2012

Integral Conditions for the Vanishing of the Cohomology of Open Sets in Cn

Fabrizio Colombo
Politecnico di Milano

M. E. Luna-Elizarrarás
IPN

Irene Sabadini
Politecnico di Milano

M. Shapiro
IPN

Daniele C. Struppa
Chapman University, struppa@chapman.edu

Recommended Citation

Colombo, F. F., Luna-Elizarrarás, M. E., Sabadini, I. I., Shapiro, M. M., & Struppa, D. C. (2012). Integral conditions for the vanishing of the cohomology of open sets in ℂn. *Advances In Geometry*, 12(2), 191-201. doi: 10.1515/advgeom.2011.053

Integral Conditions for the Vanishing of the Cohomology of Open Sets in Cn

Comments

This article was originally published in *Advances in Geometry*, Volume 12, issue 2, in 2012. DOI: 10.1515/advgeom.2011.053

Adv. Geom. **12** (2012), 191–201
DOI 10.1515/ADVGEOM.2011.053

Advances in Geometry

Integral conditions for the vanishing of the cohomology of open sets in $\mathbb{C}^n$

F. Colombo, M. E. Luna-Elizarrarás, I. Sabadini,
M. Shapiro and D. C. Struppa

(Communicated by G. Gentili)

Abstract. In this paper we develop and extend some techniques introduced in [1] to find integral conditions for the vanishing of the cohomology of open bounded sets in $\mathbb{C}^n$ with values in the sheaf of holomorphic functions.

1 Introduction

In this paper we study the vanishing of the holomorphic cohomology of open sets in $\mathbb{C}^n$. While a variety of conditions under which such cohomology vanishes are known in the literature, our paper offers new integral conditions that insure such vanishing.

As it is well known, the case of one complex variable is easily dealt with by the Mittag–Leffler theorem, which can be reformulated by saying that all cohomology groups of order at least one vanish for every open set in $\mathbb{C}$ (see for example [6], [7], [8]). This is also sometimes expressed by saying that every open set in $\mathbb{C}$ is a domain of holomorphy. The case of several variables is clearly more interesting. An important theorem of Malgrange shows that if U is an open set in $\mathbb{C}^n$, then its cohomology groups $H^j(U, \mathcal{O})$ vanish for all $j \geq n$. At the same time, it is known that if U is pseudoconvex (and a variety of characterizations for such open sets are known), then its cohomology groups vanish for all $j \geq 1$.

In a recent paper [1], we used some ideas from quaternionic analysis to study the vanishing of the cohomology groups for open sets in $\mathbb{C}^2$. Quaternions, however, are not helpful if one wants to use similar ideas to discuss the case of dimension higher than two. The use of several quaternionic variables, as well, does not seem to be of any help. But fortunately, some recent advances in Clifford analysis can help finding criteria for the vanishing of cohomology in higher dimensions. That this is indeed possible is due to the introduction in [12] of a more general definition of hyperholomorphicity. Specifically, we will consider a complex vector space of differential forms defined on open sets in $\mathbb{C}^n$, which are in the kernel of the Hodge–Dirac operator. This vector space contains, as a particular case, the space of holomorphic functions in n variables. A crucial fact is that the Hodge–Dirac operator gives rise to a one-dimensional theory (in a sense which will be specified below), and it admits a right inverse on open bounded sets with piecewise smooth boundary. Using these two crucial facts, we can deduce integral conditions for the vanishing of a holomorphic r-cocycle.

Acknowledgements. The authors are grateful to the anonymous referee for her/his careful reading of the manuscript and for pointing out the alternative formulation of Theorem 2.1.1 given in Theorem 2.1.1'.

2 Cohomology vanishing in two complex variables

In his classical works [3], [4] Fueter built a theory of holomorphic functions on the space $\mathbb{H}$ of quaternions new at that time. As a byproduct, he was able to find new proofs for known results in the theory of holomorphic functions in several variables (but see [10] for a thorough analysis of Fueter's work). Specifically, he was able to obtain a new proof for Hartogs' theorem of removability of compact singularities. The possibility of such a new proof rests fundamentally on the fact that every holomorphic function in $\mathbb{C}^2$ can be seen as a hyperholomorphic function in a way that we will make precise below. In our recent paper [1] we exploited this fact to obtain a new integral characterization of domains of holomorphy in $\mathbb{C}^2$; we will briefly summarize our results in this section.

2.1. Let $v \in \mathscr{C}^1(U)$ where U is a bounded domain in $\mathbb{C}^2$ with piecewise smooth boundary. The following formulas

$$\mathcal{T}_1[v](w_1, w_2) := \frac{1}{2\pi^2} \int_U \frac{\bar{z}_1 - \bar{w}_1}{(|z_1 - w_1|^2 + |z_2 - w_2|^2)^2}\, v(z_1, z_2)\, dV,$$
$$\mathcal{T}_2[v](w_1, w_2) := \frac{1}{2\pi^2} \int_U \frac{\bar{z}_2 - \bar{w}_2}{(|z_1 - w_1|^2 + |z_2 - w_2|^2)^2}\, \bar{v}(z_1, z_2)\, dV, \tag{1}$$

where dV is the differential form of the four-dimensional volume in $\mathbb{C}^2$, define two linear operators acting from $\mathscr{C}(U)$ into $\mathscr{C}^1(U)$, which will play an important role in what follows.

Theorem 2.1.1 ([1]). *Let U be any bounded open set in $\mathbb{C}^2$ with piecewise smooth boundary, and let $\xi = \{g_{ij}\}$ be a cocycle in $H^1(U, \mathcal{O})$, where $\mathcal{O}$ denotes the sheaf of holomorphic functions. Let $\{\varphi_i\}$ be a partition of the unity associated to a Leray covering $\mathcal{U} = \{U_i\}$ of U, set $h_i := \sum_j \varphi_j g_{ji} \in C^\infty(U_i)$, and let $k_1 := \{2\frac{\partial h_i}{\partial \bar{z}_1}\}$, $k_2 := \{2\frac{\partial h_i}{\partial \bar{z}_2}\}$. Then $\xi = 0$ if and only if $\mathcal{T}_1(k_2) + \mathcal{T}_2(k_1) = 0$.*

As a consequence, we have the following immediate characterization of the pseudoconvex domains in $\mathbb{C}^2$ with piecewise smooth boundary in terms of the operators $\mathcal{T}_1$ and $\mathcal{T}_2$:

Corollary 2.1.2 ([1]). *A bounded open set U in $\mathbb{C}^2$ with piecewise smooth boundary is pseudoconvex if and only if for every ξ in $H^1(U,\mathcal{O}) = H^1(\mathcal{U},\mathcal{O})$, one has $\mathcal{T}_1(k_2) + \mathcal{T}_2(k_1) = 0$.*

2.2. The origin of the operators $\mathcal{T}_1$ and $\mathcal{T}_2$, as well as the basic ideas of the proofs in [1], lie in the theory of quaternionic hyperholomorphic (synonyms: regular, monogenic, etc.) functions; since we will need this theory later on, we briefly recall here its main ideas and results.

Let z_1, z_2 be two complex numbers with $\mathbf{i}$ being the imaginary unit. Then a quaternion is defined as

$$q = z_1 + z_2\mathbf{j}$$

where $\mathbf{j}$ is another imaginary unit such that the relations $\mathbf{j}^2 = -1$ and $\mathbf{ij} + \mathbf{ji} = 0$ hold.

In particular, we have $z_1\mathbf{j} = \mathbf{j}\bar{z}_1$ where $\bar{z}_1$ denotes the complex conjugate of z_1. The set of quaternions is a real, non-commutative division algebra denoted by $\mathbb{H}$. We define the conjugate of a quaternion q by $\bar{q} := \bar{z}_1 - z_2\mathbf{j}$. Since the quaternion $q = z_1 + z_2\mathbf{j}$ can be identified with the pair of complex numbers (z_1, z_2) we shall interpret $\mathbb{H}$ as $\mathbb{C}^2$ with an additional multiplicative structure making it a skew-field. The reader familiar with the more usual definition of quaternions as 4-tuples of real numbers

$$q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$$

can easily see that the definition we have given above is equivalent to the usual one.

For $\mathbb{H}$-valued functions defined on an open set $U \subseteq \mathbb{H}$, it is possible to generalize the notion of holomorphy, the so called hyperholomorphy, see e.g. [11], by the use of a suitable modification $\mathcal{D}$ of the usual Cauchy–Fueter operator. Specifically, for any function $f \in \mathscr{C}^1(U;\mathbb{H})$ we define

$$\frac{1}{2}\mathcal{D}[f](q) := \frac{\partial f}{\partial \bar{z}_1}(q) + \mathbf{j}\frac{\partial f}{\partial \bar{z}_2}(q) = \left(\frac{\partial}{\partial \bar{z}_1} + \mathbf{j}\frac{\partial}{\partial \bar{z}_2}\right) f(q).$$

Since f can be decomposed as $f = f_1 + f_2\mathbf{j}$ we can write

$$\frac{1}{2}\mathcal{D}[f](q) = \left(\frac{\partial f_1}{\partial \bar{z}_1} - \frac{\partial \bar{f}_2}{\partial z_2}\right)(q) + \mathbf{j}\left(\frac{\partial f_1}{\partial \bar{z}_2} + \frac{\partial \bar{f}_2}{\partial z_1}\right)(q). \tag{2}$$

Definition 2.2.1. Let U be an open set in $\mathbb{C}^2$. A function $f \in \mathscr{C}^1(U;\mathbb{H})$ is said to be hyperholomorphic in U if, for every $q \in U$,

$$\mathcal{D}[f](q) = 0. \tag{3}$$

The local nature of this definition immediately implies the fact that hyperholomorphic functions form a sheaf which we will denote by $\mathcal{R}$.

Remark 2.2.2. Due to (2), equality (3) is equivalent to the system

$$\frac{\partial f_1}{\partial \bar{z}_1} = \overline{\frac{\partial f_2}{\partial \bar{z}_2}}; \quad \frac{\partial f_1}{\partial \bar{z}_2} = -\overline{\frac{\partial f_2}{\partial \bar{z}_1}},$$

which is satisfied if both f_1 and f_2 are the usual holomorphic functions in two variables, but not only for this situation; thus there exist hyperholomorphic functions which are not pairs of holomorphic ones.

On the other hand, if we consider a complex-valued function f, i.e. a quaternionic function with $f_2 = 0$, then f is in the kernel of the operator $\mathcal{D}$ if and only if $f = f_1$ is holomorphic in the two complex variables z_1 and z_2.

Remark 2.2.3. The operator $\mathcal{D}$ can be obtained from the classical Cauchy–Fueter operator

$$\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3}$$

by changing the sign in front of the imaginary unit $\mathbf{k}$ and setting $z_1 = x_0 + \mathbf{i}x_1$ and $z_2 = x_2 + \mathbf{i}x_3$. Note that the operator $\frac{\partial}{\partial \bar{q}}$ cannot serve for our purposes since holomorphic mappings in two complex variables are not contained in the space of its null-solutions. This explains the reason for our use of a slightly modified operator.

Let $\mathcal{K}$ be the modified Cauchy–Fueter kernel

$$\mathcal{K}(q) = \frac{1}{2\pi^2}\frac{\bar{z}_1 - \bar{z}_2\mathbf{j}}{(|z_1|^2 + |z_2|^2)^2}.$$

The operator $\mathcal{D}$ admits a right inverse operator T, see e.g. [13], defined by

$$T[v](p) := \int_U \mathcal{K}(q-p)v(q)\,dV, \tag{4}$$

where dV denotes the volume form, i.e., for any $u \in \mathscr{C}(U;\mathbb{H})$ one has that $\mathcal{D} \circ T[u] = u$.

A direct computation shows that if $v = v_1 + v_2\mathbf{j}$ is an arbitrary continuous function then

$$T[v] = \mathcal{T}_1[v_1] - \mathcal{T}_2[v_2] + (\mathcal{T}_1[v_2] + \mathcal{T}_2[v_1])\mathbf{j}, \tag{5}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are the operators defined by (1).

We now provide an alternative statement and proof of Theorem 2.1.1, which has been suggested by the anonymous referee. To this purpose, we will denote by $H^{p,q}(U)$ the Dolbeault cohomology groups of U. As it is well known, see e.g. [5], there is a canonical isomorphism

$$H^n(U,\mathcal{O}) \cong H^{0,n}(U) \tag{6}$$

thus, if we denote by λ the isomorphism in the case $n = 1$, an element ξ in $H^1(U,\mathcal{O})$ can be identified with $\eta = \lambda(\xi) \in H^{0,1}(U)$. Let h be a representative element of η, then $h = k_1 d\bar{z}_1 + k_2 d\bar{z}_2$ and $\left(\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2}\right)h = 0$, i.e. h is closed. Obviously, $h = 0$ in $H^{0,1}(U)$ if and only if $h = \left(\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2}\right)w = \frac{\partial w}{\partial \bar{z}_1}d\bar{z}_1 + \frac{\partial w}{\partial \bar{z}_2}d\bar{z}_2$ i.e. if and only if h is exact. In this language, Theorem 2.1.1 can be formulated as follows:

Theorem 2.1.1'. *Let U be any bounded open set in $\mathbb{C}^2$ with piecewise smooth boundary. Let*

$$\lambda : \; H^1(U,\mathcal{O}) \longrightarrow H^{0,1}(U)$$

be the canonical isomorphism. Let $\xi \in H^1(U, \mathcal{O})$, $\eta = \lambda(\xi)$ *and let* $h = k_1 d\bar{z}_1 + k_2 d\bar{z}_2$ *be a representative of* η. *Then* $\xi = 0$ *if and only if* $\mathcal{T}_1(k_2) + \mathcal{T}_2(k_1) = 0$.

Proof. By construction, the coefficients k_1, k_2 of the closed differential form h are infinitely differentiable functions. Now we have that $\eta = 0$ is equivalent to the fact that any representative h is an exact form so $h = \frac{\partial w}{\partial \bar{z}_1} d\bar{z}_1 + \frac{\partial w}{\partial \bar{z}_2} \bar{z}_2$ where $w : U \to \mathbb{C}$ is an infinitely differentiable function, i.e. $k_1 = \frac{\partial w}{\partial \bar{z}_1}$ and $k_2 = \frac{\partial w}{\partial \bar{z}_2}$. This fact is equivalent to say that the function $k = k_1 + k_2 \mathbf{j}$ satisfies $k = \mathcal{D}[w]$. Using the right inverse T and Formula (5) we conclude that this is equivalent to $w = T[k]$ and w is a complex-valued function if and only if $\mathcal{T}_1(k_2) + \mathcal{T}_2(k_1) = 0$. □

We conclude this section by pointing out that, by their nature, both the quaternions and the quaternionic hyperholomorphic function theory are well suited for the study of holomorphic functions in $\mathbb{C}^2$, while they are clearly inadequate for the study of functions in $\mathbb{C}^n$ with $n > 2$. In this case, one needs to resort to some ideas from Clifford analysis; the goal of this paper is to explicitly show how this can be accomplished.

3 Cohomology vanishing in several complex variables

3.1. In this section we will introduce the necessary notation and ideas from Clifford analysis in the language of complex differential forms, which are necessary to extend the results of Section 2 to the case of holomorphic functions on $\mathbb{C}^n$ with $n > 2$.

Definition 3.1.1. Let U be an open set in $\mathbb{C}^n$. For any $k = 0, 1, \dots, n$, $p \in \mathbb{N} \cup \{0, \infty\}$ we denote by $\overline{\mathcal{G}}_p^k(U)$ the set of all $(0,k)$-forms on U with $\mathscr{C}^p$ coefficients. Moreover, we set $\overline{\mathcal{G}}_p(U) := \bigcup_{k=0}^n \overline{\mathcal{G}}_p^k(U)$. For simplicity, we will denote by $\overline{\mathcal{G}}^k(U)$ the set of all $(0,k)$-forms with $\mathscr{C}^\infty$ coefficients and $\overline{\mathcal{G}}(U) := \bigcup_{k=0}^n \overline{\mathcal{G}}^k(U)$.

According to [12], we can give the following definition:

Definition 3.1.2. Let U be a bounded open set in $\mathbb{C}^n$ with piecewise smooth boundary Γ and let $g \in \overline{\mathcal{G}}_1(U) \cap \overline{\mathcal{G}}_0(U \cup \Gamma)$, $g(\zeta, d\bar{z}) = \sum_{\mathbf{j}} g_{\mathbf{j}}(\zeta)\, d\bar{z}^{\mathbf{j}}$, where $\mathbf{j} = (j_1, \dots, j_r)$, $d\bar{z}^{\mathbf{j}} = d\bar{z}^{j_1} \wedge \dots \wedge \bar{z}^{j_r}$. We define the operator T by

$$T[g](\zeta, d\bar{z}) = \sum_{\mathbf{j}} \sum_{p=1}^{n} \left(\overline{T}_p[g_{\mathbf{j}}](\zeta) d\bar{z}^p \wedge d\bar{z}^{\mathbf{j}} + T_p[g_{\mathbf{j}}](\zeta) \widehat{d\bar{z}}^p \wedge d\bar{z}^{\mathbf{j}} \right) \tag{7}$$

where $\overline{T}_q, T_j$ are given by

$$\overline{T}_q[F](z) := \frac{1}{2a_{2n}} \int_U \frac{\tau_q - z_q}{|t - z|^{2n}} F(\tau) dV_\tau,$$
$$T_j[F](z) := -\frac{1}{2a_{2n}} \int_U \frac{\bar{\tau}_j - \bar{z}_j}{|t - z|^{2n}} F(\tau) dV_\tau,$$

a_{2n} is the area of the $(2n-1)$-dimensional unit sphere in $\mathbb{R}^{2n}$, dV_τ is the differential form of the volume element in $\mathbb{R}^{2n}$ and $F : U \to \mathbb{C}$. The contraction operator $\widehat{d\bar{z}_q}$ is the endomorphism on $\overline{\mathcal{G}}(U)$ defined by its action on the generator: specifically, if $q = j_p$, then

$$\widehat{d\bar{z}^q} \wedge d\bar{z}^{\mathbf{j}} = (-1)^{p-1} d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_{p-1}} \wedge d\bar{z}^{j_{p+1}} \wedge \cdots \wedge d\bar{z}^{j_r},$$

while if $q \notin \{j_1, \ldots, j_r\}$, then

$$\widehat{d\bar{z}^q} \wedge d\bar{z}^{\mathbf{j}} = 0.$$

Remark 3.1. Note that we are denoting by the symbol T_j operators which do not coincide with T_1, T_2 given in (1).

Theorem 3.2. *Let U be an open bounded set in $\mathbb{C}^n$ with piecewise smooth boundary and let $\mathcal{U} = \{U_i\}$ be a Leray covering of U. Let $\zeta = \{g_{i_1 i_2 \ldots i_{r+1}}\} \in H^r(\mathcal{U}, \mathcal{O})$ be an r-cocycle, $1 \le r \le n-1$. Let $\{\varphi_i\}$ be any partition of unity associated to $\mathcal{U}$ and $h_{i_1 i_2 \ldots i_r} := \sum \varphi_\ell g_{i_1 i_2 \ldots i_r \ell}$. Set $v = \sum_{q=1}^n v_q d\bar{z}^q$ where $v_q = \{\frac{\partial}{\partial \bar{z}_q} h_{i_1 \ldots i_r}\}$. Then $\zeta = 0$ if and only if v satisfies the conditions*

$$\overline{T}_p[v_q] - \overline{T}_q[v_p] = 0, \quad \textit{for all } p, q = 1, \ldots, n,\ p \ne q. \tag{8}$$

We postpone the proof of this result to part 3.9 of this section, after some additional technical tools have been introduced.

Corollary 3.3. *A bounded open set U in $\mathbb{C}^n$ with piecewise smooth boundary is pseudo-convex if and only if for any $1 \le r \le n-1$ and for every ξ in $H^r(U, \mathcal{O}) = H^r(\mathcal{U}, \mathcal{O})$, one has $\overline{T}_p[v_q] - \overline{T}_q[v_p] = 0$, for all $p, q = 1, \ldots, n$, $p \ne q$.*

Since Malgrange's theorem shows that the cohomology of any open bounded set in $\mathbb{C}^n$ vanishes in dimension greater than or equal to n, we immediately obtain the following result:

Corollary 3.4. *Let U be a bounded open set U in $\mathbb{C}^n$ with piecewise smooth boundary. Then for every ξ in $H^r(U, \mathcal{O}) = H^r(\mathcal{U}, \mathcal{O})$, $r \ge n$, one has $\overline{T}_p(v_q) - \overline{T}_q(v_p) = 0$, for all $p, q = 1, \ldots, n$, $p \ne q$.*

3.5. Our proofs are based on the theory of hyperholomorphic matrix-valued differential forms (of type $(0, k)$) developed in [12], so we recall now some fragments of it.

First of all, it is immediate to note that $\overline{\mathcal{G}}(U)$ is (under the operations of addition, exterior multiplication $\wedge$, and multiplication by complex scalars) a non-commutative, associative, distributive complex algebra with identity. If $F = \sum F_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ is a differential form, define

$$\bar{\partial} F := \sum_{\mathbf{j}} \sum_{q=1}^n \frac{\partial F_{\mathbf{j}}}{\partial \bar{z}_q} d\bar{z}^q \wedge d\bar{z}^{\mathbf{j}}$$

and

$$\overline{\partial}^* F := \sum_{\mathbf{j}} \sum_{q=1}^{n} \frac{\partial F_{\mathbf{j}}}{\partial \overline{z}_q} \widehat{d\overline{z}^q} \wedge d\overline{z}^{\mathbf{j}}.$$

Definition 3.5.1. The operator $\overline{\partial} + \overline{\partial}^*$ is called the Hodge–Dirac operator.

This operator will be our main tool. Following [12], we study such an operator within a more general context. Let $\overline{G}(U)$ denote the set of 2×2 matrices with entries in $\overline{\mathcal{G}}(U)$, and define

$$\overline{\mathcal{D}} = \begin{pmatrix} \overline{\partial} & \overline{\partial}^* \\ \overline{\partial}^* & \overline{\partial} \end{pmatrix} \quad \text{and} \quad \overline{\mathcal{D}}^* = \begin{pmatrix} \overline{\partial}^* & \overline{\partial} \\ \overline{\partial} & \overline{\partial}^* \end{pmatrix}.$$

It turns out that, see Section 2.3 of [12],

$$\overline{\mathcal{D}} \cdot \overline{\mathcal{D}}^* = \overline{\mathcal{D}}^* \cdot \overline{\mathcal{D}} = \Delta_{\mathbb{C}^n} \cdot I, \tag{9}$$

where I denotes the 2×2 identity matrix. Thus $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}^*$ factor, as matrices, the Laplacian $\Delta_{\mathbb{C}^n} = \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \overline{z}_k}$ on $\mathbb{C}^n$. Of course, property (9) proves to be crucial in constructing a function theory for the operator $\overline{\mathcal{D}}$ (or $\mathcal{D}$); observe also that (9) is related to the fact that the Hodge–Dirac operator is a "square root" of the "scalar" Laplacian:

$$(\overline{\partial} + \overline{\partial}^*)(\overline{\partial} + \overline{\partial}^*) = \overline{\partial}\overline{\partial} + \overline{\partial}^*\overline{\partial}^* + \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^* = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^* = \Delta_{\mathbb{C}^n}.$$

The action of $\overline{\mathcal{D}}$ on a 2×2 matrix of differential forms $[F_{ij}]$, $i, j = 1, 2$, can be translated in a system of 4 equations which can be reduced, see Section 7.4 in [12], to the system of 2 equations:

$$\begin{cases} \overline{\partial}[F_{11}] + \overline{\partial}^*[F_{21}] = 0, \\ \overline{\partial}^*[F_{11}] + \overline{\partial}[F_{21}] = 0. \end{cases} \tag{10}$$

In order to obtain the null solutions to the Hodge–Dirac operator we study the null solutions of system (10).

Proposition 3.5.2 ([12]). *Each solution of the homogeneous Hodge–Dirac equation can be obtained from* (10) *by setting* $F_{11} = F_{21}$.

The class of elements in $\overline{\mathcal{G}}(U)$, U open set in $\mathbb{C}^n$, in the kernel of the Hodge–Dirac operator contains the class of holomorphic functions in n complex variables, as proved in the next result:

Proposition 3.5.3. *A solution $f \in \overline{\mathcal{G}}(U)$ to the Hodge–Dirac equation is a* (*necessarily holomorphic*) *function in n variables on $U \subseteq \mathbb{C}^n$ if and only if $f \in \overline{\mathcal{G}}^0(U)$.*

Proof. We consider the solutions to the Hodge–Dirac operator as the special case of solutions to $\overline{\mathcal{D}}[F] = 0$ with $F_{11} = F_{21} = f = \phi + \sum_{k=1}^{n} \sum_{|\mathbf{j}|=k} \psi_{\mathbf{j}} d\overline{z}^{\mathbf{j}}$ and $F_{12} = F_{22} = 0$. By writing explicitly the equation $\overline{\mathcal{D}}[F] = 0$ as done in [12], we obtain that f can be identified with a holomorphic function in n variables if and only if $\psi_{\mathbf{j}} = 0$ for all multi-indices $\mathbf{j}$, i.e. if and only if $f \in \overline{\mathcal{G}}^0(U)$. □

3.5.4. The null solutions of the operator $\overline{\mathcal{D}}$ are called matrix-valued hyperholomorphic differential forms while the null solutions to the Hodge–Dirac operator are called hyperholomorphic differential forms; thus, the latter are the elements of the set $\mathcal{N}(U) := \ker(\overline{\partial} + \overline{\partial}^*)$ where $U \subset \mathbf{C}^n$ is an open set. Note that, in addition to the usual holomorphic functions, $\mathcal{N}(U)$ contains subsets isomorphic to linear spaces of functions which are holomorphic in a set of variables $z_1, \dots, z_p$ and anti-holomorphic on the remaining variables $z_{p+1}, \dots, z_n$, for any p between zero and n.

3.5.5. Since $\overline{\mathcal{D}}$ admits a right inverse, see [12], it can be shown that the operator T introduced in (7) is in fact a right inverse for the Hodge–Dirac operator. This implies immediately the following result:

Proposition 3.5.6 (See Section 7.5, [12]). *On an open bounded set $U \subset \mathbb{C}^n$ with piecewise smooth boundary, a solution f to the equation*

$$(\overline{\partial} + \overline{\partial}^*)[f] = g$$

is of the form

$$f = T[g] + h$$

where h is an arbitrary solution to the homogeneous Hodge–Dirac system.

We are ready now to study deeper properties of $\mathcal{N}(U)$.

Proposition 3.6. *The collection $\{\mathcal{N}(U) \mid U \text{ open}\}$ is a sheaf of vector spaces.*

Proof. If we fix an order on the differential forms $d\bar{z}^{\mathbf{J}}$, we can identify a differential form with a vector with 2^n components. The proof follows from standard arguments since $\mathcal{N}$ can be identified with the sheaf of $\mathscr{C}^\infty$ vectors which are solutions to a system of linear partial differential equations with constant coefficients. □

The vanishing of the first cohomology group for $\mathcal{N}$ is a simple but interesting result which essentially follows from the existence of a right inverse to the Hodge–Dirac operator. Note that when we consider open coverings we will always suppose that they are locally finite since we work in paracompact spaces.

Theorem 3.7. *For any open bounded set $U \subset \mathbb{C}^n$ with piecewise smooth boundary, the first cohomology group $H^1(U, \mathcal{N})$ vanishes.*

Proof. Let $\xi = \{F_{ij}\}$ be a 1-cocycle in $H^1(U, \mathcal{N})$, i.e. if $\mathcal{U} = \{U_i\}$ is a covering of the open set U, the elements F_{ij} are hyperholomorphic differential forms on the intersections $U_i \cap U_j$ and satisfy

$$F_{ij} - F_{ki} + F_{jk} = 0.$$

Let $\{\varphi_i\}$ be a partition of unity associated to the covering $\mathcal{U}$. Then we can construct the new differential forms given by $h_j := \sum_i \varphi_i F_{ij}$ and defined on U_j, and it is immediate to observe that for every i and j such that $U_i \cap U_j \neq \emptyset$, we have

$(\overline{\partial}+\overline{\partial}^*)[h_i] = (\overline{\partial}+\overline{\partial}^*)[h_j]$. This implies that the elements of the collection $\{(\overline{\partial}+\overline{\partial}^*)h_i\}$ are differential 1-forms on all of U. Setting $k := \{(\overline{\partial}+\overline{\partial}^*)[h_i]\}$ we can apply to k the right-inverse operator T obtaining $v = T[k]$. Setting

$$F_j := h_j - v$$

we get a differential form on U_j such that $F_{ij} := F_j - F_i$ belongs to $\mathcal{N}(U_i \cap U_j)$ and the statement follows by taking an inductive limit. □

Theorem 3.8. *For any open set $U \subseteq \mathbb{C}^n$, the cohomology groups $H^j(U, \mathcal{N})$ vanish for any $j \geq 2$.*

Proof. The statement can be proved directly but it also follows by looking more closely at the system of linear partial differential equations associated to it. Indeed, the Hodge–Dirac system translates into a square system of linear partial differential equations whose associated matrix $P(D)$ is non-singular, as the operator factorizes the Laplacian. Thus the resolution having its first map equal to the symbol of $P(D)$ ends after the first step and the statement follows (see [2]). □

We are now ready to prove Theorem 3.2.

3.9 Proof of Theorem 3.2. We begin with the case of $r = 1$, i.e. we will prove that the conditions in the hypotheses are sufficient for the vanishing of the first cohomology group. Let $\mathcal{U} = \{U_i\}$ be a Leray covering of the open set U (for example a covering of open balls), so that $H^1(U, \mathcal{O}) = H^1(\mathcal{U}, \mathcal{O})$. Let $\xi = \{g_{ij}\}$ be a 1-cocycle in $H^1(\mathcal{U}, \mathcal{O})$, i.e. the functions g_{ij} are holomorphic (and therefore in the kernel of the Hodge–Dirac operator) on the intersections $U_i \cap U_j$ and satisfy $g_{ij} + g_{ji} = 0$, $g_{ij} - g_{ik} + g_{jk} = 0$. Let $\{\varphi_i\}$ be a complex-valued partition of unity associated to the covering $\mathcal{U}$. Then we can construct a family of $\mathscr{C}^\infty$ functions $h_j := \sum_i \varphi_i g_{ij}$. It is immediate to observe that for every i and j such that $U_i \cap U_j \neq \emptyset$, we have $(\overline{\partial}+\overline{\partial}^*)[h_i] = \overline{\partial}[h_j]$. This implies that the collection

$$\{\overline{\partial}[h_i]\} = \left\{ \sum_{q=1}^{n} \frac{\partial}{\partial \bar{z}_q} h_i d\bar{z}^q \right\} = \left\{ \sum_{q=1}^{n} v_q^i d\bar{z}^q \right\}$$

defines a 1-form $v = \sum_{q=1}^n v_q d\bar{z}^q$. If we now apply T, i.e. the right-inverse of the Hodge–Dirac operator, we obtain a differential form $u := T[v]$. Setting $g_j := h_j - u$ we get elements in $\mathcal{N}(U_j)$ such that $g_{ij} := g_i - g_j$ belongs to $\mathcal{N}(U_i \cap U_j)$. To guarantee that $\xi = 0$ in $H^1(U, \mathcal{O})$ we now have to require that the g_j's can be identified with holomorphic functions g_j. The right inverse T, see Formula (7), is such that $T[v]$ is made by two pieces: one is a two form; the second is a zero form. The condition that $u = T[v] \in \mathcal{G}_1^0$, i.e. u is a holomorphic function, translates into the Conditions (8) in the statement. We now show how this same idea can be applied for the second cohomology group ($r = 2$). Let $\zeta = \{g_{ijk}\}$ be a 2-cocycle in $H^2(\mathcal{U}, \mathcal{O})$, i.e. the functions g_{ijk} are holomorphic on the intersections $U_i \cap U_j \cap U_k$ and satisfy $g_{ijk} - g_{ij\ell} + g_{ik\ell} - g_{jk\ell} = 0$ whenever $U_i \cap U_j \cap U_k \cap U_\ell \neq \emptyset$. Let $\{\varphi_i\}$ be a complex-valued partition of unity associated to the covering $\mathcal{U}$. Then we can construct a family $h_{ij} := \sum_\ell \varphi_\ell g_{ij\ell}$, and it is immediate to observe that for every i, j, k such that $U_i \cap U_j \cap U_k \neq \emptyset$, we have

$$(\overline{\partial} + \overline{\partial}^*)[h_{ij} + h_{jk} - h_{ik}] = \overline{\partial}\left[\sum_\ell \varphi_\ell g_{ijk}\right] = 0.$$

Thus that the collection $\{\overline{\partial}[h_{ij}]\} = \{\sum_{q=1}^n \frac{\partial}{\partial \bar{z}_q} h_{ij} d\bar{z}^q\}$ defines $v^{ij} = \sum_{q=1}^n v_q^{ij} d\bar{z}^q$ and so we can set $v = \{v^{ij}\}$. If we now apply the right-inverse operator T to v we obtain a differential form $u := T[v]$. Setting $g_{ij} := h_{ij} - u$ we get differential forms on U_j such that $g_{ijk} := g_{ij} + g_{jk} + g_{ki}$ belong to $\mathcal{N}(U_i \cap U_j \cap U_k)$. To guarantee that $\zeta = 0$ in $H^2(\mathcal{U}, \mathcal{O})$ we now have to require that the g_{ij}'s can be identified with holomorphic functions g_{ij}. Thus, we need to impose that $u = T[v]$ is a 0-form and so a holomorphic function, and this is exactly the condition in the statement. It should then be clear how the result extends to all values of r. □

Remark 3.10. Note that, when $n = 2$, the conditions in Theorem 2.1.1 and their counterparts (8) are, at least formally, of the same type. However, the former reflects the fact that a function is complex-valued, while the latter reflects the fact that a differential form is a function.

Remark 3.11. We conclude by pointing out that it is not possible to reformulate Theorem 3.2 with the use of the Dolbeault cohomology, in analogy with what we did for Theorem 2.1.1. Indeed, if one tries to mimic the proof of Theorem 2.1.1', one realizes that the key argument there was the fact that a complex-valued function is in the kernel of the Cauchy–Fueter operator if and only if it is in the kernel of the Cauchy–Riemann operator. The analogue of this result is, in our case, Proposition 3.5.3, which however is weaker, and therefore is not sufficient here.

References

[1] F. Colombo, M. E. Luna-Elizarrarás, I. Sabadini, M. Shapiro, D. C. Struppa, A new characterization of a class of pseudoconvex domains in $\mathbb{C}^2$. *C. R. Math. Acad. Sci. Paris* **344** (2007), 677–680. MR2334674 (2008f:32041) Zbl 1153.32022

[2] F. Colombo, I. Sabadini, F. Sommen, D. C. Struppa, *Analysis of Dirac systems and computational algebra*, volume 39 of *Progress in Mathematical Physics*. Birkhäuser 2004. MR2089988 (2005m:30052) Zbl 1064.30049

[3] R. Fueter, Die Funktionentheorie der Differentialgleichungen $\Theta u = 0$ und $\Theta\Theta u = 0$ mit vier reellen Variablen. *Comment. Math. Helv.* **7** (1934), 307–330. MR1509515 JFM 61.1131.05 Zbl 0012.01704

[4] R. Fueter, Über einen Hartogs'schen Satz. *Comment. Math. Helv.* **12** (1939), 75–80. MR0000693 (1,115c) JFM 65.0363.03 Zbl 0022.05802

[5] P. Griffiths, J. Harris, *Principles of algebraic geometry*. Wiley-Interscience 1978. MR507725 (80b:14001) Zbl 0408.14001

[6] R. C. Gunning, *Lectures on Riemann surfaces*. Princeton Univ. Press 1966. MR0207977 (34 #7789) Zbl 0175.36801

[7] L. Hörmander, *An introduction to complex analysis in several variables*. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London 1966. MR0203075 (34 #2933) Zbl 0138.06203

[8] A. Kaneko, *Introduction to hyperfunctions*, volume 3 of *Mathematics and its Applications* (*Japanese Series*). Kluwer 1988. MR1026013 (90m:32017) Zbl 0687.46027

[9] S. G. Krantz, *Function theory of several complex variables*. Wiley-Interscience 1982. MR635928 (84c:32001) Zbl 0471.32008

[10] H. R. Malonek, Rudolf Fueter and his motivation for hypercomplex function theory. *Adv. Appl. Clifford Algebras* **11** (2001), 219–229. MR2075353 (2005b:01048) Zbl 1221.01131

[11] I. M. Mitelman, M. V. Shapiro, Differentiation of the Martinelli-Bochner integrals and the notion of hyperderivability. *Math. Nachr.* **172** (1995), 211–238. MR1330631 (96e:32005) Zbl 0832.30028

[12] R. Rocha-Chávez, M. Shapiro, F. Sommen, *Integral theorems for functions and differential forms in* $\mathbb{C}^m$. Chapman & Hall/CRC, Boca Raton, FL 2002. MR1889406 (2003a:32004) Zbl 0991.32002

[13] M. V. Shapiro, N. L. Vasilevski, On the Bergman kernel function in hyperholomorphic analysis. *Acta Appl. Math.* **46** (1997), 1–27. MR1432469 (98a:32036) Zbl 0877.30025

Received 25 November, 2009; revised 7 June, 2010

F. Colombo, I. Sabadini, Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, 20133 Milano, Italy
Email: fabrizio.colombo@polimi.it, irene.sabadini@polimi.it

M. E. Luna-Elizarrarás, M. Shapiro, Departamento de Matemáticas E.S.F.M. del I.P.N., 07338 México D.F., México
Email: eluna@esfm.ipn.mx, shapiro@esfm.ipn.mx

D. C. Struppa, Schmid College of Science, Chapman University, 92866 Orange, California
Email: struppa@chapman.edu