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Taxes and Share Valuation in Competitive Markets

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Table 1 below lists the series used, their source and the results of the spectral estimates. The results of the harmonic trend version, as with the Canadian data, are generally favourable to the hypothesis with long cycles indicated in about three-fifth of the series, including most of the more important ones such as GNP, NNP, investment, employment, construction, prices, etc. The indication of a peak at 40 years in population and immigration, two very important variables in theoretical discussions on long cycles, likely reflects a very strong trend in these series which has been only partially removed.

Again, the growth-rate version is extremely sympathetic toward the hypothesis, indicating long swings in 34 of the 44 series. This is likely the most significant result since taking first differences is currently the most popular method of trend elimination among long-swing students. It will also be noted that the period of the indicated swing in growth rates is shorter than that for deviations from trend. This conforms with results obtained by other workers, in the United States and Canada, using less sophisticated techniques. Those working with growth rates using some sort of moving average have found long cycles to be about 14 years in duration while those applying the same sort of method to deviations from trend have found long swings which average about 22 years.

Thus, the overall results suggest that there are, in fact, long swings in the deviations from trend of a significant number of important time series, contrary to the findings of earlier spectral analysts like Adelman and Hatanaka and Howrey who found no evidence of long cycles in the series they tested.

Given the results of Bird et al.[8] which cast serious doubt on the reliability of the methods which have traditionally been used for the analysis of long swings, spectral analysis offers the most reliable and sophisticated method for investigating the existence of this phenomenon. While many may feel that their belief in long cycles has been vindicated by this note, we might add a note of caution that any chronologies of long swings based on the old techniques are still useless, given the biases uncovered by Bird et al. Not only existence, but chronology, will have to be the object of new techniques.

[8] Roger C. Bird, Meghnad J. Desai, Jared J. Enzler and Paul Taubman, "'Kuznets Cycles' in Growth Rates: The Meaning," *Indian Economic Review*, VI (May 1965), pp. 229–239.

TAXES AND SHARE VALUATION IN COMPETITIVE MARKETS

Vernon L. Smith

This paper extends the fundamental theorem of share (or capital) valuation [1, 2, 3], under conditions of certainty and purely competitive markets, to allow for the distinction between capital gains and income in the taxation of personal income. The objective is to develop the theorem for the tax case in a form general enough to allow for corporations both currently and not currently paying a dividend. However, the general derivation is sufficiently tedious to warrant a presentation which begins with less general cases. Accordingly, we will first develop the share valuation equation for a continuous discount version of the taxless case for corporations either paying or not paying a dividend. Then we turn to the effect of income and capital gains taxes for corporations currently paying a dividend; and finally the more general case. The derivations will be simplified by assuming a constant rate of interest over time, but all the theorems can be extended to deal with foreseen changes in interest over time.

I Share Valuation in the Absence of Taxes

In a world without risk or taxes, the competitive market value of a corporation's outstanding common stock $V(t)$, at time t, must equal the present worth or capitalized value at t of the future dividend payments of the corporation. Suppose the corporation is not currently paying a dividend, but it is known that at time t^* (i.e., $t^* - t$ years in the future) dividends will begin, and be paid at an annual rate $D(\tau) > 0$, $\tau \geqq t^*$. If r' is the continuous equilibrium force of interest (equal also to the marginal productivity of capital), then the market value of the corporation is

$$V(t) = \begin{cases} e^{-r'(t^*-t)} \displaystyle\int_{t^*}^{\infty} D(\tau)\, e^{-r'(\tau-t^*)}\, d\tau, & \text{if } t < t^* \\ \displaystyle\int_{t}^{\infty} D(\tau)\, e^{-r'(\tau-t)}\, d\tau, & \text{if } t \geqq t^*. \end{cases} \quad (1)$$

Differentiating (1), to evaluate $\dot{V}(t) \equiv \frac{dV(t)}{dt}$, and then substituting from (1), gives

$$\dot{V}(t) = \begin{cases} r'\, V(t), & \text{if } t < t^* \\ r'\, V(t) - D(t), & \text{if } t \geqq t^* \end{cases} \quad (2)$$

Equation (2) constitutes a continuous form of the fundamental theorem of share valuation, derived by Samuelson [3], for the case $t \geqq t^*$, in the context of capital-income theory and by Modigliani and Miller [2, p. 412] in the context of share valuation theory. Thus, for a corporation not yet paying a dividend, its market value rises at a percentage rate equal to the instantaneous return on capital (equal to the market rate of interest) until dividend payments begin. Thereafter, share value increases at a percentage rate equal to the difference between the return on capital and the percentage dividend yield i.e., $r'(t) - \frac{D(t)}{V(t)}$.[1]

II Taxes and Share Values for Dividend-Paying Corporations

Assume next that all investors in the shares of a given corporation are subject to a personal income tax on dividends, and a tax at a different rate on capital gains realized over holding periods of at least T years.[2] Let λ_i be the proportion of the corporation's stock held by investors in the i^{th} marginal income tax bracket. Let the marginal income tax rate in the i^{th} bracket be a_i, and the marginal capital gains (or loss) tax rate be b_i. After tax dividend income is thus $\lambda_i D(\tau) - \lambda_i a_i D(\tau)$ at time τ for investors in the i^{th} bracket. If there are n brackets the total after tax dividend income of the corporation's stock holders is $\alpha D(\tau)$, where

$$\alpha = \sum_{i=1}^{n} \lambda_i(1 - a_i) = 1 - \sum_{i=1}^{n} \lambda_i a_i.$$

Similarly, total after tax capital gains income, on a gain of $V(t+T) - V(t)$, is $\beta[V(t+T) - V(t)]$, where

$$\beta = \sum_{i=1}^{n} \lambda_i(1 - b_i) = 1 - \sum_{i=1}^{n} \lambda_i b_i.$$

We assume no change in the income distribution of stock holders over time so that α and β are independent of t.

If the after tax force of interest is r, the valuation formula corresponding to (1) for a dividend paying corporation is

$$V(t) = \int_t^{t+T} \alpha D(\tau) e^{-r(\tau - t)}\, d\tau + \beta[V(t+T) - V(t)]e^{-rT} + V(t)e^{-rT}. \quad (3)$$

In equilibrium share value is the present worth of the net return from dividends and capital gains plus capital recovery (the three terms on the right).

When the equality in (3) holds, an investor would be indifferent between holding a share in $V(t)$ and lending the equivalent sum at interest, r. If $V(t)$ exceeded the after tax discounted value of dividends and capital, then shares would be overvalued, and investors could gain by selling their holdings and lending at interest (or would elect to lend at interest in place of purchasing shares). This would depress share values, and lower the interest rate. The process would continue causing shares to be more attractive and lending to be less attractive until the equality held in (3). Similarly, if $V(t)$ was below the discounted value on the right, shares would be purchased, and funds borrowed, increasing share values and interest rates until the equality held.

In a perfect capital market, with foreseen dividends and capital gains, it is also the case, in equilibrium, that no investor would desire to realize a capital gain after a holding period of less than T. If he did, the gain would be taxed as income at a higher rate than if the holding period is T or more. He would therefore find it preferable to borrow to satisfy any current cash needs in excess of current income. For holding periods in excess of T, he would be indifferent between selling shares, and borrowing to raise money, as the two alternatives would have identical effects upon his asset position. It follows, that equilibrium requires the market to continuously discount the after tax capital gain $\beta[V(t+T) - V(t)]$, which is potentially realizable T years in the future.

We first collect terms in $V(t)$, and write (3) in the form

[1] Equations (1) and (2) generalize very easily when the interest rate $r'(u)$, is a forseen function of time, u. Then

$$(1')\quad V(t) = \begin{cases} e^{-\int_t^{t^*} r'(u)du} \int_{t^*}^{\infty} D(\tau) e^{-\int_{t^*}^{\tau} r'(u)du}\, d\tau, & \text{if } t < t^* \\ \int_t^{\infty} D(\tau) e^{-\int_t^{\tau} r'(u)du}\, d\tau, & \text{if } t \geqq t^* \end{cases}$$

and

$$(2')\quad \dot{V}(t) = \begin{cases} r'(t)\, V(t) & , \text{if } t < t^* \\ r'(t)\, V(t) - D(t), & \text{if } t \geqq t^*. \end{cases}$$

If the number of shares, S, is constant over time then of course share price change is just $\dot{P}(t) = \dot{V}(t)/S$. If the number of shares outstanding, $S(t)$, varies over time we let $V(t) = P(t)\, S(t)$, and $D(\tau) = d(\tau)\, S(\tau)$, where $d(\tau)$ is per share dividend yield. Since $\dot{V}(t) = P(t)\, \dot{S}(t) + \dot{P}(t)\, S(t)$, (2') takes the more general form

$$(2'')\quad \dot{P}(t) = \begin{cases} \left[r(t) - \frac{\dot{S}(t)}{S(t)} \right] P(t) & , \text{if } t < t^* \\ \left[r(t) - \frac{\dot{S}(t)}{S(t)} \right] P(t) - d(t), & \text{if } t \geqq t^*, \end{cases}$$

where $\dot{S}(t)/S(t)$ is the annual percentage stock dividend, which of course, "waters" the per share price but has no effect, in a perfect market, on the market value of the firm.

[2] In the United States dividend receipts are taxed at progressive rates, while capital gains realized over holding periods of six months or more ($T = 1/2$, if the time unit is a year) are taxed at one-half the income tax rate up to a maximum of 25 per cent.

$$V(t)\,[1-(1-\beta)e^{-rT}] = \int_t^{t+T} aD(\tau)e^{-r(\tau-t)}\,d\tau + \beta V(t+T)e^{-rT}. \qquad (4)$$

Now, by iteration of (4) we can derive an expression for $V(t)$ as an infinite sum (or finite, if we had assumed a finite market discount horizon [2, pp. 421–422]) entirely in terms of the future dividend stream. This is because, ultimately, $V(t)$ is determined only by the discounting of future dividend yield. Thus, at $t+(k-1)T$, $k=1,2,3,\ldots$, the market value of the firm must be given by $V[t+(k-1)T]$ in the expression

$$V[t+(k-1)T][1-(1-\beta)e^{-rT}] = \int_{t+(k-1)T}^{t+kT} aD(\tau)e^{-r[\tau-t-(k-1)T]}\,d\tau + \beta V(t+kT),\ k=1,2,3,\ldots.$$

By iterative substitution for $V(t+T)$, $V(t+2T)$, ... into (4), we get

$$V(t) = \sum_{k=1}^{\infty} a\beta^{k-1}[1-(1-\beta)e^{-rT}]^{-k}\,e^{-r(k-1)T} \int_{t+(k-1)T}^{t+kT} D(\tau)e^{-r[\tau-t-(k-1)T]}\,d\tau, \qquad (5)$$

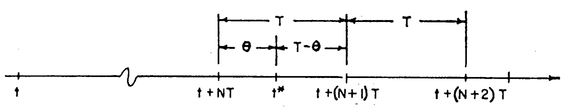
if the sum converges. Differentiating and then substituting from (5) gives

$$\dot V(t) = \sum_{k=1}^{\infty} a\beta^{k-1}[1-(1-\beta)e^{-rT}]^{-k}\,e^{-r(k-1)T} \left\{D[t+kT]e^{-rT} - D[t+(k-1)T]\right\} + rV(t), \qquad (6)$$

which provides the fundamental valuation theorem for arbitrary tax law parameters a, β and T, where $0 < a \leqq 1$, $0 < \beta \leqq 1$, and $T > 0$. If $a = \beta = 1$ (no taxes), equation (6) reduces to (2) for the dividend paying corporation.

III Extension to Non Dividend-Paying Corporations

Suppose the corporation at time t is not paying a dividend, but it is known that dividends will begin at $t^* = t + NT + \theta$, $T > 0$, $0 \leqq \theta \leqq T$, N an integer. That is, measured in terms of multiples of the holding period T (which separates "short" from "long-term" capital gains) dividends will begin between $t + NT$ and $t + (N+1)T$, at a point which divides that interval into subintervals θ with no dividend, and $1 - \theta$ with positive dividends. Diagrammatically:



It follows that share value is determined at t by the discounting of the potential capital gains (loss) over the N periods of length T, and the discounting of both dividends and potential capital gains (loss) thereafter.

The expression in $V(t)$ corresponding to (4) is now

$$V(t)\,[1-(1-\beta)e^{-rT}] = \beta V(t+T)e^{-rT}. \qquad (7)$$

By iteration,

$$V(t+T)\,[1-(1-\beta)e^{-rT}] = \beta V(t+2T)e^{-rT}$$

$$\vdots$$

$$V[t+(N-1)T]\,[1-(1-\beta)e^{-rT}] = \beta V(t+NT)e^{-rT}$$

$$V[t+NT]\,[1-(1-\beta)e^{-rT}] = e^{-r(t^*-NT-t)} \int_{t^*}^{t+(N+1)T} aD(\tau)e^{-r(\tau-t^*)}\,d\tau + \beta V[t+(N+1)T]e^{-rT}$$

$$\vdots$$

$$V[t+(N+k)T]\,[1-(1-\beta)e^{-rT}] = \int_{t+(N+k)T}^{t+(N+k+1)T} aD(\tau)e^{-r[\tau-t-(N+k)T]}\,d\tau + \beta V[t+(N+k+1)T]e^{-rT}.$$

By iterative substitution for $V(t+T)$, $V(t+2T)$, ..., $V[t+(N+k)T]$, ... we can solve for $V(t)$:

$$V(t) = a\beta^N\,[1-(1-\beta)e^{-rT}]^{-(N+1)} \int_{t^*}^{t+(N+1)T} D(\tau)e^{-r(\tau-t)}\,d\tau + \sum_{k=1}^{\infty} a\beta^{N+k}\,[1-(1-\beta)e^{-rT}]^{-(N+k+1)} \int_{t+(N+k)T}^{t+(N+k+1)T} D(\tau)e^{-r(\tau-t)}\,d\tau, \qquad (8)$$

provided that the sum converges. Differentiating and substituting as before:

$$\dot V(t) = a\beta^N\,[1-(1-\beta)e^{-rT}]^{-(N+1)} D[t+(N+1)T]e^{-r(N+1)T} + \sum_{k=1}^{\infty} a\beta^{N+k}\,[1-(1-\beta)e^{-rT}]^{-(N+k+1)} e^{-(N+k)rT}\left\{D[t+(N+k+1)T]e^{-rT} - D[t+(N+k)T]\right\} + rV(t). \qquad (9)$$

For $t < t^*$ (current dividends zero), the tax-generalized fundamental theorem of share valuation is given by (9); for $t \geqq t^*$ (current dividends positive) it is given by (6). Mathematically, what we have done in (8) and (9) is treat explicitly the problem of a discontinuous dividend stream. If we let $f(\tau) \geqq 0$ be the dividend stream without any continuity restrictions, then it is a straightforward matter (albeit tedious) to write conditions like (8) and (9) for any given $f(\tau)$ function. By a suitable partitioning of the domain of integration, these expressions can be written for any stuttering flow of dividends; for example,

$$f(\tau) = \begin{cases} 0, \ t \leq \tau < t^* \\ D^*(\tau) > 0, \ t^* \leq \tau < \hat{t} \\ 0, \ \hat{t} \leq \tau < t^{**} \\ D^{**}(\tau) > 0, \ t^{**} \leq \tau. \end{cases}$$

IV Share Valuation Under Constant Growth

The above formulas are not very practical tools until we postulate special forms for $D(\tau)$. If the corporation's dividend growth rate is a constant, g, then the dividend functions in (5) and (6) are

$$D(\tau) = D(t)\, e^{g(\tau - t)}$$
$$D[t + kT] = D(t)\, e^{gkT}$$
$$D[t + (k-1)T] = D(t)\, e^{g(k-1)T}$$

Substituting in (5), and evaluating the sum, we obtain

$$V(t) = \frac{QRD(t)}{g - r}, \quad r > g, \qquad (10)$$

where

$$Q = \frac{\alpha[1 - e^{-(g-r)T}]\, e^{(g-r)T}}{1 - (1-\beta)e^{-rT}},$$

$$R = \frac{1}{1 - \beta[1 - (1-\beta)e^{-rT}]^{-1}\, e^{(g-r)T}}, \text{ and}$$
$$\beta[1 - (1-\beta)e^{-rT}]^{-1} e^{(g-r)T} < 1.$$

Similarly, evaluating (6),

$$\dot{V}(t) = +QRD(t) + rV(t) = \frac{QRgD(t)}{g - r}. \qquad (11)$$

Observe that in a taxless world, with $\alpha = \beta = 1$, $QR = -1$, and the expression (10) is just the present worth of a stream of receipts growing at the constant rate $g < r$ from the initial value $D(t)$.

Making the above substitution for $D(\tau) = D(t^*)\, e^{g(\tau - t^*)}$ in (8),

$$V(t) = \frac{D(t^*)\, \beta^N [1 - (1-\beta)e^{-rT}]^{-N}\, e^{(g-r)NT}}{g - r} \Big\{ Q\, [e^{(g-r)T} - 1]^{-1} [e^{g(t+T-t^*) - rT} - e^{-(g-r)NT - r(t^*-t)}] + Qe^{g(t-t^*)}(R - 1) \Big\}. \qquad (12)$$

Note that when $N = 0$ and $t = t^*$, (12) reduces to (10).

Finally, making the growth function substitutions in (9) we have

$$\dot{V}(t) = D(t^*)\, \beta^N [1 - (1-\beta)e^{-rT}]^{-N} e^{(g-r)NT - g(t-t^*)}\, Q \Big\{ [1 - e^{-(g-r)T}]^{-1} + R - 1 \Big\} + rV(t) \qquad (13)$$

where $V(t)$ is given by (12).

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COMMENT ON THE "H" CONCENTRATION MEASURE AS A NUMBERS-EQUIVALENT

M. A. Adelman

The H concentration measure (so designated because it was independently devised by O. C. Herfindahl and A. O. Hirschmann) is defined as the sum of squared percentages of market. (Thus with three firms of 0.5, 0.3 and 0.2, H would be 0.25 + 0.09 + 0.04 = 0.38.) It has not had a very wide use. In this note we (1) derive it from some general premises, and (2) show how it can be interpreted as a "numbers-equivalent" [1] in any given market. (3) An important virtue both for computing and for using it is the quick convergence to a limit. Explaining its virtues will also show (4) its principal weakness.

I

Consider the very familiar cumulative concentration curve. On the vertical (y) axis we have from zero to 100 per cent of the industry. On the

[1] Robert L. Bishop, "Elasticities, Cross-Elasticities, and Market Relationships," *American Economic Review*, 42, pp. 779–803, especially 788–789; and "Comment," 43, pp. 918–19. Strictly speaking, Bishop's paper was concerned with the relationship of only two firms. I consider it as the classic statement of the oligopoly problem involving any number of firms.