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On Clifford analysis for holomorphic mappings

M. E. Luna-Elizarrarás, M. Shapiro and D. C. Struppa

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Abstract. In the classical theory of several complex variables, holomorphic mappings are just n -tuples of holomorphic functions in m variables, with arbitrary n and m , and no relations between these functions are assumed. Some 30 years ago John Ryan introduced complex, or complexified, Clifford analysis which is, in a sense, the study of certain classes of holomorphic mappings where the components are not independent, and instead obey the relations generated by the Cauchy–Riemann and Dirac-type operators. In this paper, we take a closer look at this theory emphasizing some additional properties that holomorphic mappings satisfy in this context. Our attention is mostly restricted to the case of low dimensions where it is possible to identify new and interesting properties and to single out the special role played by bicomplex analysis.

1 Introduction

Given an integer n , denote by $\mathcal{C}_{\ell_{0,n}}(\mathbb{C})$ the complex Clifford algebra generated by the imaginary units $\mathbf{e}_1, \dots, \mathbf{e}_n$ satisfying $\mathbf{e}_k^2 = -1$ for $k \in \{1, \dots, n\}$, and $\mathbf{e}_p \mathbf{e}_q + \mathbf{e}_q \mathbf{e}_p = 0$ for $q \neq p$. Any complex Clifford number, i.e. any element of $\mathcal{C}_{\ell_{0,n}}(\mathbb{C})$, is of the form

$$a = \sum_A a_A \mathbf{e}_A, \quad \mathbf{e}_A := \mathbf{e}_{p_1} \dots \mathbf{e}_{p_k}, \quad 1 \leq p_1 < p_2 < \dots < p_k \leq n$$

with $a_A \in \mathbb{C}$. We denote the imaginary unit of $\mathbb{C}$ by $\mathbf{i}$ and assume by definition that it commutes with the imaginary units $\mathbf{e}_p$. We will also denote by $\mathbf{e}_0$ the unit 1.

For each n consider a domain $\Omega_n \subset \mathbb{C}^n$ and a domain $\Omega_{n+1} \subset \mathbb{C}^{n+1}$; furthermore, let $\text{Hol}(\Omega_n, \mathcal{C}_{\ell_{0,n}}(\mathbb{C}))$ and $\text{Hol}(\Omega_{n+1}, \mathcal{C}_{\ell_{0,n}}(\mathbb{C}))$ denote the sets of holomorphic $\mathcal{C}_{\ell_{0,n}}(\mathbb{C})$ -valued functions of n and $n+1$ complex variables respectively: this means that if f is a holomorphic function on Ω_n or on Ω_{n+1} , and if we represent it as $f = \sum_A f_A \mathbf{e}_A$, then all its components f_A are holomorphic functions of n or $n+1$ variables in the classical sense. This implies, in particular, that each f_A admits complex partial derivatives $\frac{\partial f_A}{\partial z_k}$ for any k , and therefore we can define a complex partial derivative of a holomorphic function f in $\text{Hol}(\Omega_n, \mathcal{C}_{\ell_{0,n}}(\mathbb{C}))$ or in $\text{Hol}(\Omega_n, \mathcal{C}_{\ell_{0,n+1}}(\mathbb{C}))$ by means of $\frac{\partial f}{\partial z_k} = \sum_A \mathbf{e}_A \frac{\partial f_A}{\partial z_k}$.

In this paper, we will see that complex valued holomorphic functions play, in complex Clifford analysis, a role similar to the one played by C^1 -functions in real Clifford analysis. Specifically, we will consider holomorphic mappings from $\mathbb{C}^n$ or from $\mathbb{C}^{n+1}$ into $\mathbb{C}^{2^n}$ but, in contrast with classic holomorphic mappings where the components are independent, we will now impose additional relations between them. This process will endow our holomorphic mappings with additional important properties. To this purpose we introduce the Cauchy–Riemann and the Dirac operators acting on holomorphic mappings (of $n + 1$ and n variables respectively) as follows:

$$\mathcal{D}_{\text{CR}} := \sum_{k=0}^n \mathbf{e}_k \frac{\partial}{\partial z_k} \quad \text{and} \quad \mathcal{D}_{\text{Dir}} := \sum_{k=1}^n \mathbf{e}_k \frac{\partial}{\partial z_k}.$$

These two operators determine two classes of hyperholomorphy: if a function f satisfies $\mathcal{D}_{\text{CR}}[f] = 0$ we will say that it is Cauchy–Riemann-hyperholomorphic, while if it satisfies $\mathcal{D}_{\text{Dir}}[f] = 0$, then we will say that f is Dirac-hyperholomorphic.

John Ryan introduced these classes of holomorphic mappings and began their study in 1982, see [19] and [20], under the name of complex, or complexified, Clifford analysis. In a series of works (see, e.g., [21], [22], [23], [25], [26], [24]) he considered many properties of Cauchy–Riemann-hyperholomorphic functions and of Dirac-hyperholomorphic functions. Some developments of this theory in a different direction can be found in [28].

In this paper, we take a closer look at his theory, and we discuss how different classes of holomorphic mappings can be considered in this setting. However, we restrict our attention to the case of low dimensions, namely $n = 1, 2, 3$, where it is possible to connect the properties of holomorphic mappings with those arising from other hypercomplex structures. In particular, we see that for $n = 2$ there are four possible theories for hyperholomorphicity in complex Clifford analysis, and we show that for $n \geq 3$ the study of the Cauchy–Riemann operator and of the Dirac operator gives rise to substantially different theories.

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2 The case $n = 1$

We begin with the simple situation of $n = 1$. In this case the elements of $\mathcal{C}\ell_{0,1}(\mathbb{C})$ have the form $a_0 + a_1 \mathbf{e}_1$ with a_0 and a_1 in $\mathbb{C}$. These Clifford numbers are also called bicomplex numbers, and the algebra $\mathbb{B}\mathbb{C} := \mathcal{C}\ell_{0,1}(\mathbb{C})$ is the complex Clifford algebra of smallest dimension. It is also the only commutative algebra among all $\mathcal{C}\ell_{0,n}(\mathbb{C})$, and it can be usefully seen as the complex linear space $\mathbb{C}^2$ endowed with a structure of a commutative complex algebra. Given a bicomplex number $Z := a_0 + a_1 \mathbf{e}_1$ its bicomplex conjugate is $Z^\dagger := a_0 - a_1 \mathbf{e}_1$, thus

$$Z \cdot Z^\dagger = a_0^2 + a_1^2 \in \mathbb{C}.$$

This means, in particular, that a bicomplex number Z is invertible if and only if $Z \cdot Z^\dagger \neq 0$, and in this case its inverse is given by $Z^{-1} = \frac{Z^\dagger}{a_0^2 + a_1^2}$. If both a_0 and a_1 are non-zero but the sum $a_0^2 + a_1^2 = 0$, then the corresponding bicomplex number is a zero divisor. All zero divisors have the form $Z = \lambda(1 \pm \mathbf{ie}_1)$ for any $\lambda \in \mathbb{C} \setminus \{0\}$. The bicomplex numbers $e := \frac{1}{2}(1 + \mathbf{ie}_1)$ and $e^\dagger := \frac{1}{2}(1 - \mathbf{ie}_1)$ are idempotent zero divisors which are linearly independent in the complex linear space $\mathbb{C}^2$ and satisfy the equation $ee^\dagger = e^\dagger e = 0$. For any bicomplex number Z we can write $Z = \alpha e + \beta e^\dagger$ where $\alpha := z_1 - \mathbf{i}z_2$, $\beta := z_1 + \mathbf{i}z_2$; this is called the idempotent representation of Z . A remarkable feature of this representation is the fact that the operations of addition, multiplication, division and taking of the inverse can be realized term by term in the idempotent representation; for instance, if $Z_1 = \alpha e + \beta e^\dagger$ and $Z_2 = \gamma e + \delta e^\dagger$ then $Z_1 + Z_2 = (\alpha + \gamma)e + (\beta + \delta)e^\dagger$; similarly the rest. We refer the reader to [11] for more details on the elementary properties of and functions on this interesting space.

When $n = 1$, the Dirac operator has the form

$$\mathcal{D}_{\text{Dir}} = \frac{\partial}{\partial z_1}$$

and it acts on functions $f_0 + f_1 \mathbf{e}_1$ which are holomorphic with respect to their only variable z_1 and which satisfy the equation

$$\frac{\partial f}{\partial z_1} = 0 \quad \Longleftrightarrow \quad \frac{\partial f_0}{\partial z_1} = \frac{\partial f_1}{\partial z_1} = 0$$

in a domain of $\mathbb{C}$. Thus, we see that this case presents no interesting new theory.

The case of the Cauchy–Riemann operator, on the other hand, is more interesting. This operator, for $n = 1$, acts by the formula

$$\mathcal{D}_{\text{CR}}[f] = \sum_{k=0}^1 \mathbf{e}_k \frac{\partial f}{\partial z_k} = \frac{\partial f}{\partial z_0} + \frac{\partial f}{\partial z_1} \mathbf{e}_1$$

where, if $f = f_0 + f_1 \mathbf{e}_1$, both f_0 and f_1 are holomorphic functions in $\Omega_2 \subset \mathbb{C}^2$. Thus, bicomplex hyperholomorphic functions (in the sense of the Cauchy–Riemann operator) are those holomorphic mappings (f_0, f_1) from $\Omega_2 \subset \mathbb{C}^2$ to $\mathbb{C}^2$ which satisfy the (bicomplex) Cauchy–Riemann conditions

$$\frac{\partial f_0}{\partial z_0} - \frac{\partial f_1}{\partial z_1} = 0 \quad \text{and} \quad \frac{\partial f_1}{\partial z_0} + \frac{\partial f_0}{\partial z_1} = 0. \quad (1)$$

It turns out that there exists another approach to the study of bicomplex holomorphicity, which is specific to the bicomplex situation, and cannot be extended to more general complex Clifford algebras. Let $f : \Omega \subset \mathbb{BC} \rightarrow \mathbb{BC}$ be a bicomplex function; there exists a definition for the derivative of a bicomplex function (see e.g. [15]), which at least formally looks quite similar to its complex counterpart: the derivative of the function f at a point $Z_0 \in \Omega$ is defined to be the limit, when it exists,

$$f'(Z_0) := \lim_{Z \rightarrow Z_0} \frac{f(Z) - f(Z_0)}{Z - Z_0} \quad (2)$$

for Z in the domain of f such that $Z - Z_0$ is an invertible bicomplex number.

If a function f admits bicomplex derivative everywhere in a domain Ω in $\mathbb{BC}$, then it is easy to see that its components admit complex partial derivatives $\frac{\partial f_0}{\partial z_0}, \frac{\partial f_0}{\partial z_1}, \frac{\partial f_1}{\partial z_0}, \frac{\partial f_1}{\partial z_1}$. This means, in particular, that they are holomorphic in Ω in the sense of two complex variables, and furthermore the existence of the bicomplex derivative implies the bicomplex Cauchy–Riemann Conditions (1).

Thus bicomplex analysis gives us a very special case of complex Clifford analysis, which displays a deep and significant analogy with one-dimensional complex analysis. In particular, the independent variable $Z = z_0 + z_1 \mathbf{e}_1$ is bicomplex hyperholomorphic, its derivative is equal to one, and bicomplex hyperholomorphic functions can be expressed by their Taylor series in the usual form:

$$f(Z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(Z_0)}{n!} (Z - Z_0)^n.$$

Note also that the bicomplex Cauchy–Riemann Conditions (1) have a very efficient bicomplex form:

$$\frac{\partial f}{\partial Z^\dagger} := \frac{1}{2} \left(\frac{\partial}{\partial z_1} + \mathbf{e}_1 \frac{\partial}{\partial z_2} \right) f = 0; \quad (3)$$

interestingly enough, this implies that $\frac{\partial}{\partial Z^\dagger}$ together with its bicomplex conjugate

$$\frac{\partial}{\partial Z} := \frac{1}{2} \left(\frac{\partial}{\partial z_1} - \mathbf{e}_1 \frac{\partial}{\partial z_2} \right) \quad (4)$$

factorize the complex Laplacian:

$$\frac{\partial}{\partial Z^\dagger} \circ \frac{\partial}{\partial Z} = \frac{\partial}{\partial Z} \circ \frac{\partial}{\partial Z^\dagger} = \frac{1}{4} \left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} \right) =: \frac{1}{4} \Delta_{\mathbb{C}^2}.$$

This shows that bicomplex analysis, within complex Clifford analysis, acts as a counterpart of the classical theory of one complex variable within real Clifford analysis (in both cases we take Clifford analysis to mean the analysis of the solutions of the Cauchy–Riemann operator, not the Dirac operator). The reader can compare this with the reasoning in [16], and one can therefore claim that bicomplex analysis is a refinement of complex harmonic analysis, namely the study of null-solutions of the complex Laplace operator, see, e.g., [7], [8], and [14]. Quite recently, there has been a significant resurgence of interest in the theory of functions of bicomplex variables, where some of these ideas are further expanded and discussed. Without any attempt to completeness, we should at least quote [3], [5], [11], [12], [13], [17] and [18].

3 The case $n = 2$

Consider now the case of $n = 2$. Then the elements (Clifford numbers) in the Clifford algebra $\mathcal{C}\ell_{0,2}(\mathbb{C})$ are of the form

$$a = a_0 + a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_{12} \mathbf{e}_{12}$$

with $\mathbf{e}_{12} := \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$ and $a_0, a_1, a_2, a_{12} \in \mathbb{C}$. These elements are called, sometimes, the complex quaternions, or biquaternions, and the notation $\mathbb{H}(\mathbb{C}) := \mathcal{C}\ell_{0,2}(\mathbb{C})$ is used. The algebra $\mathcal{C}\ell_{0,2}(\mathbb{C})$ contains several copies of bicomplex numbers. For example, if we denote by $\mathbb{C}(\mathbf{i})$ the complex plane generated by $\mathbf{i}$ over $\mathbb{R}$, and by $\mathbb{C}(\mathbf{e}_I)$ the complex plane generated by $\mathbf{e}_I$ over $\mathbb{R}$, for $I = 1, 2$, or 12 , we can consider the sets $\mathbb{C}(\mathbf{i}) \otimes \mathbb{C}(\mathbf{e}_1) := \{a_0 + a_1\mathbf{e}_1\}$, $\mathbb{C}(\mathbf{i}) \otimes \mathbb{C}(\mathbf{e}_2) := \{a_0 + a_2\mathbf{e}_2\}$, $\mathbb{C}(\mathbf{i}) \otimes \mathbb{C}(\mathbf{e}_{12}) := \{a_0 + a_{12}\mathbf{e}_{12}\}$. Then we have

$$\begin{aligned} a &= (a_0 + a_{12}\mathbf{e}_{12}) + (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) \\ &= (a_0 + a_{12}\mathbf{e}_{12}) + \mathbf{e}_1(a_1 - a_2\mathbf{e}_{12}) \\ &= (a_0 + a_{12}\mathbf{e}_{12}) + (a_1 + a_2\mathbf{e}_{12})\mathbf{e}_1, \end{aligned}$$

which implies that

$$\mathbb{H}(\mathbb{C}) = \mathbb{B}\mathbb{C} \oplus \mathbb{B}\mathbb{C} \cdot \mathbf{e}_1 =: \mathbb{H}(\mathbb{C})^+ \oplus \mathbb{H}(\mathbb{C})^-.$$

In this last sum, the direct sum is understood in the sense of real or complex (over $\mathbb{C}(\mathbf{i})$) spaces. An isomorphism between $\mathbb{H}(\mathbb{C})^+$ and $\mathbb{H}(\mathbb{C})^-$ is realized by the multiplication by $\mathbf{e}_1$ on the right: $\mathbb{H}(\mathbb{C})^+ = \mathbb{H}(\mathbb{C})^- \cdot \mathbf{e}_1$.

Theorem 3.1. *Let $\Omega \subset \mathbb{C}^2$, and let $f : \Omega \rightarrow \mathcal{C}\ell_{0,2}(\mathbb{C})$. Then $f = F_1 + F_2\mathbf{e}_1$ belongs to the kernel of $\mathcal{D}_{\text{Dir}}$ if and only if F_1, F_2 belong to the kernel of $\frac{\tilde{\partial}}{\partial \bar{Z}}$.*

Proof. Every function $f : \Omega \rightarrow \mathcal{C}\ell_{0,2}(\mathbb{C}) = \mathbb{H}(\mathbb{C}) = \mathbb{B}\mathbb{C} + \mathbb{B}\mathbb{C} \cdot \mathbf{e}_1$ can be written as

$$\begin{aligned} f &= f_0 + \mathbf{e}_1 f_1 + \mathbf{e}_2 f_2 + \mathbf{e}_{12} f_{12} \\ &= (f_0 + f_{12}\mathbf{e}_{12}) + (f_1 + f_2\mathbf{e}_{12})\mathbf{e}_1 \\ &:= F_1 + F_2\mathbf{e}_1, \end{aligned}$$

where F_1, F_2 are bicomplex functions with $\mathbb{B}\mathbb{C} = \mathbb{C}(\mathbf{i}) \otimes \mathbb{C}(\mathbf{e}_{12})$. The Dirac operator can therefore be written as

$$\mathcal{D}_{\text{Dir}} = \mathbf{e}_1 \frac{\partial}{\partial z_1} + \mathbf{e}_2 \frac{\partial}{\partial z_2} = \mathbf{e}_1 \left(\frac{\partial}{\partial z_1} - \mathbf{e}_{12} \frac{\partial}{\partial z_2} \right)$$

acting on $\text{Hol}(\Omega, \mathbb{H}(\mathbb{C}))$; thus, one introduces the operators

$$\frac{\tilde{\partial}}{\partial \bar{Z}} := \frac{1}{2} \left(\frac{\partial}{\partial z_1} - \mathbf{e}_{12} \frac{\partial}{\partial z_2} \right) \quad \text{and} \quad \frac{\tilde{\partial}}{\partial Z^\dagger} := \frac{1}{2} \left(\frac{\partial}{\partial z_1} + \mathbf{e}_{12} \frac{\partial}{\partial z_2} \right)$$

which also act on $\text{Hol}(\Omega, \mathbb{H}(\mathbb{C}))$ and yield:

$$\begin{aligned} \mathcal{D}_{\text{Dir}}[f] &= \mathcal{D}_{\text{Dir}}[F_1 + F_2\mathbf{e}_1] \\ &= 2\mathbf{e}_1 \frac{\tilde{\partial}}{\partial \bar{Z}} [F_1 + F_2\mathbf{e}_1] = 2\mathbf{e}_1 \frac{\tilde{\partial} F_1}{\partial \bar{Z}} + 2\mathbf{e}_1 \frac{\tilde{\partial} F_2}{\partial \bar{Z}} \mathbf{e}_1 \\ &= 2\mathbf{e}_1 \frac{\tilde{\partial} F_1}{\partial \bar{Z}} - 2 \left(\frac{\tilde{\partial} F_2}{\partial \bar{Z}} \right)^\dagger = 2\mathbf{e}_1 \frac{\tilde{\partial} F_1}{\partial \bar{Z}} - 2 \frac{\tilde{\partial} F_2^\dagger}{\partial Z^\dagger}. \end{aligned} \tag{5}$$

The thesis now follows immediately from Equation (5). $\square$

Remark 3.2. There is an important difference between the operators in (3) and (4) on one hand, and $\frac{\partial}{\partial \bar{Z}}$ and $\frac{\partial}{\partial \bar{Z}^\dagger}$ on the other: while the latter act on $\mathbb{H}(\mathbb{C})$ -valued functions, the former are defined on bicomplex functions. Nevertheless, the restrictions of the latter onto $\mathbb{H}(\mathbb{C})^+$ coincide with the former. On the other hand, we should note that Formula (5) establishes a direct relation between Clifford analysis of Dirac operator for $\mathcal{C}\ell_{0,2}(\mathbb{C})$ and bicomplex analysis: a formula for $\mathcal{D}_{\text{Dir}}$ can be obtained from the corresponding formula of bicomplex analysis, and viceversa.

We illustrate this observation with a couple of simple examples.

Example 3.3. In bicomplex analysis the identity function $f_0(Z) = Z$ is $\mathbb{BC}$ -holomorphic and its bicomplex conjugate $f_0^\dagger(Z) = Z^\dagger$ is $\mathbb{BC}$ -antiholomorphic. If we take $F_1(Z) = F_2(Z) = Z^\dagger$, then

$$f(Z) := F_1(Z) + F_2(Z)\mathbf{e}_1 = Z^\dagger(1 + \mathbf{e}_1).$$

On the other hand, the solutions of the Dirac operator can be written in terms of its Fueter variables. Specifically, in this situation, there is only one such variable given by $\mathbf{e}_{12}Z^\dagger = \mathbf{e}_1\mathbf{e}_2Z^\dagger$. We will come back to this comment later on in Section 4.

Example 3.4. The notion of the derivative f' for Dirac-hyper-holomorphic functions can be introduced as in (2) and one can show that there is a direct relation between such derivative and the bicomplex derivatives of its bicomplex components:

$$f'(Z) = F'_1(Z) + F'_2(Z)\mathbf{e}_1.$$

The discussion above, and in particular Theorem 3.1 shows that the Dirac version of the complex Clifford analysis for $\mathcal{C}\ell_{0,2}(\mathbb{C})$ is, in fact, equivalent to bicomplex analysis. The Cauchy–Riemann version, on the other hand, is induced by the Cauchy–Riemann operator

$$\mathcal{D}_{\text{CR}} = \frac{\partial}{\partial z_0} + \mathbf{e}_1 \frac{\partial}{\partial z_1} + \mathbf{e}_2 \frac{\partial}{\partial z_2}$$

and there is no way to reduce it to a known situation.

However, similarly to what happens in real Clifford analysis, one can think about two more classes of holomorphic mappings with values in $\mathbb{C}^4$ which are defined by the following analogues of the Fueter operator

$$\mathcal{D}_F := \frac{\partial}{\partial z_0} + \mathbf{e}_1 \frac{\partial}{\partial z_1} + \mathbf{e}_2 \frac{\partial}{\partial z_2} + \mathbf{e}_{12} \frac{\partial}{\partial z_3}$$

and of the Moisil–Theodoresco operator

$$\mathcal{D}_{\text{MT}} := \mathbf{e}_1 \frac{\partial}{\partial z_1} + \mathbf{e}_2 \frac{\partial}{\partial z_2} + \mathbf{e}_{12} \frac{\partial}{\partial z_3}.$$

J. Ryan treats $\mathcal{D}_F$ in his papers, but does not deal with the study of $\mathcal{D}_{\text{MT}}$. The function theories for the solutions of such operators can be called “complex quaternionic analysis”,

or “biquaternionic analysis”. A fine point here is that the definitions of both operators involve not only the Cliffordian imaginary units $\mathbf{e}_1$ and $\mathbf{e}_2$ but also their product $\mathbf{e}_{12}$ which is a bivector in $\mathbb{H}(\mathbb{C})$. This fact has significant repercussions from an algebraic point of view and it implies analytic consequences as well: in domains of $\mathbb{C}^3$ we have now not only the complex Clifford analysis of $\mathbb{H}(\mathbb{C})$ -valued functions (function theory for the Cauchy–Riemann operator) but also quaternionic analysis of $\mathbb{H}(\mathbb{C})$ -valued functions (function theory for the biquaternionic Moisil–Theodoresco operator); while these theories are rather similar, they do not coincide. Indeed, the Cauchy–Riemann operator $\mathcal{D}_{\text{CR}}$ together with its (quaternionic) conjugate $\overline{\mathcal{D}}_{\text{CR}}$ factorize the complex Laplacian in $\mathbb{C}^3$: $\mathcal{D}_{\text{CR}} \circ \overline{\mathcal{D}}_{\text{CR}} = \Delta_{\mathbb{C}^3} = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial z_3^2}$, while the Moisil–Theodoresco operator $\mathcal{D}_{\text{MT}}$ is a square root of the opposite of that same Laplacian: $\mathcal{D}_{\text{MT}}^2 = -\Delta_{\mathbb{C}^3}$. Moreover, the only complex Clifford algebra $\mathcal{C}\ell_{0,n}(\mathbb{C})$ that admits the same dimension for the domain and for the range of functions (for the corresponding Clifford analysis) is $\mathcal{C}\ell_{0,1}(\mathbb{C}) = \mathbb{B}\mathbb{C}$; the case of $\mathbb{H}(\mathbb{C})$ -functions becomes similar if we complement the complex Clifford analysis with the biquaternionic analysis for the biquaternionic Fueter operator. Notice that the set $\ker D_{\text{MT}}$ contains the “holomorphic solenoidal and irrotational vector fields”: if $\vec{f} := f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$ with holomorphic functions f_1, f_2, f_3 and

$$\operatorname{div}_{\mathbb{C}} \vec{f} := \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + \frac{\partial f_3}{\partial z_3}, \quad \operatorname{rot}_{\mathbb{C}} \vec{f} := \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_3} \\ f_1 & f_2 & f_3 \end{vmatrix},$$

then the system of the two equations $\operatorname{div}_{\mathbb{C}} \vec{f} = 0$, $\operatorname{rot}_{\mathbb{C}} \vec{f} = 0$ defines a proper subclass in $\ker D_{\text{MT}}$. A quaternionic approach to the study of the real version of this system has been done in [1], [4], [9], [10], and [27].

4 The case $n = 3$

One of the most important consequences of the analysis we have carried out above is the fact that functions with values in $\mathcal{C}\ell_{0,n}(\mathbb{C})$ for $n \geq 3$ admit two types of complex Clifford analysis: the study of the Cauchy–Riemann and of the Dirac operators. For $n = 2$ there are four “complex Clifford analysis like” theories: the two versions of properly complex Clifford analysis together with the two versions of biquaternionic analysis. What is more, for no value of n any version of complex Clifford analysis coincides with complex quaternionic analysis.

Our next aim, in this work, is to show that in an exact analogy with the case of real Clifford analysis there is a relation between complex Clifford analysis for $\mathcal{C}\ell_{0,3}(\mathbb{C})$ and both versions of biquaternionic analysis.

We begin this section by describing in detail the complex Clifford algebra over three imaginary units, $\mathcal{C}\ell_{0,3}(\mathbb{C})$. We do so mostly for the sake of completeness and to ensure that the paper is self-contained. The reader should know, however, that an analogous description is carefully carried out in [6], to which we refer for more details.

In $\mathcal{C}\ell_{0,3}(\mathbb{C})$, the imaginary units and their products are

$$\mathbf{e}_0 = 1; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3.$$

If we denote by

$$\mathcal{C}\ell_{0,3}^{(r)}(\mathbb{C}) := \text{span}_{\mathbb{C}}\{\mathbf{e}_A \mid |A| = r\}$$

the subspace of $\mathcal{C}\ell_{0,3}(\mathbb{C})$ whose elements are called r -vectors, we have that

$$\mathcal{C}\ell_{0,3}(\mathbb{C}) = \bigoplus_{r=0}^3 \mathcal{C}\ell_{0,3}^{(r)}(\mathbb{C}). \quad (6)$$

The Cliffordian conjugation $a \mapsto \bar{a}$ in $\mathcal{C}\ell_{0,3}(\mathbb{C})$ is defined by $\mathbf{e}_\ell \mapsto \bar{\mathbf{e}}_\ell := -\mathbf{e}_\ell$ for $\ell \in \{1, 2, 3\}$, so that $\overline{ab} = \bar{b}\bar{a}$ for $a, b \in \mathcal{C}\ell_{0,3}(\mathbb{C})$. For ease of notation we will write $\bar{a} =: \mathcal{Z}(a)$. Inside $\mathcal{C}\ell_{0,3}(\mathbb{C})$, the even subalgebra $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$ is given by

$$\mathcal{C}\ell_{0,3}^+(\mathbb{C}) := \text{span}_{\mathbb{C}}\{1; \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2\},$$

i.e., all linear combinations of scalars and bivectors. The bivectors $\mathbf{e}_{23} := \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_{31} := \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_{12} := \mathbf{e}_1\mathbf{e}_2$ behave exactly as quaternionic imaginary units, and this leads to an isomorphism between $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$ and $\mathbb{H}(\mathbb{C})$. Note that the Cliffordian conjugation on $\mathcal{C}\ell_{0,3}(\mathbb{C})$ restricted to $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$ coincides with the quaternionic conjugation induced from $\mathbb{H}(\mathbb{C})$.

One also has

$$\mathcal{C}\ell_{0,3}^-(\mathbb{C}) := \text{span}_{\mathbb{C}}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \oplus \{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\} \cong \mathbb{C}^3 \oplus \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbb{C}$$

and

$$\mathcal{C}\ell_{0,3}(\mathbb{C}) = \mathcal{C}\ell_{0,3}^+(\mathbb{C}) \oplus \mathcal{C}\ell_{0,3}^-(\mathbb{C}). \quad (7)$$

The center of $\mathcal{C}\ell_{0,3}(\mathbb{C})$ is given by

$$\mathfrak{C}(\mathcal{C}\ell_{0,3}(\mathbb{C})) = \mathbb{C} \oplus \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbb{C}; \quad (8)$$

while

$$\mathcal{C}\ell_{0,3}^-(\mathbb{C}) = \mathcal{C}\ell_{0,3}^+(\mathbb{C}) \cdot \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \cdot \mathcal{C}\ell_{0,3}^+(\mathbb{C}), \quad (9)$$

or equivalently,

$$\mathcal{C}\ell_{0,3}^+(\mathbb{C}) = \mathcal{C}\ell_{0,3}^-(\mathbb{C}) \cdot \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \cdot \mathcal{C}\ell_{0,3}^-(\mathbb{C}). \quad (10)$$

The following “multiplication rules” between the sets $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$ and $\mathcal{C}\ell_{0,3}^-(\mathbb{C})$ can be verified directly:

- (a) $\mathcal{C}\ell_{0,3}^+(\mathbb{C}) \cdot \mathcal{C}\ell_{0,3}^+(\mathbb{C}) = \mathcal{C}\ell_{0,3}^+(\mathbb{C})$.
- (b) $\mathcal{C}\ell_{0,3}^-(\mathbb{C}) \cdot \mathcal{C}\ell_{0,3}^-(\mathbb{C}) = \mathcal{C}\ell_{0,3}^+(\mathbb{C})$.
- (c) $\mathcal{C}\ell_{0,3}^+(\mathbb{C}) \cdot \mathcal{C}\ell_{0,3}^-(\mathbb{C}) = \mathcal{C}\ell_{0,3}^-(\mathbb{C}) \cdot \mathcal{C}\ell_{0,3}^+(\mathbb{C}) = \mathcal{C}\ell_{0,3}^-(\mathbb{C})$.

Take $f \in \text{Hol}(\Omega_3 \subset \mathbb{C}^3; \mathcal{C}\ell_{0,3}(\mathbb{C}))$, then in accordance with (7) one has that

$$f = F_+ + F_-,$$

where $F_{\pm}$ is $\mathcal{C}\ell_{0,3}^{\pm}(\mathbb{C})$ -valued. Consider the $\mathbb{C}$ -linear operators

$$P^{\pm} : \text{Hol}(\Omega_3; \mathcal{C}\ell_{0,3}(\mathbb{C})) \rightarrow \text{Hol}(\Omega_3; \mathcal{C}\ell_{0,3}^{\pm}(\mathbb{C}))$$

defined by the relations

$$f \mapsto P^{\pm}[f] := F_{\pm}.$$

Since $(P^{\pm})^2 = P^{\pm}$; $P^+ + P^- = \text{Id}$, the identity operator, and $P^+ \circ P^- = P^- \circ P^+ = 0$, the projectors P^+ and P^- are mutually complementary. Therefore

$$\begin{aligned} \mathcal{D}_{\text{Dir}} &= \text{Id} \circ \mathcal{D}_{\text{Dir}} \circ \text{Id} = (P^+ + P^-) \circ \mathcal{D}_{\text{Dir}} \circ (P^+ + P^-) \\ &= P^+ \circ \mathcal{D}_{\text{Dir}} \circ P^+ + P^- \circ \mathcal{D}_{\text{Dir}} \circ P^+ + P^+ \circ \mathcal{D}_{\text{Dir}} \circ P^- + P^- \circ \mathcal{D}_{\text{Dir}} \circ P^-. \end{aligned}$$

Remark 4.1. We have defined the projectors implicitly through their actions on functions. However, one could also give an explicit definition of such operators in terms of the two mutually annihilating idempotents $\frac{1}{2}(1 \pm \mathbf{e})$ where $\mathbf{e}$ is the pseudoscalar. We refer the interested reader to [6].

Taking into account the Formulas (a)-(c), we have:

$$P^+ \circ \mathcal{D}_{\text{Dir}} \circ P^+ = P^- \circ \mathcal{D}_{\text{Dir}} \circ P^- = 0.$$

Thus

$$\mathcal{D}_{\text{Dir}} = P^+ \circ \mathcal{D}_{\text{Dir}} \circ P^- + P^- \circ \mathcal{D}_{\text{Dir}} \circ P^+. \quad (11)$$

Now set:

$$\begin{aligned} \mathcal{D}_{\text{MT}}^+ &:= \mathbf{e}_{23} \frac{\partial}{\partial z_1} + \mathbf{e}_{31} \frac{\partial}{\partial z_2} + \mathbf{e}_{12} \frac{\partial}{\partial z_3}, \\ \mathcal{D}_{\text{MT},r}^+ &:= M^{\mathbf{e}_{23}} \circ \frac{\partial}{\partial z_1} + M^{\mathbf{e}_{31}} \circ \frac{\partial}{\partial z_2} + M^{\mathbf{e}_{12}} \circ \frac{\partial}{\partial z_3}, \end{aligned}$$

where $M^a[f] := fa$, $a \in \mathcal{C}\ell_{0,3}(\mathbb{C})$.

The operators $\mathcal{D}_{\text{MT}}^+$ and $\mathcal{D}_{\text{MT},r}^+$ act on $\text{Hol}(\Omega_3; \mathcal{C}\ell_{0,3}(\mathbb{C}))$, but the fact that the Clifford numbers $\mathbf{e}_{23}$, $\mathbf{e}_{31}$, $\mathbf{e}_{12}$ behave like the quaternionic imaginary units explains, first of all, the subindex MT (these are the Moisil–Theodoresco-like operators although acting on $\mathcal{C}\ell_{0,3}(\mathbb{C})$ -valued functions), and suggests that the restrictions to $\text{Hol}(\Omega_3; \mathcal{C}\ell_{0,3}(\mathbb{C}))$ should have important peculiarities.

It is immediate to check that

$$\mathcal{D}_{\text{Dir}} \circ P^+[f] = \mathbf{e}_{123} \mathcal{Z} \circ \mathcal{D}_{\text{MT},r}^+ \circ \mathcal{Z} \circ P^+[f], \quad (12)$$

$$\mathcal{D}_{\text{Dir}} \circ P^-[f] = -\mathbf{e}_{123} \mathcal{D}_{\text{MT}}^+ \circ P^-[f], \quad (13)$$

hence, Formula (11) yields:

$$\mathcal{D}_{\text{Dir}} = P^+(-\mathbf{e}_{123}\mathcal{D}_{\text{MT}}^+ \circ P^-) + P^- (\mathbf{e}_{123}\mathcal{Z} \circ \mathcal{D}_{\text{MT},r}^+ \circ \mathcal{Z} \circ P^+). \quad (14)$$

Again recalling the Formulas (a)-(c) we obtain that

$$\mathcal{D}_{\text{Dir}} = \mathbf{e}_{123}(\mathcal{Z} \circ \mathcal{D}_{\text{MT},r}^+ \circ \mathcal{Z} \circ P^+ - \mathcal{D}_{\text{MT}}^+ \circ P^-). \quad (15)$$

There is a relation between $\mathcal{D}_{\text{MT}}^+$ and $\mathcal{D}_{\text{MT},r}^+$:

$$\mathcal{D}_{\text{MT}}^+ = -\mathcal{Z} \circ \mathcal{D}_{\text{MT},r}^+ \circ \mathcal{Z} \quad (16)$$

leading to

$$\mathcal{D}_{\text{Dir}} = -\mathbf{e}_{123}(\mathcal{D}_{\text{MT}}^+ \circ P^+ + \mathcal{D}_{\text{MT}}^+ \circ P^-). \quad (17)$$

Of course Formula (17) could also be obtained directly, and in a shorter way, by

$$\begin{aligned} \mathcal{D}_{\text{Dir}} &= \mathbf{e}_1 \frac{\partial}{\partial z_1} + \mathbf{e}_2 \frac{\partial}{\partial z_2} + \mathbf{e}_3 \frac{\partial}{\partial z_3} \\ &= \mathbf{e}_{123}^2 \left(\mathbf{e}_1 \frac{\partial}{\partial z_1} + \mathbf{e}_2 \frac{\partial}{\partial z_2} + \mathbf{e}_3 \frac{\partial}{\partial z_3} \right) \\ &= -\mathbf{e}_{123} \left(\mathbf{e}_{23} \frac{\partial}{\partial z_1} + \mathbf{e}_{31} \frac{\partial}{\partial z_2} + \mathbf{e}_{12} \frac{\partial}{\partial z_3} \right) \\ &= -\mathbf{e}_{123} \mathcal{D}_{\text{MT}}^+ \circ \text{Id} = -\mathbf{e}_{123}(\mathcal{D}_{\text{MT}}^+ \circ P^+ + \mathcal{D}_{\text{MT}}^+ \circ P^-), \end{aligned}$$

but we believe that the previous reasoning is instructive and clarifying.

The decomposition (17) of the operator $\mathcal{D}_{\text{Dir}}$, together with the decomposition (15), allow us to obtain a simple result;

Theorem 4.2. *Consider a function $f \in \text{Hol}(\Omega_3; \mathcal{C}\ell_{0,3}(\mathbb{C}))$ and its projections $F_{\pm} = P^{\pm}[f]$ in $\text{Hol}(\Omega_3; \mathcal{C}\ell_{0,3}^{\pm}(\mathbb{C}))$. Then f is left-Dirac-hyperholomorphic if and only if F_+ and $\mathbf{e}_{123}F_-$ are biquaternion-valued MT-hyperholomorphic functions.*

Proof. In fact, take any $f \in \text{Hol}(\Omega_3; \mathcal{C}\ell_{0,3}(\mathbb{C}))$ with $F_{\pm} = P^{\pm}[f]$ in $\text{Hol}(\Omega_3; \mathcal{C}\ell_{0,3}^{\pm}(\mathbb{C}))$. Then by (17) we have

$$\begin{aligned} \mathcal{D}_{\text{Dir}}[f] &= -\mathbf{e}_{123}(\mathcal{D}_{\text{MT}}^+ \circ P^+[f] + \mathcal{D}_{\text{MT}}^+ \circ P^-[f]) \\ &= -\mathbf{e}_{123}\mathcal{D}_{\text{MT}}^+[F_+] - \mathbf{e}_{123}\mathcal{D}_{\text{MT}}^+[F_-]. \end{aligned} \quad (18)$$

Note that the function $\mathcal{D}_{\text{MT}}^+[F_+]$ takes values in $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$; so $\mathbf{e}_{123}\mathcal{D}_{\text{MT}}^+[F_+]$ is $\mathcal{C}\ell_{0,3}^-(\mathbb{C})$ -valued, while the function $\mathcal{D}_{\text{MT}}^+[F_-]$ is $\mathcal{C}\ell_{0,3}^-(\mathbb{C})$ -valued, and hence $\mathbf{e}_{123}\mathcal{D}_{\text{MT}}^+[F_-]$ takes values in $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$. This concludes the proof. $\square$

The calculations above show, in fact, that the whole function theory for $\mathcal{D}_{\text{Dir}}$, that is, complex Clifford analysis for $\mathcal{C}\ell_{0,3}(\mathbb{C})$ for the Dirac operator, is somehow the “direct sum” of the two copies of biquaternionic analysis for the biquaternionic Moisil–Theodoresco operator.

We can illustrate this last remark by making reference to the analog of the Fueter variables and Taylor series for both theories. Specifically, when one considers the Cauchy–Fueter operator, one knows that the polynomials in $q \in \mathbb{H}$ are not solutions of the operator. Rather, one has to consider what are known as the Fueter variables, namely the basis for the space of homogeneous polynomial solutions of degree one for the Cauchy–Fueter equation; it is easy to see that such variables are given by

$$\zeta_1 = x_1 - \mathbf{i}x_0; \quad \zeta_2 = x_2 - \mathbf{j}x_0; \quad \zeta_3 = x_3 - \mathbf{k}x_0. \quad (19)$$

Similarly, for the Moisil–Teodorescu operator the corresponding variables are given by

$$\tilde{\zeta}_2 := x_1\mathbf{k} + x_2, \quad \tilde{\zeta}_3 := -x_1\mathbf{j} + x_3, \quad (20)$$

and for the Dirac operator in $\mathcal{C}\ell_{0,3}$ they are given by

$$\zeta_2 := x_1\mathbf{e}_1\mathbf{e}_2 + x_2; \quad \zeta_3 := x_1\mathbf{e}_1\mathbf{e}_3 + x_3. \quad (21)$$

More information on this topic can be found in [2].

Now, we can “complexify” these variables as follows: for the Dirac operator the variables are

$$\zeta_2 := z_1\mathbf{e}_1\mathbf{e}_2 + z_2; \quad \zeta_3 := z_1\mathbf{e}_1\mathbf{e}_3 + z_3; \quad (22)$$

for the “genuine” biquaternionic Moisil–Teodoresco operator the variables are

$$\tilde{\zeta}_2 := z_1\mathbf{k} + z_2, \quad \tilde{\zeta}_3 := -z_1\mathbf{j} + z_3,$$

which leads to the variables

$$\tilde{\zeta}_2 := z_1\mathbf{e}_{12} + z_2, \quad \tilde{\zeta}_3 := -z_1\mathbf{e}_{31} + z_3, \quad (23)$$

for the operator $\mathcal{D}_{\text{MT}}^\dagger$. Thus the Formulas (22) and (23) coincide and since $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$ is an algebra, the powers and symmetrized products remain inside $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$; hence all of them do not have an F_- -part, a fact which is coherent with (13).

Note that this does not mean that the Taylor series for a function in the kernel of D_{Dir} does not have an F_- -part. Indeed, the variable ζ_2 may enter into the series with a coefficient $c_2 = c_2^+ + c_2^- \in \mathcal{C}\ell_{0,3}(\mathbb{C})$: $\zeta_2 c_2 = \zeta_2 c_2^+ + \zeta_2 c_2^-$, where $\zeta_2 c_2^+ \in \mathcal{C}\ell_{0,3}^+(\mathbb{C})$, $\zeta_2 c_2^- \in \mathcal{C}\ell_{0,3}^-(\mathbb{C})$. Finally, we see that the Taylor series for f consists of the F_+ - and F_- -parts but the “building blocks” for both are the elements of $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$ only.

We proceed now to the comparison of the complex Cliffordian Cauchy–Riemann operator with the biquaternionic Fueter operator introduced in Section 3, namely

$$\mathcal{D}_{\text{CR}} = \frac{\partial}{\partial z_0} + \mathbf{e}_1 \frac{\partial}{\partial z_1} + \mathbf{e}_2 \frac{\partial}{\partial z_2} + \mathbf{e}_3 \frac{\partial}{\partial z_3}, \quad (24)$$

with

$$\mathcal{D}_F^+ = \frac{\partial}{\partial z_0} + \mathbf{e}_{23} \frac{\partial}{\partial z_1} + \mathbf{e}_{31} \frac{\partial}{\partial z_2} + \mathbf{e}_{12} \frac{\partial}{\partial z_3}. \quad (25)$$

Theorem 4.3. Let $\psi := (e_{123}, e_1, e_2, e_3)$, and consider the modified Cauchy–Riemann operator defined by

$$\mathcal{D}_{\text{CR}}^\psi = \mathbf{e}_{123} \frac{\partial}{\partial z_0} + \mathbf{e}_1 \frac{\partial}{\partial z_1} + \mathbf{e}_2 \frac{\partial}{\partial z_2} + \mathbf{e}_3 \frac{\partial}{\partial z_3}.$$

Then $\mathcal{D}_{\text{CR}}^\psi = \frac{\partial}{\partial z_0} - \mathcal{D}_{\text{MT}}^+$.

Proof. The reasoning that is necessary is similar to the one employed in the case of real Clifford algebras, and we describe it briefly. Both the operators $\mathcal{D}_{\text{CR}}$ and $\mathcal{D}_F^+$ are defined on $\text{Hol}(\Omega_4 \subset \mathbb{C}^4; \mathcal{C}\ell_{0,3}(\mathbb{C}))$ but there is a deep algebraic difference between them: $\mathcal{D}_F^+$ maps $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$ -valued functions into the functions of the same type, while $\mathcal{D}_{\text{CR}}$ does not. This is because all the coefficients of $\mathcal{D}_F^+$ are in $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$ and the coefficients of $\mathcal{D}_{\text{CR}}$ are partly in $\mathcal{C}\ell_{0,3}^+(\mathbb{C})$ and partly in $\mathcal{C}\ell_{0,3}^-(\mathbb{C})$. Thus an analog of the Formula (12) does not exist here. Instead, we can modify the operator (24) or the operator (25). First, consider the operator $\mathcal{D}_{\text{CR}}^\psi$ together with its Clifford conjugate

$$\overline{\mathcal{D}_{\text{CR}}^\psi} = \mathbf{e}_{123} \frac{\partial}{\partial z_0} - \mathbf{e}_1 \frac{\partial}{\partial z_1} - \mathbf{e}_2 \frac{\partial}{\partial z_2} - \mathbf{e}_3 \frac{\partial}{\partial z_3}.$$

Since these two operators, together, factorize the complex Laplacian

$$\mathcal{D}_{\text{CR}}^\psi \circ \overline{\mathcal{D}_{\text{CR}}^\psi} = \overline{\mathcal{D}_{\text{CR}}^\psi} \circ \mathcal{D}_{\text{CR}}^\psi = \Delta_{\mathbb{C}^4} = \frac{\partial^2}{\partial z_0^2} + \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial z_3^2}$$

we are able to construct the whole theory of complex Clifford analysis for $\mathcal{D}_{\text{CR}}^\psi$ and for $\overline{\mathcal{D}_{\text{CR}}^\psi}$. Now notice that, since all the coefficients of $\mathcal{D}_{\text{CR}}^\psi$ are all in $\mathcal{C}\ell_{0,3}^-(\mathbb{C})$,

$$\begin{aligned} \mathcal{D}_{\text{CR}}^\psi &= \mathbf{e}_{123} \frac{\partial}{\partial z_0} + \mathcal{D}_{\text{Dir}} = (\text{see}(12)) \\ &= \mathbf{e}_{123} \frac{\partial}{\partial z_0} - \mathbf{e}_{123} \mathcal{D}_{\text{MT}}^+ = \mathbf{e}_{123} \overline{\mathcal{D}_F^+} \end{aligned}$$

with $\overline{\mathcal{D}_F^+} := \frac{\partial}{\partial z_0} - \mathcal{D}_{\text{MT}}^+$. This concludes the proof. $\square$

Thus, the conclusions we made before about the relation between the two theories carry over to this new situation where we used the modification of the Cliffordian Cauchy–Riemann operator.

It is clear that one could also modify the operator (25), i.e., one could introduce the modification

$$\mathbf{D}_F^+ := \mathbf{e}_{123} \frac{\partial}{\partial z_0} + \mathcal{D}_{\text{MT}}^+$$

of the biquaternionic Fueter operator, together with its conjugate

$$\overline{\mathbf{D}_F^+} := \mathbf{e}_{123} \frac{\partial}{\partial z_0} - \mathcal{D}_{\text{MT}}^+$$

so that $\mathbf{D}_F^+ \circ \overline{\mathbf{D}_F^+} = \overline{\mathbf{D}_F^+} \circ \mathbf{D}_F^+ = \Delta_{\mathbb{C}^4}$. Using again Formula (12) we obtain

$$\mathbf{D}_F^+ := \mathbf{e}_{123} \frac{\partial}{\partial z_0} - \mathbf{e}_{123} \mathcal{D}_{\text{Dir}},$$

hence $\mathbf{D}_F^+ := \mathbf{e}_{123} \overline{\mathcal{D}_{\text{CR}}}$, where again $\overline{\mathcal{D}_{\text{CR}}}$ means the Clifford conjugate to the Cauchy–Riemann operator:

$$\overline{\mathcal{D}_{\text{CR}}} := \frac{\partial}{\partial z_0} - \mathbf{e}_1 \frac{\partial}{\partial z_1} - \mathbf{e}_2 \frac{\partial}{\partial z_2} - \mathbf{e}_3 \frac{\partial}{\partial z_3}.$$

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