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STRONGLY COMPLETE LOGICS FOR COALGEBRAS

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ABSTRACT. Coalgebras for a functor model different types of transition systems in a uniform way. This paper focuses on a uniform account of finitary logics for set-based coalgebras. In particular, a general construction of a logic from an arbitrary set-functor is given and proven to be strongly complete under additional assumptions. We proceed in three parts.

Part I argues that sifted colimit preserving functors are those functors that preserve universal algebraic structure. Our main theorem here states that a functor preserves sifted colimits if and only if it has a finitary presentation by operations and equations. Moreover, the presentation of the category of algebras for the functor is obtained compositionally from the presentations of the underlying category and of the functor.

Part II investigates algebras for a functor over ind-completions and extends the theorem of Jónsson and Tarski on canonical extensions of Boolean algebras with operators to this setting.

Part III shows, based on Part I, how to associate a finitary logic to any finite-sets preserving functor T. Based on Part II we prove the logic to be strongly complete under a reasonable condition on T.

Contents

1. Introduction

This paper can be read as consisting of three independent parts or it can be read with a unifying story in mind. Since the three parts may be of interest to different readers, and require somewhat different prerequisites, we keep them separated and try to make them reasonably self-contained. On the other hand, the story will be of interest to some readers and so we sketch it in this introduction. We begin with a brief overview of the three parts.

Part I presents an investigation in categorical universal algebra. In universal algebra, a variety is a category that has a finitary presentation by operations and equations. We investigate functors on varieties that have finitary presentations by operations and equations. We will show that an endofunctor on a variety has such a presentation if and only if the functor preserves sifted colimits.

Part II studies a topic in Stone duality. Given a small finitely complete and cocomplete category $\mathcal{C}$, one obtains a dual adjunction between the ind-completions $\mathsf{Ind}\mathcal{C}$ and $\mathsf{Ind}(\mathcal{C}^{\mathrm{op}})$. We prove a Jónsson-Tarski style representation theorem showing how to represent algebras over $\mathsf{Ind}\mathcal{C}$ as duals of coalgebras over $\mathsf{Ind}(\mathcal{C}^{\mathrm{op}})$.

Part III investigates how to associate in a uniform way to a set-functor T a logic for T-coalgebras. Using the results from Part I and Part II, we show that, under a mild additional condition, any finite set-preserving functor T has a strongly complete finitary modal logic.

Our story starts with the idea of universal coalgebra as a general theory of systems, due to Rutten [50], which allows us to deal with issues such as bisimilarity, coinduction, etc in a uniform way. A natural question then is whether something similar can be done for logics of coalgebras. The first answer to this was Moss's seminal Coalgebraic Logic [44]. Moss's logic is parametric in T, the basic idea being to take T itself as an operation to construct formulas: if Φ is a set of formulas, then $T(\Phi)$ is a set of formulas.

Following on from Moss [44], attention turned to the question of how to set up logics for coalgebras in which formulas are built according to a more conventional scheme: If $\Box$ is a

unary operation symbol, or 'modal operator', and φ is a formula, then $\Box\varphi$ is a formula. After some work in this direction, see eg [33, 49, 23], Pattinson [46] showed that such languages arise from modal operators given by natural transformations $2^X \to 2^{TX}$, which are called predicate liftings of T. Schröder [51] investigated the logics given by all predicate liftings of finite arity and showed that these logics are expressive for finitary functors T.

Another approach is based on Stone duality. In the context of coalgebraic logic, it was first advocated by [11], but is based on the ideas of domain theory in logical form [1, 3]. We will explain it briefly here.

We think of Stone duality [24, 3] as relating a category of algebras $\mathcal{A}$, representing a propositional logic, to a category of topological spaces $\mathcal{X}$, representing the state-based models of the logic. The duality is provided by two contravariant functors P and S,

$$\mathcal{X} \underset{S}{\overset{P}{\rightleftarrows}} \mathcal{A} \ . \tag{1.1}$$

P maps a space X to a propositional theory and S maps a propositional theory to its 'canonical model'. For the moment let us assume that $\mathcal{A}$ and $\mathcal{X}$ are dually equivalent, as it is the case when $\mathcal{X}$ is the category Stone of Stone spaces and $\mathcal{A}$ is the category BA of Boolean algebras. This means that, from an abstract categorical point of view, the two categories are the same, up to reversal of arrows. But this ignores the extra structure which consists of both categories having a forgetful functor to Set, with $\mathcal{X} \to$ Set not being dual to $\mathcal{A} \to$ Set. An object of $\mathcal{X}$ specifies a *set* of states that serves as our semantic domain; an object of $\mathcal{A}$ specifies a *set* of propositions, which we use to specify properties of spaces. More specifically, we will assume that $\mathcal{A}$ is a variety, that is, $\mathcal{A}$ is isomorphic to a category of algebras given by operations and equations, the operations being our logical connectives and the equations the logical axioms. Or, equivalently, we can say that $\mathcal{A} \to$ Set is finitary and monadic.

To extend a basic Stone duality as above to coalgebras over $\mathcal{X}$, we consider, as Abramsky did in his Domain Theory in Logical Form [1], the dual L of T:

$$T \circlearrowright \mathcal{X} \underset{S}{\overset{P}{\rightleftarrows}} \mathcal{A} \circlearrowleft L \qquad\qquad LP \cong PT \tag{1.2}$$

Then the category of L-algebras is dual to the category of T-coalgebras and the initial L-algebra provides a propositional theory characterising T-bisimilarity. Moreover, if L can be presented by generators and relations, or rather operations and equations, one inherits a proof system from equational logic which is sound and strongly complete. Thus, *logics for T-coalgebras arise from presentations of the dual of T by operations and equations.* Part I characterises those functors L on varieties $\mathcal{A}$ that have a finitary presentation by operations and equations.

The approach indicated in Diagram 1.2 can be applied to set-coalgebras, but as the dual of Set is the category CABA of *complete* atomic Boolean algebras, the corresponding logics would become infinitary. Hence, being interested in finitary logics, we are led to consider

two Stone dualities

$$\text{Stone} \rightleftarrows \text{BA} \,\circlearrowright\, L \tag{1.3}$$

$$T \,\circlearrowleft\, \text{Set} \rightleftarrows \text{CABA}$$

The upper row is the duality between Stone spaces and Boolean algebras, accounting for (classical finitary) propositional logic. The lower row is the duality where our set-based T-coalgebras live. How can the two be related?

The crucial observation is the following. BA is the ind-completion of the category BA_ω of finite Boolean algebras, that is, the completion of finite Boolean algebras under filtered colimits; Set is the ind-completion of the category Set_ω of finite sets; and finite sets are dually equivalent to finite Boolean algebras. In other words, Set^{op} is the pro-completion of finite Boolean algebras, that is, the completion of finite Boolean algebras under cofiltered limits.

$$L \,\circlearrowright\, \text{BA} \underset{S}{\overset{P}{\rightleftarrows}} \text{Set}^{\text{op}} \,\circlearrowright\, T^{\text{op}} \tag{1.4}$$
$$(\hat{-}) \searrow \qquad \swarrow (\bar{-})$$
$$\text{BA}_\omega \simeq \text{Set}^{\text{op}}_\omega$$

This gives us a systematic link between Boolean propositional logic and its set-theoretic semantic. It is extended to modal logics and their coalgebraic semantics by a natural transformation

$$\delta : LP \to PT$$

lifting P to a functor $\tilde{P} : \text{Coalg}(T) \to \text{Alg}(L)$. Not in general, but in a large number of important examples, we also obtain a (not necessarily natural) transformation

$$h : SL \to TS$$

giving rise to a map on objects $\tilde{S} : \text{Alg}(L) \to \text{Coalg}(T)$. Part II shows that then every L-algebra A can be embedded into the P-image of a T-coalgebra, namely, into $\tilde{P}\tilde{S}A$. This extends the Jónsson-Tarski theorem from Kripke frames [10, Thm 5.43] to coalgebras and shows that the logics L are canonical in the sense that all formulas hold in the underlying frame of the canonical model $\tilde{S}I$, where I is the Lindenbaum-algebra of L.

Starting from an arbitrary functor $T : \text{Set} \to \text{Set}$, Part III shows first how to define a suitable L. From Part I, we know that L has a presentation by operations and equations and therefore corresponds to a 'concrete' modal logic in the traditional sense. This is detailed in Section 8. Section 9 then applies the Jónsson-Tarski theorem from Part II to obtain strong completeness results for modal logics for coalgebras.

Further related work Unary predicate liftings and a criterion for weak completeness go back to Pattinson [46]. The observation that all logics given by predicate liftings correspond to a functor L on BA was made in [28], with the approach of functorial modal logic going back to [11]. The logic of all predicate liftings of finite arity was introduced in Schröder [51]. Our notion of a presentation of a functor and the fact that such presentations give rise to modal logics is from [12]. The Jónsson-Tarski theorem of this paper generalises the corresponding theorem in [29]. The process of taking a finite set preserving functor and extending it to BA, and hence to Stone, is related to a construction in Worrell [56] where a set-functor is

lifted to complete ultrametric spaces. Klin [27] generalises the expressivity result of [51] working essentially with the same adjunction as in Diagram 1.2.

Since an earlier version of the paper was made available in June 2006, the field continued to develop quickly. For example, Schröder [52] improves on [46] by showing that complete axiomatisations always exist and, moreover, that completeness holds wrt finite models. [47] uses Stone duality and algebraic techniques to derive conditions for the finite model property of logics with additional non-rank-1 axioms. Schröder and Pattinson [54] push the strong completeness result of this paper further and add several important examples.

2. Introduction to Part I: Algebras and Varieties

There is a general agreement that algebras over a category $\mathcal{A}$ are described by means of a monad $M : \mathcal{A} \to \mathcal{A}$ (see [43]). In the case when $\mathcal{A}$ is the category Set of sets, the category Set_M of algebras over a monad $M : \mathsf{Set} \to \mathsf{Set}$ always has a *presentation* in the sense that there exists a signature Σ (allowing infinite arities) and equations E such that Set_M is concretely isomorphic to the category $\mathsf{Alg}(\Sigma, E)$ of (Σ, E)-algebras. Both Σ and E can be proper classes but the characteristic property is that free (Σ, E)-algebras always exist. This allows compact Hausdorff spaces and complete atomic Boolean algebras but eliminates complete Boolean algebras. The important special case is when a monad M has a rank which means that it preserves λ-filtered colimits for some regular cardinal λ. It corresponds to the case when Σ is a set (and λ is greater than arities of all Σ-operations). In particular, algebras over a monad preserving filtered colimits (such functors are called *finitary*) then correspond to classical universal algebras.

Alternatively and equivalently, classical universal algebras can be described by algebraic theories (see Lawvere [38]). Recall that an *algebraic theory* is a category $\mathcal{T}$ whose objects are integers $0, 1, 2, \ldots$ and such that n is a product of n copies of 1 for each $n = 0, 1, \ldots$. It means that $\mathcal{T}$ has finite products and, in particular, a terminal object 0. Intuitively, arrows $n \to 1$ represent n-ary terms and commuting diagrams represent equations. An algebraic theory $\mathcal{T}$ determines the category $\mathsf{Alg}(\mathcal{T})$ which is the full subcategory of $\mathsf{Set}^{\mathcal{T}}$ consisting of all functors $\mathcal{T} \to \mathsf{Set}$ preserving finite products. The underlying functor $\mathsf{Alg}(\mathcal{T}) \to \mathsf{Set}$ is given by the evaluation at 1. A category is called a *variety* if it is equivalent to $\mathsf{Alg}(\mathcal{T})$ for some algebraic theory $\mathcal{T}$.

Given a category $\mathcal{A}$ with finite products and an algebraic theory $\mathcal{T}$, one can still define $\mathcal{T}$-algebras in $\mathcal{A}$ as finite product preserving functors $\mathcal{T} \to \mathcal{A}$ and consider the category $\mathsf{Alg}_{\mathcal{A}}(\mathcal{T})$ of these algebras. In general, there is no guarantee that this category is monadic over $\mathcal{A}$ because free algebras do not need to exist. But, whenever $\mathcal{A}$ is locally presentable (see [7]), $\mathsf{Alg}_{\mathcal{A}}(\mathcal{T})$ is always monadic and the corresponding monad $M : \mathcal{A} \to \mathcal{A}$ has a rank. In particular, when $\mathcal{A}$ is locally finitely presentable, M is finitary. But one cannot expect that each finitary monad is determined by an algebraic theory. A typical example is when $\mathcal{A}$ is the category Vect_P of vector spaces over a field P. Important binary operations are not given by linear maps $V \times V \to V$ but by bilinear ones. So, linear universal algebra deals with operations $V \otimes V \to V$. The corresponding "tensor algebraic theories" were introduced by

Mac Lane [42] under the names of PROP's and PACT's and led to the concept of an *operad*. We recommend [41] to learn about linear universal algebra. One can describe algebras over a monad $M : \mathcal{A} \to \mathcal{A}$ for an arbitrary category $\mathcal{A}$ by means of "operations and equations" (see [39]). However, arities of operations are not natural numbers but objects of $\mathcal{A}$. For instance, by taking the category of posets, binary operations whose arity is a two-element chain are defined only for pairs of comparable elements. It is just the special feature of Set that, besides being cartesian closed, finite sets are coproducts of 1 which makes algebraic theories powerful enough to cover finitary monads.

Filtered categories are precisely categories $\mathcal{D}$ such that colimits over $\mathcal{D}$ commute with finite limits in Set (see, e.g., [7]). There is also a characterization of filtered categories independent of sets – a category $\mathcal{D}$ is filtered if and only if the diagonal functor $\triangle : \mathcal{D} \to \mathcal{D}^{\mathcal{I}}$ is final for each finite category $\mathcal{I}$ (see [17]). It makes filtered colimits belong more to the "doctrine of finite limits" than to that of finite products. It appears as the fact that algebras over a finitary monad do not need to "look like algebras". For example, the category of torsion free abelian groups is the category of algebras for the monad $M : \mathsf{Ab} \to \mathsf{Ab}$ on the category Ab of abelian groups given by the reflection to torsion free ones. This monad is finitary because torsion free groups are closed under filtered colimits in Ab but to be torsion free is not equationally definable. It would be more appropriate to consider categories $\mathcal{D}$ such that colimits over $\mathcal{D}$ commute with finite products. These categories are called sifted and are characterized by the property that the diagonal functor $\triangle : \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is final (see [8]).

Explicitely, a category $\mathcal{D}$ is **sifted**, if it is non-empty and for all objects $A, B \in \mathcal{D}$ the category $\mathsf{Cospan}_{\mathcal{D}}(A, B)$ of cospans is connected. Here, $\mathsf{Cospan}_{\mathcal{D}}(A, B)$ has as objects pairs of arrows $(A \to C, B \to C)$ and arrows $(A \xrightarrow{a} C, B \xrightarrow{b} C) \to (A \xrightarrow{a'} C', B \xrightarrow{b'} C')$ are given by arrows $f : C \to C'$ such that $f \circ a = a', f \circ b = b'$. A category is connected if it is non-empty and cannot be decomposed into a disjoint union (coproduct) of two non-empty subcategories.

Each filtered category is sifted but there are sifted categories which are not filtered – the most important example are reflexive pairs (a parallel pair of morphisms f, g is reflexive if there is t with $ft = gt = id$). Another special feature of sets is that any finitary functor Set $\to$ Set preserves sifted colimits. But it is not true for Ab – the torsion free monad above does not preserve sifted colimits. The consequence is that torsion free groups do not form a variety of universal algebras.

We can define algebras over an arbitrary functor $L : \mathcal{A} \to \mathcal{A}$; an *L-algebra* (A, α) is a pair consisting of an object A and a morphism $\alpha : LA \to A$. In the case when L is a monad, these L-algebras are more general than algebras over a monad L because the latter have to satisfy some equations. Morphisms $(A, \alpha) \to (A', \alpha')$ are morphisms $f : A \to A'$ such that $f \circ \alpha = \alpha' \circ Lf$. The resulting category of L-algebras is denoted by $\mathsf{Alg}(L)$. Like in the case of general equational theories (Σ, E), free L-algebras do not need to exists. If they exist, the category $\mathsf{Alg}(L)$ is monadic over $\mathcal{A}$ with respect to the forgetful functor $\mathsf{Alg}(L) \to \mathcal{A}$ sending an L-algebra (A, α) to A. Given a functor $L : \mathcal{A} \to \mathcal{A}$, algebras over $L^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$ are called *L-coalgebras*.

3. Sifted Colimit Preserving Functors

The concept of a locally finitely presentable category stems from that of a filtered colimit, i.e., belongs to the doctrine of finite limits. Recall that an object A of a category $\mathcal{A}$ is finitely presentable if its hom-functor $\hom(A, -) : \mathcal{A} \to \mathsf{Set}$ preserves filtered colimits. A category $\mathcal{A}$ is locally finitely presentable if it is cocomplete and has a set $\mathcal{X}$ of finitely presentable objects such that each object of $\mathcal{A}$ is a filtered colimit of objects from $\mathcal{X}$. By changing the doctrine from finite limits to finite products, we replace filtered colimits by sifted ones. We say that an object A is *strongly finitely presentable* if its hom-functor $\hom(A, -) : \mathcal{A} \to \mathsf{Set}$ preserves sifted colimits. A category $\mathcal{A}$ is *strongly locally finitely presentable* if it is cocomplete and has a set $\mathcal{X}$ of strongly finitely presentable objects such that each object of $\mathcal{A}$ is a sifted colimit of objects from $\mathcal{X}$. These categories were introduced in [8] where it was shown that they are precisely categories of algebras over many-sorted algebraic theories in Set. Recall that a *many-sorted algebraic theory* $\mathcal{T}$ is a small category with finite products and a $\mathcal{T}$-algebra in Set is a functor $\mathcal{T} \to \mathsf{Set}$ preserving finite products. In particular, each variety is strongly locally finitely presentable. (Note that our usage differs from that of [7] where varieties are given by many-sorted algebraic theories.) While finitely presentable objects in $\mathsf{Alg}(\mathcal{T})$ are algebras finitely presentable in a usual sense, i.e., given by a finite set of generators subjected to a finite set of equations, strongly finitely presentable algebras are precisely retracts of finitely generated free algebras, i.e., finitely presentable projective algebras. An important fact is that each finitely presentable algebra is a reflexive coequalizer of strongly finitely presentable ones (see [8, 2.3.(2)]). Every strongly locally finitely presentable category $\mathcal{A}$ is locally finitely presentable and has, up to isomorphism, only a set of strongly finitely presentable objects. We will denote by $\mathcal{A}_{sfp}$ the corresponding full subcategory of $\mathcal{A}$. In the same way, $\mathcal{A}_{fp}$ denotes a representative small full subcategory of finitely presentable objects.

Given a strongly locally finitely presentable category $\mathcal{A}$ and a category $\mathcal{B}$ having sifted colimits then a sifted colimit preserving functor $H : \mathcal{A} \to \mathcal{B}$ is fully determined by its values on strongly finitely presentable objects. In fact, $\mathcal{A}$ is a free completion of $\mathcal{A}_{sfp}$, i.e., each functor $\mathcal{A}_{sfp} \to \mathcal{B}$ extends to a functor $\mathcal{A} \to \mathcal{B}$ (see [8]). In particular, it applies to functors $\mathcal{A} \to \mathcal{A}$. In analogy to [7], Remark 2.75, we can prove the following basic result.

Theorem 3.1. *Let $\mathcal{A}$ be a strongly locally finitely presentable category and $L : \mathcal{A} \to \mathcal{A}$ preserve sifted colimits. Then $\mathsf{Alg}(L)$ is strongly locally finitely presentable.*

Proof. Since $\mathcal{A}$ is strongly locally finitely presentable and L preserves filtered colimits, $\mathsf{Alg}(L)$ is locally finitely presentable (see [7, Remark 2.75]). Hence the forgetful functor $U : \mathsf{Alg}(L) \to \mathcal{A}$ has a left adjoint F (see [7], 1.66). Thus U is monadic, i.e., each L-algebra X admits a regular epimorphism $e : FA \to X$ from some free L-algebra. Since $\mathcal{A}$ is strongly locally finitely presentable, A is a sifted colimit of strongly finitely presentable objects. Hence FA is a sifted colimit of free algebras over strongly finitely presentable objects. Thus there is a regular epimorphism from a coproduct of free algebras over strongly finitely presentable objects to FA. Consequently, there is a strong epimorphism from such a coproduct to X, which means that free algebras over strongly finitely presentable objects form a strong generator of $\mathsf{Alg}(L)$. By [13], $\mathsf{Alg}(L)$ is strongly locally finitely presentable. $\square$

Theorem 3.2. *Let $\mathcal{A}$ be a variety and $L : \mathcal{A} \to \mathcal{A}$ preserve sifted colimits. Then $\mathsf{Alg}(L)$ is a variety.*

Proof. Since $\mathcal{A}$ is a variety, there is an object $S_1 \in \mathcal{A}_{sfp}$ such that each object of $\mathcal{A}_{sfp}$ is a retract of a finite coproduct of copies of S_1 (S_1 is a free algebra with one generator). Hence each object of $F(\mathcal{A}_{sfp})$ is a retract of finite coproducts of copies of $F(S_1)$. By [13], the closure of $F(\mathcal{A}_{sfp})$ under finite coproducts is the set $\mathcal{X}$ from the definition of a strongly locally finitely presentable category. Since a coproduct of retracts is a retract of coproducts, each object of $\mathcal{X}$ is a retract of finite coproducts of copies of $F(S_1)$. Hence $\mathsf{Alg}(L)$ is a variety. $\square$

In some simple but important varieties like sets or vector spaces, every finitely presentable algebra is projective.

Proposition 3.3. *Let $\mathcal{A}$ be a variety such that every finitely presentable algebra is projective. Then any functor $L : \mathcal{A} \to \mathcal{A}$ preserving filtered colimits preserves sifted colimits.*

Proof. Let $L : \mathcal{A} \to \mathcal{A}$ preserve filtered colimits. Then L is uniquely determined by its restriction L_0 to $\mathcal{A}_{fp}$. Since $\mathcal{A}_{fp} = \mathcal{A}_{sfp}$, there is a unique extension $L' : \mathcal{A} \to \mathcal{A}$ of L_0 preserving sifted colimits. Since $L = L'$, L preserves sifted colimits. $\square$

The previous proposition can be extended to Boolean algebras. In fact, the trivial Boolean algebra $\mathbb{1}$ is the only finitely presentable Boolean algebra that is not projective. $\mathbb{1}$ is the reflexive coequalizer

$$F1 \underset{o}{\overset{i}{\rightrightarrows}} F0 \overset{s}{\longrightarrow} \mathbb{1} \tag{3.1}$$

where F is the left adjoint to the forgetful functor $\mathsf{BA} \to \mathsf{Set}$, i maps the generator to the top, and o maps the generator to the bottom. If $L : \mathsf{BA} \to \mathsf{BA}$ preserves filtered colimits and the above coequalizer, then L preserves sifted colimits.

Proposition 3.4. *For any filtered colimit preserving functor $L : \mathsf{BA} \to \mathsf{BA}$ there is a sifted colimit preserving functor $L' : \mathsf{BA} \to \mathsf{BA}$ such that L and L' are isomorphic when restricted to the full subcategory of BA without $\mathbb{1}$. Moreover, $\mathsf{Alg}(L) = \mathsf{Alg}(L')$.*

Proof. If $A \neq \mathbb{1}$ then there is no arrow $\mathbb{1} \to A$. Thus A is a filtered colimit of objects from BA_{sfp}. Let L_0 be the restriction of L to BA_{sfp} and $L' : \mathsf{BA} \to \mathsf{BA}$ be the sifted colimit preserving extension of L_0. Then L' is isomorphic to L on the full subcategory of BA without $\mathbb{1}$. The rest is evident. $\square$

The proposition shows that as far as we are concerned with algebras over BA, we can assume any finitary functor to preserve sifted colimits. It also gives a category theoretic reason for sometimes restricting attention to non-trivial Boolean algebras.

4. PRESENTING FUNCTORS ON VARIETIES

Given a strongly locally finitely presentable category $\mathcal{A}$, this section shows that a functor $L : \mathcal{A} \to \mathcal{A}$ has a finitary presentation by operations and equations iff L preserves sifted colimits. We start by investigating the category $\mathcal{S}(\mathcal{A})$ of all functors $\mathcal{A} \to \mathcal{A}$ preserving sifted colimits. Morphisms are natural transformations. It is a legitimate category because it is equivalent to $\mathcal{A}^{\mathcal{A}_{sfp}}$.

Proposition 4.1. *Let $\mathcal{A}$ be a strongly locally finitely presentable category. Then $\mathcal{S}(\mathcal{A})$ is strongly locally finitely presentable.*

Proof. There is a small category $\mathcal{T}$ with finite products such that $\mathcal{A}$ is equivalent to the category of all functors $\mathcal{T} \to \mathsf{Set}$ preserving finite products. Since

$$(\mathsf{Set}^{\mathcal{T}})^{\mathcal{A}_{sfp}} \cong \mathsf{Set}^{\mathcal{T} \times \mathcal{A}_{sfp}},$$

the category $\mathcal{A}^{\mathcal{A}_{sfp}}$ is equivalent to the category of all functors

$$\mathcal{T} \times \mathcal{A}_{sfp} \to \mathsf{Set}$$

sending cones $((p_i, id_S) : (T_1, S) \times \cdots \times (T_n, S) \to (T_i, S))_{i=1}^n$ to products; here $(p_i : T_1 \times \cdots \times T_n \to T_i)_{i=1}^n$ is a finite product in $\mathcal{T}$. Hence $\mathcal{A}^{\mathcal{A}_{sfp}}$ is equivalent to the category of models of a finite product sketch and thus it is strongly locally finitely presentable (see [7], 3.17). Hence $\mathcal{S}(\mathcal{A})$ is strongly locally finitely presentable. $\square$

A functor $H : \mathcal{A} \to \mathcal{B}$ between strongly locally finitely presentable categories is called *algebraically exact* provided that it preserves limits and sifted colimits. Then H has a left adjoint and the reason for this terminology is that such functors dually correspond to morphisms of many-sorted algebraic theories (see [6]). In more detail, given a finite product preserving functor $M : \mathcal{T}_1 \to \mathcal{T}_2$ between categories with finite products, then the corresponding algebraically exact functor H sends a $\mathcal{T}_2$-algebra A to the composition AM.

Let $\mathcal{A}$ be a variety, $U : \mathcal{A} \to \mathsf{Set}$ the forgetful functor and F its left adjoint. We get functors

$$\Psi : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathsf{Set})$$

and

$$\Phi : \mathcal{S}(\mathsf{Set}) \to \mathcal{S}(\mathcal{A})$$

by means of $\Psi(L) = ULF$ and $\Phi(G) = FGU$. The definition is correct because U preserves sifted colimits and F preserves all colimits.

Proposition 4.2. *Let $\mathcal{A}$ be a variety. Then the functor Ψ is algebraically exact and Φ is its left adjoint.*

Proof. The functor Ψ is equivalent to the composition

$$\mathcal{A}^{\mathcal{A}_{sfp}} \xrightarrow{U^{\mathcal{A}_{sfp}}} \mathsf{Set}^{\mathcal{A}_{sfp}} \xrightarrow{\mathsf{Set}^{F_{sfp}}} \mathsf{Set}^{\mathsf{Set}_{sfp}}$$

where F_{sfp} denotes the restriction of F to strongly finitely presentable objects. Clearly, Ψ preserves limits and sifted colimits. It remains to show that Φ is left adjoint to Ψ. This left adjoint is equivalent to the composition

$$\mathsf{Set}^{\mathsf{Set}_{sfp}} \xrightarrow{\Phi_1} \mathsf{Set}^{\mathcal{A}_{sfp}} \xrightarrow{F^{\mathcal{A}_{sfp}}} \mathcal{A}^{\mathcal{A}_{sfp}}$$

where Φ_1 is left adjoint to $\mathsf{Set}^{F_{sfp}}$. Since Set_{sfp} is the category Fin of finite sets and each functor $\mathsf{Fin} \to \mathsf{Set}$ is a colimit of hom-functors $\hom(k, -)$ where k is a finite cardinal, it suffices to show that the left adjoint to Ψ coincides with Φ on hom-functors $\hom(k, -)$. But it follows from

$$\Phi_1(\hom(k, -)) = \hom(Fk, -) \cong \hom(k, U-) = \hom(k, -)U.$$

$\square$

Remark 4.3. The adjoint transpose of $\tau : G \to ULF$ is $FGU \xrightarrow{F\tau U} FULFU \xrightarrow{\varepsilon LFU} LFU \xrightarrow{L\varepsilon} L$, which we can also write as $(GU \xrightarrow{\tau U} ULFU \xrightarrow{UL\varepsilon} UL)^\dagger$ where $\dagger$ denotes the adjoint transpose wrt $F \dashv U$.

As it is well-known, presentations can be obtained as follows.

Proposition 4.4. *Let $H : \mathcal{A} \to \mathcal{B}$ be an algebraically exact functor between strongly locally finitely presentable categories with a left adjoint F such that the counit ε is a pointwise regular epimorphism. Then each object A in $\mathcal{A}$ has a presentation as a coequalizer*

$$FR \overset{r_1^\sharp}{\underset{r_2^\sharp}{\rightrightarrows}} FB \xrightarrow{e} A \tag{4.1}$$

where B is an object of $\mathcal{B}$, R a subobject of $HFB \times HFB$ and $r_1^\sharp, r_2^\sharp$ are adjoint transposes of the projections $r_1, r_2 : RB \to HFB$.

Proof. It suffices to take a regular epimorphism $e : FB \to A$ (such as the counit ε_A), its kernel pair $e_1, e_2 : C \to FB$ and to put $r_i = He_i$, $i = 1, 2$. The claim then follows because in strongly locally finitely presentable categories any regular epi is the coequalizer of its kernel pair and because, by assumption, the counit $FHC \to C$ is regular epi. $\square$

We will need the following modification.

Lemma 4.5. *Let $H_1 : \mathcal{A} \to \mathcal{B}$ and $H_2 : \mathcal{B} \to \mathcal{C}$ be algebraically exact functors between strongly locally finitely presentable categories with left adjoints F_1 and F_2, respectively, such that both counits $\varepsilon_1, \varepsilon_2$ are pointwise regular epimorphisms. Then each object A in $\mathcal{A}$ has a presentation as a coequalizer*

$$F_1R \overset{r_1^\sharp}{\underset{r_2^\sharp}{\rightrightarrows}} F_1F_2C \xrightarrow{e} A \tag{4.2}$$

where C is an object of $\mathcal{C}$, R a subobject of $H_1F_1F_2C \times H_1F_1F_2C$ and $r_1^\sharp, r_2^\sharp$ are adjoint transposes of the projections $r_1, r_2 : R \to H_1F_1F_2C$.

Proof. Consider the composition $e = e_1F_1(e_2)$, where $e_1 : F_1B \to A$ and $e_2 : F_2C \to B$ are regular epis. Note that F_1 preserves regular epis since it is a left-adjoint and that in a strongly locally finitely presentable category regular epis are closed under composition since, in many-sorted varieties, regular epis are precisely sort-wise surjective homomorphisms. Now we follow Prop 4.4 by taking $r_i = H_1p_i$ and p_1, p_2 to be the kernel pair of e. $\square$

Let $\mathcal{A}$ be a variety and $\Psi : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathsf{Set})$ the algebraically exact functor from 4.2. Since $\mathcal{S}(\mathsf{Set})$ is equivalent to $\mathsf{Set}^{\mathsf{Fin}}$, it is an $\mathbb{N}$-sorted variety where $\mathbb{N}$ is the set of non-negative integers. Hence there is another algebraically exact functor $H_2 : \mathcal{S}(\mathsf{Set}) \to \mathsf{Set}^{\mathbb{N}}$. Its left adjoint F_2 sends an ω-sorted set $(G_k)_{k<\omega}$ to the functor $G : \mathsf{Set} \to \mathsf{Set}$ given as

$$GX = \coprod_{k<\omega} G_k \times X^k.$$

We are going to show that Lemma 4.5 leads to the presentation of a functor $L : \mathcal{A} \to \mathcal{A}$ as in [12].

Definition 4.1 ([12])**.** A finitary presentation by operations and equations of a functor is a pair $\langle G, E \rangle$ where $G : \mathsf{Set} \to \mathsf{Set}$, $GX = \coprod_{k<\omega} G_k \times X^k$ and $E = (E_V)_{V \in \omega}$, $E_V \subseteq (UFGUFV)^2$. The functor L presented by $\langle G, E \rangle$ is the multiple coequalizer

$$FE_V \begin{array}{c} \xrightarrow{\pi_1^\dagger} \\ \xrightarrow{\pi_2^\dagger} \end{array} FGUFV \xrightarrow{FGUv} FGUA \xrightarrow{q_A} LA \tag{4.3}$$

where $\pi_{V,i}^\dagger$ are the adjoint transposes of the projections $E_V \to UFGUFV$; V ranges over finite cardinals and v over morphisms (valuations of variables) $FV \to A$.

Example 4.2. A modal algebra, or Boolean algebra with operator (BAO), consists of a Boolean algebra A and a meet-preserving operation $A \to A$. Equivalently, a BAO is an algebra for the functor $L : \mathsf{BA} \to \mathsf{BA}$, where LA is defined by generators $\Box a$, $a \in A$, and relations $\Box\top = \top$, $\Box(a \wedge a') = \Box a \wedge \Box a'$. That is, in the notation of the definition, $GX = X$, $E_V = \emptyset$ for $V \neq 2$, $E_2 = \{\Box\top = \top, \Box(v_0 \wedge v_1) = \Box v_0 \wedge \Box v_1\}$.

In [46, 28, 51] 'modal axioms of rank 1' play a prominent role. These are exactly those which, considered as equations, are of the form $E_V \subseteq (UFGUFV)^2$.

Definition 4.3 (rank 1)**.** Let $\mathcal{A}$ be a variety with equational presentation $\langle \Sigma_\mathcal{A}, E_\mathcal{A} \rangle$. Consider a collection Σ of additional operation symbols and a set E of equations in variables V over the combined signature $\Sigma_\mathcal{A} + \Sigma$. We say that the equations E are of *rank 1* if every variable is under the scope of precisely one operation symbol from Σ, or more formally, $E \subseteq (UFGUFV)^2$ where $G : \mathsf{Set} \to \mathsf{Set}$ is the endofunctor associated with Σ and $F \dashv U$ is the adjunction associated with $U : \mathcal{A} \to \mathsf{Set}$.

Remark 4.6.
(1) The generators appear as a functor G. This expresses that the same generators (the $\Box$ in the example above) are used for all LA where A ranges over BA. Similarly, the coequalizer (4.3) is expressed using equations in variables V, that is, the same relations are used for all LA. In $E_V \subseteq (UFGUFV)^2$ the inner UF allows for the conjunction in $\Box(v_0 \wedge v_1)$ whereas the outer UF allows for the conjunction in $\Box v_0 \wedge \Box v_1$.
(2) It is often useful to analyse algebras over algebras using monads and distributive laws. But in our situation, as will be clear from the following proof, $UFGUFV$ arises not from applying the monad UF to V and then to $GUFV$, but from applying to G the functor $\Phi = F - U$ followed by $\Psi = U - F$.

Theorem 4.7. *An endofunctor on a variety has a finitary presentation by operations and equations if and only if it preserves sifted colimits.*

Proof. For 'if', let $\mathcal{A}$ be a variety, $U : \mathcal{A} \to \mathsf{Set}$ the forgetful functor, F its left adjoint, and let $L : \mathcal{A} \to \mathcal{A}$ preserve sifted colimits. We apply Lemma 4.5 to $H_1 = \Psi$ and $H_2 : \mathcal{S}(\mathsf{Set}) \to \mathsf{Set}^\omega$ described above. It presents L as a coequalizer

$$\Phi R \begin{array}{c} \xrightarrow{r_1^\sharp} \\ \xrightarrow{r_2^\sharp} \end{array} \Phi G \xrightarrow{e} L \tag{4.4}$$

where $(G_k)_{k<\omega}$ is an ω-sorted set and $G = \coprod_{k<\omega} G_k \times (-)^k$. It yields a finitary presentation of L with E_V given by RV. To verify this in detail consider

$$
\begin{array}{ccc}
FRUA \xrightarrow{\quad r_i^\sharp A \quad} & FGUA \xrightarrow{\ q_A\ } & LA \\
\end{array}
\tag{4.5}
$$

where the upper row is Diagram 4.4, and the dotted arrows are Diagram 4.3 († denotes adjoint transpose wrt $F \dashv U$ and $\sharp$ wrt $\Phi \dashv \Psi$). To check that the two triangles commute, recall from Lemma 4.5 that $R = UPF$ arises from the kernel $p_1, p_2 : P \to FGU$ of $FGU \to L$. We put $r_i = Up_iF$ and $\pi_{V,i} = Up_iFV$. We have $r_i^\sharp FV = FGU\varepsilon FV \circ \varepsilon FGUFUFV \circ FUpFUFV$ (Remark 4.3) and $\pi_{V,i}^\dagger = \varepsilon FGUFV \circ FUpFV$. Commutativity of the triangles now follows from the naturality of FUp and εFGU. To finish the argument, recall that the upper row is a coequalizer. Because A is a sifted colimit of finitely generated free algebras FV and because FRU preserves sifted colimits, it follows that q_A is the multiple coequalizer obtained from the middle row. Since $FUP\varepsilon FV$ is epi (εF is split by $F\eta$) the dotted arrows also form a multiple coequalizer.

Conversely, every functor $L : \mathcal{A} \to \mathcal{A}$ with a presentation preserves sifted colimits because such a functor is a coequalizer of two natural transformations between functors preserving sifted colimits. In more detail, suppose that L has a presentation as in (4.3). Let $c_i : A_i \to A$ be a sifted colimit. We have to show that Lc_i is a sifted colimit. Given a cocone $d_i : LA_i \to L'$ we have to show that there is a unique k as depicted in

$$
\begin{array}{ccccccc}
FE_V \underset{\pi_2^\sharp}{\overset{\pi_1^\sharp}{\rightrightarrows}} FGUFV & \longrightarrow & FGUA_i & \xrightarrow{q_{A_i}} & LA_i & & \\
& & \downarrow{\scriptstyle FGUc_i} & & \downarrow{\scriptstyle Lc_i} & \searrow^{d_i} & \\
& & FGUA & \xrightarrow{q_A} & LA & \dashrightarrow{k} & L'
\end{array}
$$

U preserves sifted colimits because $\mathcal{A}$ is a variety, G preserves sifted colimits because they commute with finite products, and F preserves all colimits. Therefore we have h with $d_i \circ q_{A_i} = h \circ FGUc_i$. Then k is obtained from the joint coequalizer q_A once we show that $h \circ FGUv^\sharp \circ \pi_1^\sharp = h \circ FGUv^\sharp \circ \pi_2^\sharp$ for all $v : V \to UA$. For this consider $v : V \to UA$. Since $\hom(V, -)$ preserves sifted colimits (V is finite) and U preserves sifted colimits, there is some A_j and some $w : V \to UA_j$ such that $v = Uc_j \circ v'$. It follows $v^\sharp = c_j \circ w^\sharp$, hence $FGUv^\sharp = FGUc_j \circ FGUw^\sharp$. □

Remark 4.8 (correspondence between functors and presentations). We summarise the constructions of the proof for future reference.

(1) Let L be a sifted colimit preserving functor. Then we obtain a finitary presentation $\langle \Sigma, E \rangle$ as follows. Given L we find a suitable G as $GX = \coprod_{n<\omega} ULFn \times X^n$, with $GX \to ULFX$ given by $(\sigma \in ULFn, v : n \to X) \mapsto ULF(v)(\sigma)$. The quotient $q : FGUA \to LA$ is then given by (see Remark 4.3) the adjoint transpose of $GUA \to ULA$,

mapping $(\sigma, v : n \to UA) \mapsto UL(v^\dagger)(\sigma)$. To summarise, the set of n-ary operations of Σ is $ULFn$ and the set of equations in n variables is the kernel of $q : FGUFn \to LFn$.

(2) Conversely, every presentation $\langle \Sigma, E \rangle$ defines a functor as in Definition 4.1.

Given a variety $\mathcal{A}$ and a functor $L : \mathcal{A} \to \mathcal{A}$ preserving sifted colimits, we know that $\mathsf{Alg}(L)$ is a variety (see 3.2). The main point is that one obtains a presentation of $\mathsf{Alg}(L)$ from a presentation of $\mathcal{A}$ and from a presentation of L.

Theorem 4.9 ([12])**.** *Let $\mathcal{A} \cong \mathsf{Alg}(\Sigma_{\mathcal{A}}, E_{\mathcal{A}})$ be a variety and $\langle \Sigma_L, E_L \rangle$ a finitary presentation of $L : \mathcal{A} \to \mathcal{A}$. Then $\mathsf{Alg}(\Sigma_{\mathcal{A}} + \Sigma_L, E_{\mathcal{A}} + E_L)$ is isomorphic to $\mathsf{Alg}(L)$, where equations in $E_{\mathcal{A}}$ and E_L are understood as equations over $\Sigma_{\mathcal{A}} + \Sigma_L$.*

Remark 4.10. The theorem shows that, for a functor L determined by its action on the finitely generated free algebras of a variety $\mathcal{A}$, the notion of L-algebra is a special case of the universal algebraic notion of algebra for operations and equations.

(1) In detail, given an algebra $\alpha : LA \to A$, we define a $(\Sigma_{\mathcal{A}} + \Sigma_L)$-algebra structure A on A via $\sigma^{\mathsf{A}}(a_1, \dots a_n) = \alpha(\sigma(a_1, \dots a_n))$ for all n-ary operations $\sigma \in \Sigma_L$ and all $(a_1, \dots a_n) \in A^n$.

(2) Conversely, given $\mathsf{A} \in \mathsf{Alg}(\Sigma_{\mathcal{A}} + \Sigma_L, E_{\mathcal{A}} + E_L)$, we can define $\alpha : LA \to A$ by $\alpha(\sigma(a_1, \dots a_n)) = \sigma^{\mathsf{A}}(a_1, \dots a_n)$ for all n-ary operations $\sigma \in \Sigma_L$ and all $(a_1, \dots a_n) \in A^n$. Since LA is freely constructed from generators this determines α on all of LA.

(3) The logical significance of the theorem is that it ensures that the Lindenbaum algebra for the signature $\Sigma_{\mathcal{A}} + \Sigma_L$ and the equations $E_{\mathcal{A}} + E_L$ is the initial L-algebra.

A consequence of Theorem 4.7 is the immediate proof of the fact that functors having a presentation are closed under composition (see [12]). Also, they preserve surjections as regular epis are reflexive coequalizers of their kernel pairs and hence sifted colimits.

A previous draft of this paper posted by the first author contained a wrong statement about the preservation of injections. Although it is true that a sifted colimit preserving functor on BA preserves injections, this does not in general extend to other varieties. For example, take the category of semi-groups and the functor L given by the presentation consisting of a unary operator $\square$ and equations $\square(a \circ b) = (\square a) \circ (\square b)$ and $\square(a \circ b) = \square(a' \circ b')$. The first equation on its own would just specify that LA is isomorphic to A. But the second equations means that all elements of A that can be decomposed must be identified. Now consider the non-negative integers $\mathbb{N}$ with addition as the semi-group operation. Then L does not preserve the injectivity of inclusion $\mathbb{N} \setminus \{0\} \to \mathbb{N}$ since $L(\mathbb{N} \setminus \{0\})$ has two elements whereas $L\mathbb{N}$ has only one element. More importantly, Rob Myers [45] has an example of an equationally presented functor on distributive lattices that does not preserve injections.

5. Introduction to Part II: Duality of Algebras and Coalgebras

This second part of the paper proves a representation theorem for functor-algebras based on Stone's representation theorem for Boolean algebras and extending the Jónsson-Tarski theorem for modal algebras (Boolean algebras with operators).

Stone's representation theorem for Boolean algebras shows that any Boolean algebra A can be represented as an algebra of subsets where the Boolean operations are interpreted set-theoretically (conjunction as intersection, etc). For the proof, one identifies a functor $\Sigma : \mathsf{BA} \to \mathsf{Set}^{\mathrm{op}}$ and a BA-morphism

$$\iota_A : A \to \Pi \Sigma A \tag{5.1}$$

into the powerset $\Pi(\Sigma A)$ and then shows that $\iota_A : A \to \Pi\Sigma A$ is injective, exhibiting A as isomorphic to a subalgebra of a powerset.

Analysing this situation from a categorical point of view one finds that

$$
\begin{array}{ccc}
\mathsf{BA} & \overset{\Pi}{\underset{\Sigma}{\rightleftarrows}} & \mathsf{Set}^{\mathrm{op}} \\
\uparrow & & \uparrow \\
\mathsf{BA}_\omega & \rightleftarrows & \mathsf{Set}^{\mathrm{op}}_\omega
\end{array}
\tag{5.2}
$$

(i) the category BA_ω of finite Boolean algebras is dually equivalent to the category Set_ω of finite sets, (ii) BA and Set are the completion under filtered colimits, or ind-completion, of BA_ω and Set_ω, (iii) the two functors Π and Σ appearing in Stone's representation theorem arise from lifting the duality between BA_ω and Set_ω to the completions BA and Set. In such a situation, Σ is left-adjoint to Π and the representation morphism (5.1) is the unit of the adjunction.

Abstracting from the particularities of finite Boolean algebras and finite sets leads us to replace BA_ω by an arbitrary small, finitely complete and co-complete category $\mathcal{C}$ and to identify BA and $\mathsf{Set}^{\mathrm{op}}$ as the so-called ind- and pro-completions of BA_ω:

$$
\begin{array}{ccc}
\mathsf{Ind}\mathcal{C} & \overset{\Pi}{\underset{\Sigma}{\rightleftarrows}} & \mathsf{Pro}\mathcal{C} \\
{\scriptstyle(\hat{-})}\nwarrow & & \nearrow{\scriptstyle(\bar{-})} \\
& \mathcal{C} &
\end{array}
\tag{5.3}
$$

We summarise what we need to know about the diagram above, details can be found in Johnstone [24, VI.1]. In the diagram $(\hat{-}) : \mathcal{C} \to \mathsf{Ind}\mathcal{C}$ is the completion of $\mathcal{C}$ under filtered colimits and $(\bar{-}) : \mathcal{C} \to \mathsf{Pro}\mathcal{C}$ under cofiltered limits. Since $\mathcal{C}$ is finitely cocomplete, we have that $\mathsf{Ind}\mathcal{C}$ is cocomplete. $\mathsf{Ind}\mathcal{C}$ is also complete. Dually $\mathsf{Pro}\mathcal{C}$ is complete and cocomplete. Then Σ is defined to be the unique extension of $(\bar{-})$ along $(\hat{-})$ preserving all colimits, and Π is the unique extension of $(\hat{-})$ along $(\bar{-})$. The functors Σ and Π, also known as Kan-extensions, restrict to isomorphisms on $\mathcal{C}$, that is,

$$
\Sigma\hat{C} \cong \bar{C} \qquad \Pi\bar{C} \cong \hat{C} \tag{5.4}
$$

Example 5.1. Let $\mathcal{A}$ be a variety and $\mathcal{A}_{fp}$ be the full subcategory of finitely presentable algebras. $\mathcal{A}_{fp}$ is closed under finite colimits. If $\mathcal{A}$ is a *locally finite variety*, that is, if finitely generated free algebras are finite, then $\mathcal{A}_{fp}$ is also closed under finite limits and $\mathcal{A}_{fp}$ is an example of a finitely complete and finitely cocomplete category $\mathcal{C}$. In particular we have the following instances of our general situation.

(1) $\mathcal{C} = \mathsf{BA}_\omega$ (finite Boolean algebras = finitely presentable Boolean algebras), $\mathsf{Ind}\mathcal{C} = \mathsf{BA}$, $\mathsf{Pro}\mathcal{C} = \mathsf{Set}^{\mathrm{op}}$. ΣA is the set of ultrafilters over A and Π is (contravariant) powerset.
(2) $\mathcal{C} = \mathsf{DL}_\omega$ (finite distributive lattices = finitely presentable distributive lattices), $\mathsf{Ind}\mathcal{C} = \mathsf{DL}$, $\mathsf{Pro}\mathcal{C} = \mathsf{Poset}^{\mathrm{op}}$. ΣA is the set of prime filters over A and Π gives the set of downsets.

The following is a well-known fact.

Proposition 5.1. Σ *is left adjoint to* Π.

Proof. Any left Kan extension such as Σ has a right adjoint R given by $RY = \mathsf{Pro}\mathcal{C}((\bar{-}), Y)$. As Π is the unique extension of $(\hat{-})$ preserving cofiltered limits, the proposition follows from Π and R agreeing on $\mathcal{C}$, which, in turn, is a consequence of the Yoneda lemma. $\qquad\square$

The Jónsson-Tarski theorem [26] extends Stone's theorem to modal algebras, or, Boolean algebras with operators. For example, the BAO of Example 4.2 is a Boolean algebra with one unary operation $\square$ to interpret the modal operator. Given a BAO A, we can associate with it a dual Kripke frame $(X, R_\square)$, where X is ΣA as above (Example 5.1). Describing ΣA as the set of ultrafilters of A, the relation $R_\square$ is given explicitly by

$$xR_\square y \;\Leftrightarrow\; \forall a \in A \; . \; \square a \in x \Rightarrow y \in a, \tag{5.5}$$

that is, $R_\square$ is the largest relation such that, in logical notation, $(xR_\square y \;\&\; x \Vdash \square a) \;\Rightarrow\; y \Vdash a$. Conversely, to any Kripke frame (X, R), one can associate the so-called complex algebra $(\Pi X, \square_R)$ where Π is powerset and $\square_R \, a = \{x \in X \mid \forall y \; . \; xRy \Rightarrow y \in a\}$. The Jónsson-Tarski theorem then states that the representation map (5.1) is not only a Boolean algebra homomorphism but also BAO-morphism

$$(A, \square) \longrightarrow (\Pi\Sigma A, \square_{R_\square}). \tag{5.6}$$

In our category theoretic reconstruction, the additional operator $\square$ corresponds to a functor H and a BAO to an algebra $HA \to A$. The relational structure corresponds to an algebra $KX \to X$ for a functor K on $\mathsf{Pro}\mathcal{C}$. (For the purpose of this part, it is notationally easier to work with algebras on $\mathsf{Pro}\mathcal{C}$ rather than with coalgebras on $(\mathsf{Pro}\mathcal{C})^{\mathrm{op}}$.)

It is interesting to note that the co-unit $\Sigma\Pi X \to X$ of the adjunction does not lift to a morphism between Kripke frames (written now in $(\mathsf{Pro}\mathcal{C})^{\mathrm{op}}$)

$$(X, R) \longrightarrow (\Sigma\Pi X, R_{\square_R})$$

as it is only a graph homomorphism lacking the backward condition of a Kripke frame morphism. This makes the Jónsson-Tarski representation theorem particularly interesting. In our category theoretic reconstruction, it means that (i) Σ does not lift to a *functor* $\mathsf{Alg}(H) \to \mathsf{Alg}(K)$ in (6.13) and (ii) $h : K\Sigma \to \Sigma H$ is not required to be natural.

In (5.6), $(\Pi\Sigma A, \square_{R_\square})$ is known as the **canonical extension** of $(A, \square)$. The theory of canonical extensions, also going back to Jónsson-Tarski [26] (but see eg [55] for a more recent overview), studies the following question: Suppose that the BAO A satisfies some equation e, does its canonical extension $\Pi\Sigma A$ then satisfy e. Investigations of this kind are beyond the scope of the paper. Our generalisation of the Jónsson-Tarski theorem only concerns algebras for a *functor*. In terms of additional equations e, this means that our result only shows that equations of rank 1 (Definition 4.3) are preserved under canonical extensions. Of course, in the case of Kripke frames this (and much more) is already known, but the point of this paper is to generalise to other functors T (or K, as they are called in Part II).

6. Representing Algebras on Ind-Completions

We want to present algebras over $\mathsf{Ind}\mathcal{C}$ by coalgebras over $\mathsf{Pro}\mathcal{C}^{\mathrm{op}}$, or equivalently, by algebras over $\mathsf{Pro}\mathcal{C}$. Therefore, with $\mathcal{C}, \Sigma, \Pi$ as in Diagram 5.3, we consider now $H : \mathsf{Ind}\mathcal{C} \to \mathsf{Ind}\mathcal{C}$

and $K : \mathsf{Pro}\mathcal{C} \to \mathsf{Pro}\mathcal{C}$

$$H \left(\mathsf{Ind}\mathcal{C} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{Pro}\mathcal{C} \right) K \qquad (6.1)$$

We write $\iota : Id \to \Pi\Sigma$ and $\varepsilon : \Sigma\Pi \to Id$ for the unit and co-unit of the adjunction and note that $\iota_{\hat{C}}$ and $\varepsilon_{\bar{C}}$ are isomorphisms for $C \in \mathcal{C}$.

We say that H *is determined by K on $\mathcal{C}$* if there is an isomorphism

$$\kappa_C \quad : \quad H\hat{C} \xrightarrow{\cong} \Pi K\Sigma\hat{C} \qquad (6.2)$$

natural in $C \in \mathcal{C}$; we say that K *restricts to $\mathcal{C}$* if the counit ε is an iso on $K\bar{C}$

$$\varepsilon_{K\bar{C}} \quad : \quad \Sigma\Pi K\bar{C} \xrightarrow{\cong} K\bar{C} \qquad (6.3)$$

Together, (6.2) and (6.3) give an isormorphism

$$(\varepsilon_{K\Sigma\hat{C}} \circ \Sigma\kappa_C)^{-1} \quad : \quad K\Sigma\hat{C} \xrightarrow{\cong} \Sigma H\hat{C} \qquad (6.4)$$

Recalling (5.4), we remark that (6.2) and (6.3) can be written more symmetrically saying that H and K agree on $\mathcal{C}$:

$$H\hat{C} \cong \Pi K\bar{C} \qquad\qquad K\bar{C} \cong \Sigma H\hat{C} \qquad (6.5)$$

Conversely, these (6.5) implies (6.2) and (6.3) if we require that the compositions $H\hat{C} \cong \Pi K\bar{C} \cong \Pi\Sigma H\hat{C}$ and $K\bar{C} \cong \Sigma H\hat{C} \cong \Sigma\Pi K\bar{C}$ give the unit ι and the counit ε.

The natural transformation $\delta : H\Pi \to \Pi K$ is obtained by extending (6.2) from $\mathcal{C}$ to $\mathsf{Ind}\mathcal{C}$ as follows. ΠX is a filtered colimit $\hat{C}_i \to \Pi X$. If H preserves filtered colimits we therefore obtain $H\Pi \to \Pi K$ as in

$$
\begin{array}{ccc}
\Pi X & \qquad H\Pi X \xrightarrow{\quad\delta_X\quad} \Pi K X & \qquad (6.6) \\[4pt]
{\scriptstyle c_i}\uparrow & \quad {\scriptstyle Hc_i}\uparrow \qquad\qquad\qquad \uparrow {\scriptstyle \Pi K c_i^\sharp} & \\[4pt]
\hat{C}_i & \quad H\hat{C}_i \xrightarrow{\quad(6.2)\quad} \Pi K\Sigma\hat{C}_i &
\end{array}
$$

where $c_i^\sharp : \Sigma\hat{C}_i \to X$ is the transpose of $c_i : \hat{C}_i \to \Pi X$. δ allows us to lift Π to a functor

$$\mathsf{Alg}(H) \xleftarrow{\quad\tilde{\Pi}\quad} \mathsf{Alg}(K) \qquad (6.7)$$

mapping a K-algebra (B, β) to the H-algebra $(\Pi B, \Pi\beta \circ \delta_B)$.

Lemma 6.1. *For all $C \in \mathcal{C}$ we have*

$$
\begin{array}{ccc}
H\Pi\Sigma\hat{C} & \xrightarrow{\quad\delta_{\Sigma\hat{C}}\quad} & \Pi K\Sigma\hat{C} \qquad (6.8) \\[4pt]
{\scriptstyle\iota_{\hat{C}}}\nwarrow & & \nearrow {\scriptstyle\kappa_C} \\[4pt]
& H\hat{C} &
\end{array}
$$

The transpose $\delta^* : \Sigma H \to K\Sigma$ of δ is defined as

$$\delta^* \;=\; \Sigma H \longrightarrow \Sigma H\Pi\Sigma \xrightarrow{\Sigma\delta\Sigma} \Sigma\Pi K\Sigma \longrightarrow K\Sigma \qquad (6.9)$$

where the unlabelled arrows arise from the unit and counit of the $\Sigma \dashv \Pi$. We will show below that H-algebras can be presented as K-algebras if there is some, not necessarily natural,

$$h : K\Sigma \to \Sigma H \qquad (6.10)$$

such that

$$h \circ \delta^* = \mathrm{id} \qquad (6.11)$$

The transformation $h : K\Sigma \to \Sigma H$ may not exist in general, but we can say more if K restricts to $\mathcal{C}$. Then we may require the existence of an h as in the following diagram

$$
\begin{array}{ccc}
A & K\Sigma A \xrightarrow{\;h_A\;} \Sigma H A & (6.12) \\
{\scriptstyle d_k}\big\uparrow & \quad\big\uparrow \qquad\qquad \big\uparrow & \\
\hat{A}_k & K\Sigma \hat{A}_k \xrightarrow{\;(6.4)\;} \Sigma H \hat{A}_k &
\end{array}
$$

where the d_k are a filtered colimit. Moreover, the transformation h does exists if K weakly preserves filtered colimits. We don't require that h_A be uniquely determined or natural. h allows us to lift Σ to a map on objects

$$\mathsf{Alg}(H) \underset{\tilde{\Sigma}}{\xrightarrow{\hspace{3cm}}} \mathsf{Alg}(K) \qquad (6.13)$$

Lemma 6.2. *If h is as in Diagram 6.12 and K restricts to $\mathcal{C}$, see (6.3), then (6.11) holds.*

Proof. We first not that the lower row of (6.12) is the inverse $(\delta^*_{A_k})^{-1}$ of the iso $\delta^*_{A_k}$. This is a direct consequence of (6.3) and (6.8). Since $\delta^*_{A_k}$ is natural and (6.12) commutes, it follows $h_A \circ \delta^*_A \circ \Sigma H d_k = \Sigma H d_k \circ (\delta^*_{A_k})^{-1} \circ \delta^*_{A_k}$, hence $h_A \circ \delta^*_A \circ \Sigma H d_k = \Sigma H d_k$. Now (6.11) follows from $\Sigma H d_k$ being a colimit. $\qquad\square$

Remark 6.3. Part III will be devoted to the logical interpretation of the developments of this section. But let us say here already that H will represent the syntax of a modal logic, K^{op} its coalgebraic models, and δ will map a formula to its denotation, that is, to a set of states. An element of ΣH will be a maximal consistent theory Φ and $h^{\mathrm{op}} : \Sigma^{\mathrm{op}} H^{\mathrm{op}} \to K^{\mathrm{op}} \Sigma^{\mathrm{op}}$ will map a theory to a state x. Then (6.11), that is, $(\delta^*)^{\mathrm{op}} \circ h^{\mathrm{op}} = \mathrm{id}$, ensures that the theory of x coincides with Φ. In other words (6.11) says that every maximal consistent one-step theory has a one-step model.

Representing H-algebras as Π-images of K-algebras. Denote by ι the unit of the adjunction $\Sigma \dashv \Pi$. Our next theorem states that for all algebras $\alpha : HA \to A$ the following diagram commutes

$$
\begin{array}{ccc}
A & \xleftarrow{\quad\alpha\quad} & HA \qquad\qquad\qquad (6.14) \\
{\scriptstyle \iota_A}\big\downarrow & & \big\downarrow{\scriptstyle H\iota_A} \\
\Pi\Sigma A \xleftarrow{\Pi\Sigma\alpha} \Pi\Sigma HA \xleftarrow{\Pi h_A} \Pi K\Sigma A & \xleftarrow{\delta_{\Sigma A}} & H\Pi\Sigma A
\end{array}
$$

Theorem 6.4. *Suppose in Diagram 6.1 that H preserves filtered colimits and that H is determined by K on $\mathcal{C}$, ie (6.2) holds.*

(1) *Assume there is $h : K\Sigma \to \Sigma H$ satisfying (6.11). Then for any H-algebra (A, α) we have that $\iota_A : A \to \Pi\Sigma A$ is an H-algebra morphism $(A, \alpha) \to \tilde{\Pi}(\Sigma A, \alpha \circ h_A)$.*

(2) *Furthermore, if K restricts to $\mathcal{C}$, see (6.3), and weakly preserves filtered colimits, then there is an h satisfying the assumption of item 1.*

(3) *If, morevoer, K preserves filtered colimits then h is uniquely determined by (6.12) and a natural transformation.*

Proof. The second item is immediate from Lemma 6.2, with the existence of h coming from K mapping the colimit Σd_k to a weak colimit. For the third item, we note that if $K\Sigma d_k$ is even a colimit, then h is uniquely determined, which in turn yields naturality. Thus it remains to prove that that Diagram 6.14 commutes. Since ι is natural $\iota_{HA} = \Pi h_A \circ \delta_{\Sigma A} \circ H\iota_A$ does suffice,

$$HA \xrightarrow{H\iota_A} H\Pi\Sigma A \xrightarrow{\delta_{\Sigma A}} \Pi K \Sigma A \xrightarrow[\Pi h_A]{} \Pi\Sigma HA \qquad (6.15)$$
$$\text{with } \iota_{HA} \text{ over the top and } \Pi\delta_A^* \text{ under the bottom}$$

for which in turn, because of (6.11), it suffices to have $\delta_{\Sigma A} \circ H\iota_A = \Pi\delta_A^* \circ \iota_{HA}$. For this we first note that in the following diagram the rectangles consisting of non-dotted arrows commute due to $\iota : Id \to \Pi\Sigma$ being natural.

$$
\begin{array}{ccccccc}
\Pi\Sigma H & \xrightarrow{\Pi\Sigma H\iota} & \Pi\Sigma H\Pi\Sigma & \xrightarrow{\Pi\Sigma\delta\Sigma} & \Pi\Sigma\Pi K\Sigma & \xrightarrow{\Pi\varepsilon K\Sigma} & \Pi K\Sigma \\
\uparrow{\iota H} & & \uparrow{\iota H\Pi\Sigma} & & \uparrow{\iota\Pi K\Sigma} & \text{id} & \\
H & \xrightarrow{H\iota} & H\Pi\Sigma & \xrightarrow{\delta\Sigma} & \Pi K\Sigma & &
\end{array}
\qquad (6.16)
$$
$$\text{(dotted arrow } \Pi\delta^* \text{ over the top)}$$

Further, with ε denoting the counit of $\Sigma \dashv \Pi$, the triangle commutes due to the definition of adjunction. We have shown $\delta_{\Sigma A} \circ H\iota_A = \Pi\delta_A^* \circ \iota_{HA}$. $\square$

The theorem does not imply that $\iota_A : A \to \Pi\Sigma A$ is a monomorphism. But this holds in case that $\mathsf{Ind}\mathcal{C}$ is BA or the category DL of distributive lattices as in the following example.

Example 6.1.

(1) We obtain the setting of Jónsson and Tarski [26] with $\mathcal{C} = \mathsf{BA}_\omega$ as in Example 5.1, H the functor L from Example 4.2 and K the powerset. With this data, our theorem states that every Boolean algebra with operators can be embedded into a complete Boolean algebra whose carrier is a powerset.

(2) We obtain the setting of Gehrke and Jónsson [18] with $\mathcal{C}$ finite distributive lattices. For H one can take, for example, the Vietoris functor of Johnstone [25], restricted to DL, and for K the convex powerset functor on posets.

Remark 6.5. Compared to the earlier version of the paper, we reorganised the proof of the theorem and made condition (6.11) explicit. This allows us to strengthen the statement of the theorem and also to compare it precisely to [29, Theorem 3], which is now the special case of Theorem 6.4.1 where $\mathcal{C} = \mathsf{BA}_\omega$. Indeed, for $\mathcal{C} = \mathsf{BA}_\omega$, the existence of an h satsifying [29, Definition 1] is equivalent to the existence of an h satisfying (6.11). The categorical formulation (6.11) of this condition using the transpose δ^* (whose importance for coalgebraic logic was shown by Klin [27] where it is called ϱ^*) is new.

7. Introduction to Part III: Functorial Coalgebraic Logic

We develop the point of view that if a coalgebra is given wrt a functor $T : \mathsf{Set} \to \mathsf{Set}$ then a (finitary, classical) modal logic is given by a functor $L : \mathsf{BA} \to \mathsf{BA}$ on Boolean algebras[1]

$$L \; \overset{\curvearrowleft}{(} \; \mathsf{BA} \underset{S}{\overset{P}{\rightleftarrows}} \mathsf{Set} \; \overset{\curvearrowright}{)} \; T \tag{7.1}$$

together with the semantics

$$\delta : LP \longrightarrow PT.$$

We call such a logic (L, δ) abstract, because no concrete syntactic description has been fixed. Such a concrete description arises from a presentation: A presentation of BA describes BA as a category $\mathsf{Alg}(\Sigma_{\mathsf{BA}}, E_{\mathsf{BA}})$ of algebras for a signature and equations in the usual sense; this gives us classical propositional logic with connectives from Σ_{BA} and axioms from E_{BA}. Moreover, if L has a presentation $\langle \Sigma_L, E_L \rangle$, this gives us the concrete logic with operation symbols from $\Sigma_{\mathsf{BA}} + \Sigma_L$ and axioms $E_{\mathsf{BA}} + E_L$, see Theorem 4.9.

Section 9 will show how to define (L_T, δ_T) from an arbitrary functor $T : \mathsf{Set} \to \mathsf{Set}$ and then give conditions on T under which the logic (L_T, δ_T) is strongly complete (for the global consequence relation). From Part I, we will use that all (L_T, δ_T) have a presentation, ie, the abstract logic indeed arises from a concrete logic. From Part II, we will use the Jónsson-Tarski theorem which will provide us, as a corollary, with the strong completeness result.

To keep this part self-contained, the remainder of this section contains preliminaries on coalgebras and the next section details carefully the relationship between abstract logics (L, δ) and concrete logics given by presentations.

Preliminaries on Coalgebras. Coalgebras for a functor provide a uniform account of different kinds of transition systems and Kripke structures.

Definition 7.1 (coalgebra). The category $\mathsf{Coalg}(T)$ of coalgebras for a functor T on a category $\mathcal{X}$ has as objects arrows $\xi : X \to TX$ in $\mathcal{X}$ and morphisms $f : (X, \xi) \to (X', \xi')$ are arrows $f : X \to X'$ such that $Tf \circ \xi = \xi' \circ f$.

The paradigmatic example are coalgebras $X \to \mathcal{P}X$ for the powerset functor. They can be considered as a set X with a relation $R \subseteq X \times X$, ie as (unlabelled) transition systems or Kripke frames. Similarly, $X \to \mathcal{P}(C \times X)$ is a transition system with transitions labelled with elements of a constant set C. If 2 denotes some two-element set, then $X \to 2 \times X^C$ is a deterministic automaton with input from C and a labelling of states as accepting/non-accepting. To cover all these examples and many more we can consider the following inductively defined class of 'type functors'.

Example 7.2 (gKPF). A generalised Kripke polynomial functor (gKPF) $T : \mathsf{Set} \to \mathsf{Set}$ is built according to

$$T ::= Id \mid K_C \mid T + T \mid T \times T \mid T \circ T \mid \mathcal{P} \mid \mathcal{H}$$

where Id is the identity functor, K_C is the constant functor that maps all sets to a finite set C, $\mathcal{P}$ is covariant powerset and $\mathcal{H}$ is 2^{2^-}.

---

[1]As opposed to Part II, this part will benefit from a notation working with contravariant functors $P : \mathsf{Set} \to \mathsf{BA}$ and $S : \mathsf{BA} \to \mathsf{Set}$ instead of covariant functors $\Pi : \mathsf{Set}^{\mathrm{op}} \to \mathsf{BA}$ and $\Sigma : \mathsf{BA} \to \mathsf{Set}^{\mathrm{op}}$.

Remark 7.1. The term 'Kripke polynomial functor' was coined in Rößiger [48]. We add the functor $\mathcal{H}$. $\mathcal{H}$-coalgebras are known as neighbourhood frames in modal logic and are investigated, from a coalgebraic point of view, in Hansen and Kupke [20].

If we can consider the carriers X of the coalgebras to have elements, ie if there is a forgetful functor $\mathcal{X} \to \mathsf{Set}$, each functor T induces a corresponding notion of bisimilarity or behavioural equivalence.

Definition 7.3 (bisimilarity)**.** Two states x_i in two coalgebras X_i are T-*bisimilar* if there is a coalgebra (X', ξ') and there are coalgebra morphisms $f_i : (X_i, \xi_i) \to (X', \xi')$ such that $f_1(x_1) = f_2(x_2)$.

Remark 7.2. In other words, two states are bisimilar if they are in the same equivalence class of the equivalence relation generated by pairs $(x, f(x))$ where f ranges over all coalgebra morphisms. More categorically, two states are bisimilar if they are in the same connected component of the category of elements of $U : \mathsf{Coalg}(T) \to \mathsf{Set}$. If U has a colimit Z, then Z classifies T-bisimilarity and is the carrier of the final coalgebra.

This notion of bisimilarity has sometimes been called *behavioural equivalence*, since only for weak pullback preserving functors it is the case that behavioural equivalence is characterised by coalgebraic bisimulations [50]. On the other hand, in cases where the functor T does not preserve weak pullbacks, coalgebraic bisimulations are not well-behaved and it has been argued since [32], but see also [21] for a study of 2^2-coalgebras, that behavioural equivalence is the better notion in such situations. We thus find it defensible to choose the more recognisable name of bisimilarity for behavioural equivalence.

In all of the examples above, coalgebraic bisimilarity coincides with the 'natural' notion of equivalence. For, $T = \mathcal{P}$ this goes back to Aczel [4, 5], for deterministic automata two states are bisimilar iff they accept the same language (Rutten [50]) and $\mathcal{H}$-coalgebras have been investigated by Hansen and Kupke [20]. We now turn to logics for T-coalgebras.

8. Functorial Modal Logics for Coalgebras

In this section we present a general framework for logics for T-coalgebras. We do this in two steps.

(1) First, abstracting from syntax, we simply consider as formulas of the logic the elements of the initial L-algebra, where L is a functor which is dual to T in a suitable sense.
(2) Second, we obtain a syntax and a proof system for the abstract logic from a presentation of the functor L. We call these logics the concrete logics of T-coalgebras.

The point of this separation is that it allows us to prove results about concrete logics in a presentation-independent way on the level of the abstract logics. An example of this is presented in the next section.

8.1. **Abstract modal logics.** We are interested in the following situation

$$
\begin{array}{ccc}
\mathsf{Coalg}(T) & \xrightarrow{\tilde{P}} & \mathsf{Alg}(L) \\
\downarrow & & \downarrow \\
T \circlearrowleft \mathcal{X} & \underset{S}{\overset{P}{\rightleftarrows}} & \mathcal{A} \circlearrowleft L \\
\downarrow & & \downarrow U \\
\mathsf{Set} & & \mathsf{Set}
\end{array}
\tag{8.1}
$$

where P and S are contravariant functors.

Example 8.1.
(1) $\mathcal{X} = \mathsf{Set}$, $\mathcal{A} = \mathsf{BA}$. PX is the powerset of X and SA is the set of ultrafilters on A. On maps both P and S act as inverse image. It is also useful to think of PX as the set $\mathsf{Set}(X, 2)$ of functions from X to a two-element set 2 and to think of SA as the set $\mathsf{BA}(A, 2)$ of algebra morphisms from A to the two-element Boolean algebra 2.
(2) $\mathcal{X}$ is the category Stone of Stone spaces (compact Hausdorff spaces that have a basis of clopens), $\mathcal{A} = \mathsf{BA}$, PX is the set of clopens of X and SA is the space of ultrafilters on A with a basis given by $\{\{u \in SA \mid a \in u\} \mid a \in A\}$. In this situation, $\mathcal{X}$ and $\mathcal{A}$ are dually equivalent.

Kripke frames arise under (1) and descriptive (general) frames [19, 10] under (2). The latter situation has been studied from a coalgebraic point of view in [30], whereas this paper will focus on the former.

Remark 8.1. Diagram 8.1 has too many possible variations to give—at this stage—an axiomatic account of the properties the data in (8.1) should satisfy in order to give rise to coalgebraic logics. We indicate some of the possible variations.
(1) In Example 8.1.(1) above, one could keep $\mathcal{X} = \mathsf{Set}$ but take $\mathcal{A}$ to be eg distributive lattices or semi-lattices. A sufficient set of conditions for this set-up is the following: $\mathcal{X} = \mathsf{Set}$ and $\mathcal{A}$ any variety such that there is $P : \mathsf{Set} \to \mathcal{A}$ with $UPX = 2^X$. It then follows that P has an adjoint $SA = \mathcal{A}(A, P1)$ but one would want to require that the unit $\iota_A : A \to PSA$ is an embedding.
(2) In Example 8.1.(2) above, one could work with other dualities such as the one of spectral spaces and distributive lattices.
(3) One could also replace Set by some other categories such as Poset.

To continue the discussion of the data in (8.1), we assume that $\mathcal{A}$ is a variety in the sense of Part I and that L is a sifted colimit preserving functor on $\mathcal{A}$, that is, L is determined by its action on finitely generated free algebras. Then the forgetful functor $U^a : \mathsf{Alg}(L) \to \mathsf{Set}$ has a left adjoint F^a and we consider $U^a F^a V$ as the set of formulas of L in propositional variables V. The semantics of L in terms of T-coalgebras is specified by choosing a natural transformation

$$
\delta : LP \to PT,
\tag{8.2}
$$

where we assume, as in Remark 8.1.(1), that P is a functor satisfying $UPX = 2^X$. Intuitively, δ takes syntax from LPX and maps it to its interpretation as a subset of TX. Technically, δ allows us to extend the functor $P : \mathcal{X} \to \mathcal{A}$ to a functor $\tilde{P} : \mathsf{Coalg}(T) \to \mathsf{Alg}(L)$,

where $\tilde{P}$ maps a coalgebra (X,ξ) to the L-algebra

$$P\xi \circ \delta_X \;:\; LPX \to PTX \to PX \tag{8.3}$$

Consequently, every formula, ie every element of the free L-algebra F^aV, has a unique interpretation as an element of PX, ie a subset of X. This is summarised in the following definition.

Definition 8.2. Let $T : \mathsf{Set} \to \mathsf{Set}$ be a functor, let $L : \mathcal{A} \to \mathcal{A}$ be a sifted colimit preserving functor on a variety $U : \mathcal{A} \to \mathsf{Set}$, and let $P : \mathsf{Set} \to \mathcal{A}$ be a contravariant functor satisfying $UPX = 2^X$. Further, let F^a be a left-adjoint of the forgetful functor $U^a : \mathsf{Alg}(L) \to \mathsf{Set}$ and let $\delta : LP \to PT$ be a natural transformation. We call (L,δ) an (abstract) logic for T-coalgebras. The formulas of the logic are the elements of U^aF^aV. Given a coalgebra (X,ξ), we write $[\![-]\!]_{(X,\xi,h)}$ for the morphism $F^aV \to \tilde{P}(X,\xi)$ determined by the valuation $h : V \to UPX$. We define

$$(X,\xi,h) \models \varphi \lesssim \psi$$

if $[\![\varphi]\!]_{(X,\xi,h)} \subseteq [\![\psi]\!]_{(X,\xi,h)}$. For a collection Γ of 'sequents' $\{\varphi_i \lesssim \psi_i \mid i \in I\}$, we write

$$\Gamma \models (\varphi \lesssim \psi) \tag{8.4}$$

for the global consequence relation, that is, if for all T-coalgebras (X,ξ) and all valuations h we have that $(X,\xi,h) \models \Gamma$ only if $(X,\xi,h) \models (\varphi \lesssim \psi)$. We also write $\models (\varphi \lesssim \psi)$ for $\emptyset \models (\varphi \lesssim \psi)$.

Remark 8.2. For $\mathcal{A}$ being semi-lattices or distributive lattices $\varphi \lesssim \psi$ can be rendered as the equation $\varphi \wedge \psi = \varphi$. In case $\mathcal{A} = \mathsf{BA}$, since Boolean algebra has implication, it is enough to consider sequents of the form $\top \lesssim \psi$. In this case we drop the '$\top \lesssim$' and write $\models \psi$, etc.

Proposition 8.3. *The logic for T-coalgebras given in Definition 8.2 respects bisimilarity.*

Proof. Let $f : (X,\xi) \to (X',\xi')$ be a coalgebra homomorphism and let $h : V \to UPX$, $h' : V \to UPX'$ be two valuations such that $UPf \circ h' = h$. According to Definition 7.3, we have to show that $x \in [\![\varphi]\!]_{(X,\xi,h')} \;\Leftrightarrow\; f(x) \in [\![\varphi]\!]_{(X',\xi',h')}$. But this is immediate from the universal property of F^aV. □

8.2. **Concrete modal logics.** We restrict our attention now to set-coalgebras and to logics over BA. Fix a set of operations Σ_{BA} and equations E_{BA} describing BA, that is, $\mathsf{BA} = \mathsf{Alg}(\Sigma_{\mathsf{BA}}, E_{\mathsf{BA}})$. We assume that the constants $\bot, \top$ are in Σ_{BA}.

Conceptually, a concrete logic is given by a finitary presentation $\langle \Sigma, E \rangle$ of a functor L in the sense of Definition 4.1 together with a natural transformation $LP \to PT$ as in (8.2). Explicitly, this means that a concrete logic is given by the following data.

Definition 8.3 (concrete logic for T-coalgebras). A concrete logic for T-coalgebras is given by a triple (Σ, E, Δ) as follows.

modal operators: A set Σ of operation symbols and a map arity $: \Sigma \to \omega$ assigning to each operation symbol a finite arity.

equations (axioms): A set E of equations $s = t$ of rank 1 as in Definition 4.3. That is, s,t are terms over $\Sigma_{\mathsf{BA}} + \Sigma$ and variables V in which each variable is in the scope of precisely one modal operator (operation symbol from Σ).

semantics: A set Δ containing for each $\heartsuit \in \Sigma$ a natural transformation, also called a 'predicate lifting',

$$[\![\heartsuit]\!] : (2^{\text{arity}(\heartsuit)})^X \to 2^{TX}. \tag{8.5}$$

The equations E are required to be sound with respect to the semantics Δ in the following sense. We lift $[\![-]\!]$ from modal operators $\heartsuit$ to terms s of rank 1,

$$[\![s]\!] : (2^{|V|})^X \to 2^{TX}.$$

In detail, given a valuation $h : V \to 2^X$ and a term s, define $[\![s]\!]_h \subseteq TX$ inductively as follows. First, h lifts to a function $\bar{h}$ on BA-terms by interpreting Boolean operations set-theoretically. Modal operators are then interpreted according to

$$[\![\heartsuit(s_1, \ldots s_{\text{arity}(\heartsuit)})]\!]_h = [\![\heartsuit]\!](\bar{h}(s_1), \ldots \bar{h}(s_{\text{arity}(\heartsuit)})) \tag{8.6}$$

Then an equation $s = t$ in variables from V is sound, if $[\![s]\!]_h = [\![t]\!]_h$ for all $h : V \to 2^X$.

More examples will be given in the next section, here we only present the fundamental one [10, Def's 1.9, 1.13], which translates into our setting as follows.

Example 8.4. The basic modal logic for $\mathcal{P}$-coalgebras (Kripke frames) is given by

modal operators: one unary operator $\square$,

equations (axioms): two equations: $\square\top = \top$ with $V = \emptyset$ and $\square(a \wedge b) = (\square a) \wedge (\square b)$ with $V = \{a, b\}$,

semantics:

$$[\![\square]\!] : 2^X \to 2^{\mathcal{P}X} \tag{8.7}$$

$$Y \mapsto \{Z \subseteq X \mid Z \subseteq Y\}. \tag{8.8}$$

Remark 8.4.

(1) The semantics (8.5) can be written, in the notation of Diagram 8.1, as a natural transformation $U(PX)^n \to UPTX$. This gives a notion of predicate lifting for other categories than Set such as Stone and what follows applies to this setting as well.

(2) The signature Σ is just a collection of n-ary modal operators in the usual sense of modal logic, called a similarity type in [10, Def 1.11].

(3) In modal logic, axioms are usually given by formulas, not by equations. The translation between the two formats is a standard procedure [10, Section 5.1]. In a nutshell, each term in operation symbols from $\Sigma_{\text{BA}} + \Sigma$ is considered as a formula. Equations $s = t$ are turned into formulas $s \leftrightarrow t$. Conversely, any formula s can be read as an equation $s = \top$.

(4) Without restricting E to rank 1 the interpretation $[\![s]\!]_h$ would not be well-defined.

(5) The coalgebraic semantics (see below) of modal operators in terms of predicate liftings goes back to Pattinson [46] and, in the n-ary case, to Schröder [51].

The definition of the language below is standard, see [10, Def 1.12]. For the proof system we use equational logic, see [10, Def B.20].

Definition 8.5 (language, proof system). Let (Σ, E, Δ) be a logic for T-coalgebras. The language $\mathcal{L}(\Sigma, E)$ is the set of terms built from operations $\Sigma_{\text{BA}} + \Sigma$ and the variables that appear in E. Terms are also called formulas. The proof system is that of equational logic plus the additional equations $E_{\text{BA}} + E$ and we write $\vdash_{(\Sigma, E)} s = t$ if an equation is derivable. We also write $\vdash_{(\Sigma, E)} \varphi$ if φ is a formula and $\vdash_{(\Sigma, E)} \varphi = \top$.

Remark 8.5. In modal logic, the standard proof system is not equational logic, but the two systems are equivalent in terms of the theorems that can be derived, see Chapter 5 and Appendix B of [10] for full details.

The next definition reformulates [10, Def 5.19] in our notation. As in Definition 8.5, the semantic component Δ is not needed here, but only in Definition 8.7.

Definition 8.6 (algebraic semantics). Let (Σ, E, Δ) be a logic for T-coalgebras. The category of modal algebras of (Σ, E, Δ) is the category $\mathsf{Alg}(\Sigma_{\mathsf{BA}} + \Sigma, E_{\mathsf{BA}} + E)$ of algebras given by the signature $\Sigma_{\mathsf{BA}} + \Sigma$ and satisfying the equations $E_{\mathsf{BA}} + E$.

In particular, there is a map $[-] : \mathcal{L}(\Sigma, E) \to UFV$ taking formulas to the carrier $U^c F^c V$ of the free $\mathsf{Alg}(\Sigma_{\mathsf{BA}} + \Sigma, E_{\mathsf{BA}} + E)$-algebra $F^c V$ over the variables V.

Next we give the coalgebraic semantics of a concrete logic.

Definition 8.7 (coalgebraic semantics). Let (Σ, E, Δ) be a logic for T-coalgebras and let (X, ξ) be a T-coalgebra. Then $[\![\varphi]\!]^{concrete}_{(X,\xi,h)}$ is defined by induction over $\varphi \in \mathcal{L}(\Sigma, E)$ with Boolean clauses as usual and

$$[\![\heartsuit(\varphi_1, \dots \varphi_{\mathrm{arity}(\heartsuit)})]\!]^{concrete}_{(X,\xi,h)} = P\xi \circ [\![\heartsuit]\!]([\![\varphi_1]\!]^{concrete}_{(X,\xi,h)}, \dots [\![\varphi_{\mathrm{arity}(\heartsuit)}]\!]^{concrete}_{(X,\xi,h)}) \qquad (8.9)$$

for each $\heartsuit \in \Sigma$.

Remark 8.6. If the semantics of an n-ary modal operator $\heartsuit$ is expressed with the help of the Yoneda lemma by a map $T(2^n) \to 2$, then (8.9) takes a list of n-ary predicates $\varphi : X \to 2^n$ and maps it to

$$X \xrightarrow{\ \xi\ } TX \xrightarrow{\ T\varphi\ } T(2^n) \longrightarrow 2.$$

Example 8.8. Going back to Example 8.4, we take now $T = \mathcal{P}$ so that (X, ξ) is a Kripke frame and $\xi(x)$ is the set of successors of x. Recalling that $P\xi = \xi^{-1}$ it follows immediately from the definitions that instantiating (8.9) with (8.7) gives

$$[\![\Box\varphi]\!]^{concrete}_{(X,\xi,h)} = \{x \in X \mid \xi(x) \subseteq [\![\varphi]\!]^{concrete}_{(X,\xi,h)}\},$$

which is the usual definition of the semantics of $\Box$.

In Definition 8.7, the $[\![\heartsuit]\!] \in \Delta$ provided the semantics of the modal operators. Alternatively, we can think of Δ as giving us a functor from coalgebras to algebras, mapping a coalgebra to its 'complex algebra' [10, Def 5.21]. That these two points of view are essentially the same is the contents of Proposition 8.7.

Definition 8.9 (complex algebra). Let (Σ, E, Δ) be a logic for T-coalgebras and let (X, ξ) be a T-coalgebra. Then the complex algebra $\tilde{P}^c(X, \xi)$ of (X, ξ) is the $\mathsf{Alg}(\Sigma_{\mathsf{BA}} + \Sigma, E_{\mathsf{BA}} + E)$-algebra with carrier PX and which interprets operations $\heartsuit \in \Sigma$ according to

$$\heartsuit^{\tilde{P}^c(X,\xi)}(a_1, \dots a_{\mathrm{arity}(\heartsuit)}) = P\xi \circ [\![\heartsuit]\!](a_1, \dots a_{\mathrm{arity}(\heartsuit)}). \qquad (8.10)$$

The relationship between algebraic and coalgebraic semantics follows the classical pattern [10, Prop 5.24, Thm 5.25], again replacing Kripke frames by coalgebras.

Proposition 8.7 (relationship of algebraic and coalgebraic semantics). *Let (Σ, E, Δ) be a logic for T-coalgebras and let (X, ξ) be a T-coalgebra. Any valuation $h : V \to 2^X$ induces a morphism $mng_h : F^c V \to \tilde{P}^c(X, \xi)$ from the free $\mathsf{Alg}(\Sigma_{\mathsf{BA}} + \Sigma, E_{\mathsf{BA}} + E)$-algebra over V to the complex algebra of (X, ξ). For all $\varphi \in \mathcal{L}(\Sigma, E)$ we have*

$$[\![\varphi]\!]^{concrete}_{(X,\xi,h)} = mng_h([\varphi]) \qquad (8.11)$$

Consequently, the equation $\varphi = \top$ holds in the algebra $\tilde{P}(X,\xi)$ iff $[\![\varphi]\!]^{concrete}_{(X,\xi,h)} = X$.

Proof. The proof is a routine induction as in [10, Prop 5.24], using (8.9) and (8.10). $\qquad\square$

Theorem 8.8 (equivalence of abstract and concrete logics). *For each abstract logic (L,δ) there is a concrete logic (Σ, E, Δ), and for each concrete logic there is an abstract logic (L,δ), such that concrete and abstract semantics agree.*

Proof. For the purposes of the proof, write $[\![-]\!]^{abstract}_{(X,\xi,h)}$ for $[\![-]\!]_{(X,\xi,h)}$ in Definition 8.2. As before, we denote by $F^c V$ the term algebra over $(\Sigma_{\mathsf{BA}}+\Sigma+V)$-terms quotiented by equations $E_{\mathsf{BA}} + E$. The two statements of the theorem say:

(1) For each abstract logic (L,δ) there is a concrete logic (Σ, E, Δ) such that, for any left-adjoint F^a of $U^a : \mathsf{Alg}(L) \to \mathsf{Set}$, there is concrete isomorphism $g : \mathsf{Alg}(L) \to \mathsf{Alg}(\Sigma_{\mathsf{BA}} + \Sigma, E_{\mathsf{BA}} + E)$, inducing an isomorphism $f : F^c V \to g(F^a V)$, so that for all coalgebras (X,ξ) and all formulas $\varphi \in \mathcal{L}(\Sigma, E)$ we have $[\![f[\varphi]]\!]^{abstract}_{(X,\xi,h)} = [\![\varphi]\!]^{concrete}_{(X,\xi,h)}$.

(2) For each concrete logic (Σ, E, Δ) there is an abstract logic (L,δ) and a left-adjoint F^a of $U^a : \mathsf{Alg}(L) \to \mathsf{Set}$ such that such that for all coalgebras (X,ξ) and all formulas $\varphi \in \mathcal{L}(\Sigma, E)$ we have $[\![[\varphi]]\!]^{abstract}_{(X,\xi,h)} = [\![\varphi]\!]^{concrete}_{(X,\xi,h)}$.

To prove (1), take the presentation $\langle \Sigma, E \rangle$ of L from Remark 4.8 and let Δ be given by $[\![\heartsuit]\!] : (2^{\mathrm{arity}(\heartsuit)})^X \to 2^{TX}, (Y_1, \ldots Y_{\mathrm{arity}(\heartsuit)}) \mapsto \delta_X(\heartsuit(Y_1, \ldots Y_{\mathrm{arity}(\heartsuit)}))$. By Theorem 4.9, we have a concrete isomorphism $g : \mathsf{Alg}(L) \to \mathsf{Alg}(\Sigma_{\mathsf{BA}} + \Sigma, E_{\mathsf{BA}} + E)$, inducing an isomorphism $f : F^c V \to g(F^a V)$. Since $g(\tilde{P}(X,\xi)) = \tilde{P}^c(X,\xi)$ where the two versions of $\tilde{P}$ refer to (8.3) and Definition 8.9 respectively, we have $[\![f[\varphi]]\!]^{abstract}_{(X,\xi,h)} = mng_h([\varphi]) = [\![\varphi]\!]^{concrete}_{(X,\xi,h)}$, where the second step is (8.11).

For (2), define L as in Remark 4.8 and let $\delta_X(\heartsuit(Y_1, \ldots Y_{\mathrm{arity}(\heartsuit)})) = [\![\heartsuit]\!](Y_1, \ldots Y_{\mathrm{arity}(\heartsuit)})$. Since L is not an 'absolutely' free algebra but quotiented wrt E, we need to check that δ is well-defined, but this follows from the equations E being sound, see Definition 8.3. By Theorem 4.9, we have a concrete isomorphism $g : \mathsf{Alg}(L) \to \mathsf{Alg}(\Sigma_{\mathsf{BA}} + \Sigma, E_{\mathsf{BA}} + E)$. Choose F^a so that $g(F^c V) = F^a V$ and finish the argument as above in item (1). $\qquad\square$

Remark 8.9. To summarise, given a functor $L : \mathsf{BA} \to \mathsf{BA}$ determined by its action on finitely generated free Boolean algebras, we can find a presentation $\langle \Sigma, E \rangle$ as described in Remark 4.8. This gives us an isomorphism between L-algebras and $(\Sigma_{\mathsf{BA}} + \Sigma, E_{\mathsf{BA}} + E)$-algebras as described in Remark 4.10. Conversely, given operations Σ and equations E of rank 1, we define a functor L as described in Remark 4.8 and this gives us, again, an isomorphism between L-algebras and $(\Sigma_{\mathsf{BA}} + \Sigma, E_{\mathsf{BA}} + E)$-algebras as described in Remark 4.10. The theorem shows that the logic arising from L and the logic arising from (Σ, E) are equivalent.

Example 8.10. Starting from the concrete logic of Example 8.4, we define (L,δ) as follows. LA is the BA generated by $\Box a, a \in A$ and quotiented with respect to the equations of Example 8.4. To give Boolean algebra homomorphisms $\delta_X : LPX \to P\mathcal{P}X$ it is enough to describe them on generators, which is exactly what $[\![\Box]\!]$ in 8.4 does.

Conversely, we could start by defining $LA = P\mathcal{P}SA$ on finite Boolean algebras. This determines L on finitely generated free algebras and hence defines a sifted colimit preserving functor. Therefore we can present L as in Remark 4.8. This canonical presentation, which is made from all (finitary) predicate liftings for $\mathcal{P}$, is different from the presentation with a single $\Box$, but it presents an isomorphic functor. This observation is at the the heart of the next section.

Summary.   The correspondence between functors and logics gives us the licence to switch at will between the abstract point of view for which a logic is a pair (L, δ) and the concrete point of view for which a logic is given by operations and equations. We will therefore, in the following, blur the distinction whenever convenient.

The good functors $L : \mathsf{BA} \to \mathsf{BA}$ for which this correspondence is available, are those functors for which one of the following equivalent conditions holds:

- L preserves sifted colimits,
- L is determined by its action on finitely generated free algebras.

This follows from the general considerations of Part I, but in the case of BA one can go further. Using Proposition 3.4, we can extend this list by saying that, up to modification of L on the one-element Boolean algebra, one of the following equivalent condition holds:

- L preserves filtered colimits,
- L preserves directed colimits,
- LA is determined by its action on the finite subalgebras of A.

9. The Finitary Modal Logic of Set-Coalgebras

The aim of this section is to associate a modal logic to an arbitrary functor $T : \mathsf{Set} \to \mathsf{Set}$. As we are interested here in classical propositional logic the logic will be given by a functor $L_T : \mathsf{BA} \to \mathsf{BA}$. That is, we are concerned with the following situation

$$L_T \; \overset{\curvearrowleft}{} \; \mathsf{BA} \underset{S}{\overset{P}{\rightleftarrows}} \mathsf{Set} \; \overset{\curvearrowright}{} \; T \tag{9.1}$$

where S maps an algebra to the set of its ultrafilters and P is the contravariant powerset. For the readers of Part II, we note that (9.1) is the instance of (6.1) with $\mathcal{C}$ being the category of finite Boolean algebras. But in this part, instead of writing arrows in $\mathsf{Pro}(\mathsf{BA}_\omega) \simeq \mathsf{Pro}(\mathsf{Set}_\omega^{\mathrm{op}}) \simeq \mathsf{Set}^{\mathrm{op}}$, we write them in Set. Further details of how to translate the notation from Part II are summarised in the next remark.

Remark 9.1. To apply the results of Part II, instantiate $H = L_T$, $\mathsf{Ind}\mathcal{C} = \mathsf{BA}$, $\mathsf{Pro}\mathcal{C} = \mathsf{Set}^{\mathrm{op}}$, $K = T^{\mathrm{op}}$. We also write $P : \mathsf{BA} \to \mathsf{Set}$ for the contravariant functor given by the covariant $\Pi : \mathsf{BA} \to \mathsf{Set}^{\mathrm{op}}$ and similarly we write $S : \mathsf{Set} \to \mathsf{BA}$ for the contravariant functor given by the covariant $\Sigma : \mathsf{Set}^{\mathrm{op}} \to \mathsf{BA}$. Accordingly, the types of the unit and counit become $\iota : Id \to PS$ and $\varepsilon : Id \to SP$. Similarly, we have $\delta : LP \to PT$, $\delta^* : TS \to SL$, $h : SL \to TS$.

The next definition generalises Example 8.10.

Definition 9.1 $((L_T, \delta_T))$**.** Let $T : \mathsf{Set} \to \mathsf{Set}$ be any set-functor. We define L_T to be $L_T A = PTSA$ on finite BAs. This determines L_T on finitely generated free algebras and hence defines a sifted colimit preserving functor. For finite X we put $\delta_T X : L_T PX = PTSPX \cong PTX$ and extend to arbitrary X as in (6.6).

Remark 9.2 (Bisimulation-somewhere-else)**.** Since BA and Stone are dually equivalent, the functor $L_T : \mathsf{BA} \to \mathsf{BA}$ has a dual $\hat{T} : \mathsf{Stone} \to \mathsf{Stone}$, which simplifies the definition of the same functor in [29, Def 7, Rmk 16]. We can associate to any T-coalgebra (X, ξ) a $\hat{T}$-coalgebra $SPX \to SPTX \to SL_T PX \to \hat{T}SPX$. Then two states $x_1, x_2 \in X$ satisfy the

same formulas of (L_T, δ_T) iff they are bisimilar (not necessarily in X but) in SPX, see [29, Thm 18].

This definition applies in particular to all gKPFs, see Example 7.2, and we are able now to supplement further examples to Section 8.2. We note that Part II of this paper does not deal with many-sorted signatures which are required for binary functors $\mathsf{BA} \times \mathsf{BA} \to \mathsf{BA}$. This has been done in Schröder and Pattinson [53] and for the functorial framework of this paper in [36].

Example 9.2. We describe functors $L : \mathsf{BA} \to \mathsf{BA}$ or $L : \mathsf{BA} \times \mathsf{BA} \to \mathsf{BA}$ by generators and relations as follows.

(1) $L_{K_C}(A)$ is the free BA given by generators $c \in C$ and satisfying $c_1 \wedge c_2 = \bot$ for all $c_1 \neq c_2$ and $\bigvee_{c \in C} c = \top$.
(2) $L_+(A_1, A_2)$ is generated by $[\kappa_1]a_1$, $[\kappa_2]a_2$, $a_i \in A_i$ where the $[\kappa_i]$ preserve finite joins and binary meets and satisfy $[\kappa_1]a_1 \wedge [\kappa_2]a_2 = \bot$, $[\kappa_1]\top \vee [\kappa_2]\top = \top$, $\neg[\kappa_1]a_1 = [\kappa_2]\top \vee [\kappa_1]\neg a_1$, $\neg[\kappa_2]a_2 = [\kappa_1]\top \vee [\kappa_2]\neg a_2$.
(3) $L_\times(A_1, A_2)$ is generated by $[\pi_1]a_1$, $[\pi_2]a_2$, $a_i \in A_i$ where $[\pi_i]$ preserve Boolean operations.
(4) $L_\mathcal{P}(A)$ is generated by $\Box a$, $a \in A$, and $\Box$ preserves finite meets.
(5) $L_\mathcal{H}(A)$ is generated by $\Box a$, $a \in A$ (no equations).

For the semantics, we define Boolean algebra morphisms δ_T

(1) $L_{K_C}PX \to PC$ by $c \mapsto \{c\}$,
(2) $L_+(PX_1, PX_2) \to P(X_1 + X_2)$ by $[\kappa_i]a_i \mapsto a_i$,
(3) $L_\times(PX, PY) \to P(X_1 \times X_2)$ by $[\pi_1]a_1 \mapsto a_1 \times X_2$, $[\pi_2]a_2 \mapsto X_1 \times a_2$,
(4) $L_\mathcal{P}PX \to PPX$ by $\Box a \mapsto \{b \subseteq X \mid b \subseteq a\}$,
(5) $L_\mathcal{H}PX \to P\mathcal{H}X$ by $\Box a \mapsto \{s \in \mathcal{H}X \mid a \in s\}$.

and extend them inductively to $\delta_T : L_T P \to PT$ for all gKPF T. To be precise, we will for the moment denote by (L'_T, δ'_T) the (L_T, δ_T) given by the presentations in this example and reserve the notation (L_T, δ_T) for the logics given by Definition 9.1. We need to show that (L'_T, δ'_T) is equivalent in the sense of Theorem 8.8 to (L_T, δ_T), in other words, that the presentations of this example indeed present the logics of Definition 9.1. This amounts to showing that $(\delta'_T)_X : L'_T PX \to PTX$ is an isomorphism for all finite sets X. It is exactly here where the machinery presented in this paper needs to be supplemented by additional work depending on the concrete presentation at hand. In our case this is essentially known: (1)-(3) are slight variations of cases appearing in Abramsky [1], (4) is in Abramsky [2], and δ_X in (5) is given by the identity on $2^{2^{2^X}}$.

For gKPFs excluding $\mathcal{H}$, the maps

$$h_A : SLA \to TSA \tag{9.2}$$

from (6.10) have been described by Jacobs [23, Definition 5.1]. We detail the definitions of the following two cases.

(4) $h_A : SL_\mathcal{P}A \to \mathcal{P}SA$ maps $v \in SL_\mathcal{P}A$ to $\{u \in SA \mid \Box a \in v \Rightarrow a \in u\}$.
(5) $h_A : SL_\mathcal{H}A \to \mathcal{H}SA$ maps $v \in SL_\mathcal{H}A$ to $\{\hat{a} \in 2^{SA} \mid \Box a \in v\}$.

Remark 9.3.

(1) In modal logic, given a modal algebra $\alpha : L_\mathcal{P}A \to A$, one defines a Kripke frame with carrier SA and accessibility relation $R_\Box$ given by $vR_\Box u \Leftrightarrow \forall a \in A.(\Box a \in v \Rightarrow a \in u)$,

see [10, Def 5.40]. To define $R_\square$ in this way is the same as to give h_A as in (4) above, only that h_A is independent of any given algebra. More precisely, we obtain $R_\square$ as $h_A \circ S\alpha : SA \to \mathcal{P}SA$.

(2) Whereas [29, Def 1] only formulates a condition on h, (6.12) gives us a systematic way of calculating h from δ. For finite $A \in \mathsf{BA}$, denoting the units of the adjunction by $\iota : Id \to PS$ and $\varepsilon : Id \to SP$, we have that h_A is given as an arrow in Set by

$$SLA \overset{(SL\iota_A)^\circ}{\longrightarrow} SLPSA \overset{(S\delta_{SA})^\circ}{\longrightarrow} SPTSA \overset{(\varepsilon_{TSA})^\circ}{\longrightarrow} TSA \tag{9.3}$$

Here we use that T preserves finite sets and hence the arrows above are isos and we can take their inverse, denoted by $^\circ$. Going through (9.3) explicitly will yield (4) and (5) above for finite A and it then turns out that (4) and (5) also work for all A.

(3) Note that (9.3) is the the inverse of

$$\delta_A^* = TSA \overset{(\varepsilon_{TSA})}{\longrightarrow} SPTSA \overset{(S\delta_{SA})}{\longrightarrow} SLPSA \overset{(SL\iota_A)}{\longrightarrow} SLA \tag{9.4}$$

which already appeared as (6.9).

We now come to the main theorem of Part III. Recall Definition 8.2 of a logic for T-coalgebras and the global consequence relation (8.4).

Theorem 9.4. *Let $T : \mathsf{Set} \to \mathsf{Set}$ preserve finiteness of sets and weakly preserve cofiltered limits. Then T has a sound and strongly complete modal logic.*

Proof. Suppose $\Gamma \not\vdash \varphi$. Let A be the free L_T algebra quotiented by Γ. By Theorem 6.4, there is a T-coalgebra on SA such that the injective $\iota_A : A \to PSA$ is an L_T-algebra morphism. ι_A maps all propositions in Γ to all of SA, but φ only to a proper subset. Therefore there is an element in SA satisfying Γ and refuting φ. $\qquad\square$

Remark 9.5.

(1) The condition of weak preservation of cofiltered limits is elegant, but going back to Theorem 6.4 we find that it is enough to ask that we can find $h_A : SLA \to TSA$ such that

$$\delta_A^* \circ h = \mathrm{id}_A \tag{9.5}$$

where δ^* is as in (9.4). It follows from Theorem 6.4 that under (9.5) strong completeness holds without the conditions of T restricting to finite sets or weakly preserving filtered colimits. This version of the theorem was first proved as [29, Theorem 3], although [29] only states the completeness, not the strong completeness consequence, of the Jónsson-Tarski-style representation theorem. Theorem 9.4 extends [29, Theorem 3] first by the construction of the logic L_T from the functor T and second by giving a sufficient condition directly in terms of T for this logic to be strongly complete.

(2) The property of logics expressed in the Jónsson-Tarski-style representation theorems [29, Theorem 3], Theorem 6.4 and Theorem 9.4, known as canonicity in modal logic, is stronger than strong completeness. It is also worth noting that these representation theorems imply strong completeness wrt the global consequence relation which is a stronger property in general than strong completeness wrt to the local consequence relation. For a comparison of these notions of canonicity and strong completeness we refer to Litak [40].

(3) Schröder and Pattinson [54] use similar but weaker conditions to prove strong completeness (but not canonicity) wrt local consequence. They give a number of important examples of such logics for functors T that do not restrict to finite sets.

(4) The weak preservation of cofiltered limits means, in particular, that all projections in the final sequence are onto. The only common example of a finite set preserving functor we are aware of that does not satisfy this condition is the finite powerset functor, see [56]. And indeed, standard modal logic is strongly complete wrt Kripke frames, but not wrt finitely branching ones.

(5) The probability distribution functor [15] does not preserve finite sets and modal logics for probabilistic transition systems, see eg [22], are not strongly complete. A similar situation occurs for $TX = K \times X$ where K is an infinite constant.

(6) In contrast, we can extend our result to functors $X \mapsto (TX)^K$ for infinite K if T preserves finite sets. Indeed, T^K is a cofiltered limit of the functors T^{K_i} where K_i ranges over the finite subsets of K. We can now apply the theorem to obtain logics $L_{T^{K_i}}$ and then extend the result to the colimit of the $L_{T^{K_i}}$ and the limit of the T^{K_i}. This allows us to include functors such as $(\mathcal{P}X)^K \cong \mathcal{P}(K \times X)$, K infinite (which give rise to labelled transition systems).

10. CONCLUSION

Summary The purpose of the paper was to associate a finitary modal logic to a functor T, so that the logic is strongly complete wrt T-coalgebras. We took up the idea, well-established in domain theory [3], that a logic for the solution of a domain equation $X \cong TX$ is given by a presentation of the dual L of T. To obtain a logic from L, one presents L by operations and equations and we characterised those functors on a variety that have a presentation (Theorem 4.7) in Part I. This result is based on the fundamental role that sifted colimits play in the category theoretic analysis of universal algebra, see [9].

To obtain strong completeness of the logic, we showed in Part II how to present L-algebras as T-coalgebras, Theorem 6.4. This can be considered as the Jónsson-Tarski Theorem for L-algebras and T-coalgebras.

Part III shows how an arbitrary $T : \mathsf{Set} \to \mathsf{Set}$ gives rise to a logic L_T. By Part I, we know that L_T has a presentation and, therefore, corresponds to a modal logic given by operations and equations. Applying the representation theorem of Part II, we obtain that under additional conditions on T, this logic is strongly complete for T-coalgebras.

An interesting point is that we do not need the assumption that T is finitary. This assumption is powerful when working with T-algebras, but it is much less so for T-coalgebras. Similarly, we do not need that T preserves weak pullbacks. Each of these assumptions would exclude fundamental examples.

Further work An important aspect of this work is that it makes use of the notion of the presentation of a functor in order to separate syntax and semantics. For example, the strong completeness proof of Theorem 9.4 is conducted—via Theorem 6.4—in terms of abstract category theoretic properties of the logic (L, δ) and is independent of a choice of concrete presentation. This approach was also used in [37], which proves a Goldblatt-Thomason style theorem for coalgebras, and in [34], which compares and translates logics given by predicate liftings and Moss's coalgebraic logic. This is based on the observation that the notion of a coalgebraic logic (L, δ) also accounts for Moss's logic and makes it amenable to

a study via Stone duality. This idea was also used in [31] to give a completeness proof for an axiomatisation of the finitary Boolean version of Moss's logic.

Another important feature of our approach, which goes back to [12], is that it is modular in the sense that the presentation of $\mathsf{Alg}(L)$ is obtained by composing a presentation of the base category with a presentation of L, see Theorem 4.9. This can be extended to a formalism that allows to compose the presentation of $L_1 L_2$ from presentations of L_1 and of L_2 [36]. This requires to move to many-sorted universal algebra and [36] also investigates further applications of the many-sorted generalisation to the semantics of first-order logic and presheaf models of name-binding.

[35] exploits that the nominal algebra [16] of Gabbay and Mathijssen and the nominal equational logic [14] of Clouston and Pitts gives rise to theories which correspond to sifted colimit preserving monads on the category Nom of nominal sets and can thus be viewed as equational theories of many-sorted set-based universal algebra.

Myers [45] extends our work on presentations of Part II to other notions of presentations of functors on varieties and, importantly, starts the systematic investigation of connecting properties of presentations with properties of algorithms checking eg for bisimilairty of process expressions.

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