mod}$ of R -modules and the class $\mathcal{L} = \text{Mono}$ of all monomorphisms. Then $\mathcal{L}^\Delta$ consists of injective R -modules and $\mathcal{R} = \text{Mono}^\square$ of all epimorphisms with an injective kernel. Then $(\mathcal{L}, \mathcal{R})$ is a weak factorization system, $\mathcal{L}$ is left cancellable and $R\text{-Mod}/\mathcal{L}^\Delta$ is the usual stable category of modules.

5. PRESHAVES OVER POSETS

An object K of a category $\mathcal{K}$ is called *indecomposable* if the hom-functor $\text{hom}(K, -) : \mathcal{K} \rightarrow \mathbf{Set}$ preserves binary coproducts.

Proposition 5.1. *Let $\mathcal{K}$ be a category with finite coproducts and $(\mathcal{L}, \mathcal{R})$ a weak factorization system such that $\mathcal{R} = \mathcal{C}^\square$ where every morphism from $\mathcal{C}$ has an indecomposable domain. Then $\mathcal{K}/\sim$ is equivalent to a poset.*

Proof. It suffices to show that $f \sim g$ for each $f, g : K \rightarrow L$. Consider a commutative square

$$\begin{array}{ccc} X & \xrightarrow{u} & K + K \\ \downarrow f & \searrow u' & \nearrow i_1 \\ & K & \\ \uparrow v & \nwarrow \text{id}_K & \\ Y & \xrightarrow{v} & K \end{array} \quad \begin{array}{c} \\ \\ \nabla \\ \end{array}$$

with $f \in \mathcal{C}$. Then X is indecomposable and thus u factorizes through one of the coproduct injections, say, $u = i_1 \cdot u'$. Then

$$i_1 \cdot v \cdot f = i_1 \cdot \nabla \cdot u = i_1 \cdot \nabla \cdot i_1 \cdot u' = i_1 \cdot u' = u$$

and $\nabla \cdot i_1 \cdot v = v$. Hence $\nabla \in \mathcal{R}$ and thus the cylinder object is $\overline{K} = K + K$ with

$$c_K = \text{id}_{K+K} : K + K \longrightarrow K + K.$$

Consequently, $f \sim g$ for any $f, g : K \rightarrow L$. $\square$

Remark 5.2. If $\mathcal{K}/\sim$ is equivalent to a poset then two objects K and L have $QK \cong QL$ iff there are morphisms both $K \rightarrow L$ and $L \rightarrow K$.

Proposition 5.3. *Let $\mathcal{K}$ have finite limits and finite coproducts and $(\mathcal{L}, \mathcal{R})$ be a weak factorization system such that $\mathcal{K}/\sim$ is equivalent to a poset. Then $\mathcal{K}[\mathcal{R}^{-1}]$ is equivalent to a poset.*

Proof. Following 2.3, $\mathcal{K}[\mathcal{R}^{-1}]$ is a quotient of the category of spans

$$K \xleftarrow{r} X \xrightarrow{f} L$$

with $r \in \mathcal{R}$. Consider two spans from K to L

$$\begin{array}{ccccc} K & \xleftarrow{r_1} & X_1 & \xrightarrow{f_1} & L \\ \uparrow r_2 & & \uparrow \bar{r}_2 & & \\ X_2 & \xleftarrow{\bar{r}_1} & X & & \\ \downarrow f_2 & & & & \\ & & & & L \end{array}$$

and a pullback of r_1 and r_2 . Following our assumption the objects QX and QL are either isomorphic in $\mathcal{K}/\sim$ or $Q(f_1 \cdot \bar{r}_2) = Q(f_2 \cdot \bar{r}_1)$. Following 3.4, the objects PX and PL are either isomorphic in $\mathcal{K}[\mathcal{R}^{-1}]$ or $P(f_1 \cdot \bar{r}_2) = P(f_2 \cdot \bar{r}_1)$. In the first case, $PK \cong PL$. In the second case, we have

$$\begin{aligned} P(f_1) \cdot P(r_1)^{-1} &= P(f_1) \cdot P(\bar{r}_2) \cdot P(\bar{r}_2)^{-1} \cdot P(r_1)^{-1} \\ &= P(f_1) \cdot P(\bar{r}_2) \cdot P(\bar{r}_1)^{-1} \cdot P(r_2)^{-1} \\ &= P(f_2) \cdot P(\bar{r}_1) \cdot P(\bar{r}_1)^{-1} \cdot P(r_2)^{-1} = P(f_2) \cdot P(r_2)^{-1}. \end{aligned}$$

Thus the starting spans yield the same morphism in $\mathcal{K}[\mathcal{R}^{-1}]$. $\square$

Remark 5.4. If all morphisms in $\mathcal{R}$ are split epimorphisms then 5.3 immediately follows from 3.9.

Observation 5.5. Let $\mathcal{P}$ be a poset and consider the category $\mathcal{K} = \mathbf{Set}^{\mathcal{P}^{\text{op}}}$ of presheaves on $\mathcal{P}$. Let $\mathcal{P}_\perp$ be the full subcategory of $\mathbf{Set}^{\mathcal{P}^{\text{op}}}$ consisting of the image of $\mathcal{P}$ in the Yoneda embedding $Y : \mathcal{P} \rightarrow \mathbf{Set}^{\mathcal{P}^{\text{op}}}$ with the initial presheaf 0 added. Then $\mathcal{P}_\perp$ is nothing else than $\mathcal{P}$ with a new initial element added. Following 2.9, we get a weak factorization system $(\square(\mathcal{P}_\perp^\square), \mathcal{P}_\perp^\square)$ in $\mathbf{Set}^{\mathcal{P}^{\text{op}}}$. Since all objects in $\mathcal{P}_\perp$ are indecomposable, it follows from 5.1 and 5.3 that both categories $\mathcal{K}/\sim$ and $\mathcal{K}[(\mathcal{P}_\perp^\square)^{-1}]$ are equivalent to posets. Bisimilarity in this situation was used in [JNW] to formalize bisimilarity of processes. $\mathcal{P}_\perp^\square$ is the class of $\mathcal{P}_\perp$ -open maps. 5.1 and 3.9 show that inverting all $\mathcal{P}_\perp$ -open maps gives the category that has the same objects as $\mathbf{Set}^{\mathcal{P}^{\text{op}}}$ and an arrow $K \rightarrow L$ iff there is some arrow (simulation) $K \rightarrow L$ in $\mathbf{Set}^{\mathcal{P}^{\text{op}}}$.

Proposition 5.6. *Let $\mathcal{P}$ be a poset. Then the following conditions are equivalent:*

- (i) $\square(\mathcal{P}_\perp^\square)$ coincides with the class *Mono* of all monomorphisms,
- (ii) for every element $x \in \mathcal{P}$, any non-empty subset of $\{y \in \mathcal{P} \mid y \leq x\}$ has a greatest element.

Proof. (i) $\Rightarrow$ (ii): Assume (i) and consider $x \in \mathcal{P}$ and a non-empty subset Z of $\{y \in \mathcal{P} \mid y \leq x\}$. Let K be a subfunctor of the representable functor $Y(x) : \mathcal{P}^{\text{op}} \rightarrow \mathbf{Set}$ given as follows

$$K(p) = \begin{cases} 1 & \text{if } p \leq z \text{ for some } z \in Z \\ 0 & \text{otherwise} \end{cases}$$

(here $0 = \emptyset$ and $1 = \{\emptyset\}$). Then the embedding $K \rightarrow Y(x)$ belongs to $\square(\mathcal{P}_\perp^\square)$ and thus, following 2.9, is a retract of a transfinite composition g_λ of a chain $g_{ij} : K_i \rightarrow K_j$, $i \leq j < \lambda$ (λ is a limit ordinal) where $K_0 = K$, g_{ii+1} is a pushout of a morphism in $\mathcal{P}_\perp$ and, for j limit, g_{ij} is a colimit of g_{ik} , $i \leq k < j$. Let i be the smallest ordinal $i \leq \lambda$ such that $g_i = g_{0i}$ factorizes through a representable functor. It makes sense because g_λ has this property. Since representable functors are finitely presentable (cf. [AR] 1.2.(7)), i is either an isolated ordinal or $i = 0$.

Assume that $i = j + 1$. Then we have a pushout below where $A, B \in \mathcal{P}_\perp$ and a factorization of g_i through a representable functor C

$$\begin{array}{ccccc}
 & & A & \xrightarrow{f} & B \\
 & & \downarrow u & & \downarrow v \\
 K & \xrightarrow{g_j} & K_j & \xrightarrow{g_{ji}} & K_i \\
 & \searrow h & & \nearrow t & \\
 & & C & &
 \end{array}$$

Then either $t = g_{ji} \cdot p$ for some p or $t = v \cdot q$ for some q . In the first case,

$$g_{ji} \cdot g_j = g_i = t \cdot h = g_{ji} \cdot p \cdot h$$

and thus $p \cdot h = g_j$ because g_{ji} is a monomorphism. Hence g_j factorizes through a representable functor, which contradicts the definition of i . In the second case,

$$g_{ji} \cdot g_j = t \cdot h = v \cdot q \cdot h$$

and thus there is a unique $w : K \rightarrow A$ such that

$$u \cdot w = g_j \quad \text{and} \quad f \cdot w = q \cdot h.$$

Since $K \neq 0$, we have $A \neq 0$ as well and thus A is representable. Hence g_j factorizes through a representable functor again; a contradiction.

Therefore $i = 0$, which means that $g_0 = \text{id}_K$ factorizes through a representable functor. Thus K is a retract of a representable functor and, since $\mathcal{P}$ is a poset, K is representable. Thus Z has the greatest element.

(ii) $\Rightarrow$ (i) The condition (ii) clearly means that subfunctors of representable functors are representable or 0. Since $\mathcal{P}$ is a poset, quotients of representable functors are representable. Following the proof of [Be] 1.12, any monomorphism in $\mathbf{Set}^{\mathcal{P}^{\text{op}}}$ belongs to $\square(\mathcal{P}_\perp^\square)$. $\square$

Corollary 5.7. *Let $\mathcal{P}$ be a poset such that, for every element $x \in \mathcal{P}$, any non-empty subset of $\{y \in \mathcal{P} \mid y \leq x\}$ has a greatest element. Then $\mathbf{Set}^{\mathcal{P}^{\text{op}}}[(\text{Mono}^\square)^{-1}]$ is equivalent to a poset.*

Example 5.8. Let $\mathcal{P}$ be a two-element chain. Then $\mathbf{Set}^{\mathcal{P}^{\text{op}}} = \mathbf{Set}^\rightarrow$ is the category of maps and $\mathcal{P}_\perp$ has three elements $o_0 : 0 \rightarrow 0$, $o_1 : 0 \rightarrow 1$

and $\text{id}_1 : 1 \rightarrow 1$. Then a morphism $(u_1, u_2) : f \rightarrow g$

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & A_2 \\ u_1 \downarrow & & \downarrow u_2 \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

belongs to $\mathcal{P}_\perp^\square = \text{Mono}^\square$ iff u_2 is surjective and for every $a \in A_2$ and $b \in B_1$ with $g(b) = u_2(a)$ there is $c \in A_1$ such that $f(c) = a$ and $u_1(c) = b$. This implies that u_1 is surjective as well. In other words, (u_1, u_2) is a (surjective) bisimulation, that is, using the transition system notation $b_0 \rightarrow b$ for $g(b) = b_0$ and $a \rightarrow c$ for $f(c) = a$, it holds that for any “transition” $b_0 \rightarrow b$ and any a with $u_2(a) = b_0$ there is c such that $a \rightarrow c$ and $u_1(c) = b$.

Following 5.7, we have that $\mathbf{Set}^\neg[(\text{Mono}^\square)^{-1}] = \mathbf{Set}^\neg / \sim$ is equivalent to a poset. The objects id_1 and $\text{id}_1 + o_1$ are isomorphic in this category because there are morphisms

$$\text{id}_1 \rightarrow \text{id}_1 + o_1 \quad \text{and} \quad \text{id}_1 + o_1 \rightarrow \text{id}_1 .$$

But the objects id_1 and $\text{id}_1 + o_1$ are not bisimilar. In fact, assume that there are $f : A \rightarrow B$ and morphisms

$$\text{id}_1 \xleftarrow{(u_1, u_2)} f \xrightarrow{(v_1, v_2)} \text{id}_1 + o_1$$

in $\text{Mono}^\square$. Then (u_1, u_2) makes f surjective and thus $\text{id}_1 + o_1$ is surjective as well; a contradiction.

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