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The Künneth Formula and Applications

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Comments

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Outline

The q-th de Rham cohomology of $\mathbb{R}^n$ is the vector space defined by the closed q-forms over the exact q-forms. Furthermore, the support of a continuous function f is the closure of the set on which f is nonzero. If we restrict the above definition of the de Rham cohomology to functions with compact support, then the resulting cohomology is called the de Rham cohomology with compact support, or the compact cohomology. The concept of cohomology can also be expanded to general manifolds through constructions such as the Mayer-Vietoris Sequence.

The Künneth Formula in differential topology relates the cohomology of the product of two manifolds to the cohomologies of the individual manifolds through the tensor product. In this project, we provide a proof of the Künneth Formula both for de Rham cohomology and compact cohomology and then show several applications.

Preliminary Definitions

Given a k-dimensional linear space X with a basis $\{e_1, ..., e_k\}$, and an m dimensional linear space Y with a basis $\{f_1, ...f_m\}$, the **tensor product** $X \otimes Y = Z$ of the spaces X and Y is defined as the set of all finite formal sums

$$\sum_{v=1}^{p} x_v \times y_v,$$

where $x_v \in X$, $y_v \in Y$.

A map $f : X \to Y$ is called a **diffeomorphism** if f is a homeomorphism, and both f and f^{-1} are smooth.

A subset $M \subset \mathbb{R}^n$ is called a **smooth manifold of dimension m** if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open subset U of the euclidean space $\mathbb{R}^m$.

Let M be a manifold of dimension m. An open cover $U = \{U_\alpha\}$ of M is called a **good cover** if all nonempty finite intersections $U_{\alpha_0} \cap ... \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^m$. If M has a good cover U which is finite, then we say that M has **finite good cover**, and that M is **of finite type**.

A **partition of unity** on a manifold M is a collection of non-negative C^∞ functions $\{\rho_\alpha\}_{\alpha \in I}$ such that

- Every point has a neighborhood in which $\sum \rho_\alpha$ is a finite sum.
- $\sum \rho_\alpha = 1$.

The q-th **de Rham cohomology** of $\mathbb{R}^n$ is the vector space

$$H^q(\mathbb{R}^n) = \{\text{closed q-forms}\}/\{\text{exact q-forms}\}.$$

We define **support** of a continuous function f on a space X as:

$$\text{Supp} f = \{p \in X \mid f(p) \neq 0\}.$$

The complex resulting from taking only the C^∞ functions with compact support is called the **de Rham complex with compact support**, and the q-th compactly supported cohomology of $\mathbb{R}^n$ is denoted by $H^q_c(\mathbb{R}^n)$.

The Künneth Formula

Theorem 1. The cohomology of the product of two manifolds M and F is the tensor product

$$H^*(M \times F) = H^*(M) \otimes H^*(F).$$

It follows that for every nonnegative integer n,

$$H^n(M \times F) = \bigoplus_{p+q=n} H^p(M) \otimes H^q(F)$$

The Leray-Hirsch Theorem

Lemma 2. Let E be a fiber bundle over M with fiber F. Suppose M has a finite good cover. If there are global cohomology classes $e_1, ..., e_r$ on E which when restricted to each fiber freely generate the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, ..., e_r\}$. That is,

$$H^*(E) \simeq H^*(M) \otimes \mathbb{R}\{e_1, ..., e_r\} \simeq H^*(M) \otimes H^*(F)$$

The Five Lemma

Lemma 3. Given a commutative diagram of Abelian groups and group homomorphisms as in Figure 1 in which the rows are exact, if the maps α, β, δ, and ε are isomorphisms, then γ is also an isomorphism.

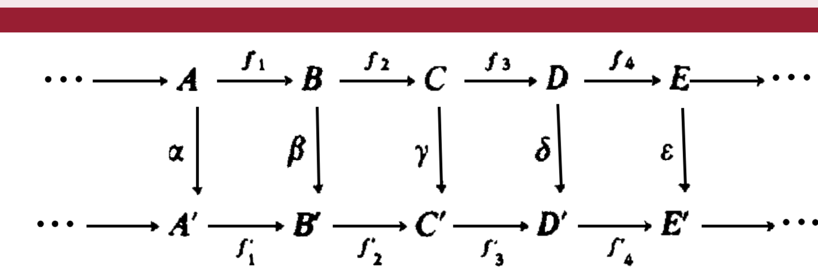

Figure 1: A commutative diagram to illustrate the Five Lemma

Proof of the Künneth Formula

The method proceeds by induction and assumes that the manifold M has finite good cover, using the Mayer-Vietoris sequence and the Five Lemma to guarantee an isomorphism between the desired spaces.

- Let π and ρ represent the natural projections from $M \times F$ to M and F respectively.
- From the projections we obtain a map on forms that takes the tensor product of two forms $\omega \otimes \phi$ into $\pi^*\omega \wedge \rho^*\phi$.
- This induces a map
$$\psi : H^*(M) \otimes H^*(F) \to H^*(M \times F).$$
- To conclude the proof, we need only show that ψ is an isomorphism.

To this end, let U and V be open sets in M, and fix n, a nonnegative integer.

- Then we use the Mayer-Vietoris sequence, which is exact, and tensor every term with $H^{n-p}(F)$.
- This gives us an exact sequence, since the tensor product preserves exactness, and then we may sum over all integers p, again maintaining exactness.

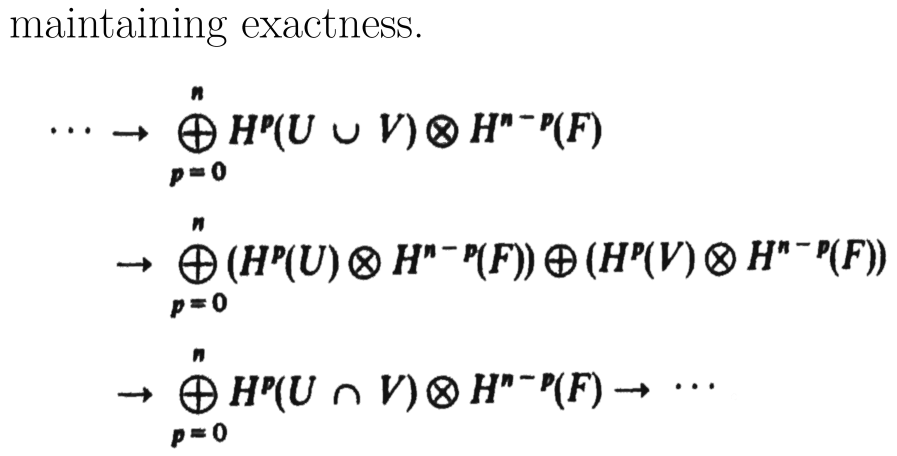

Figure 2: The sequence obtained through calculation on the Mayer-Vietoris sequence

- Applying ψ at each step, we obtain the commutative diagram Figure 3
- A brief computation using the pullback functions to form a partition of unity proves the commutativity of the diagram, allowing us to apply the Five Lemma:

Proof (Continued)

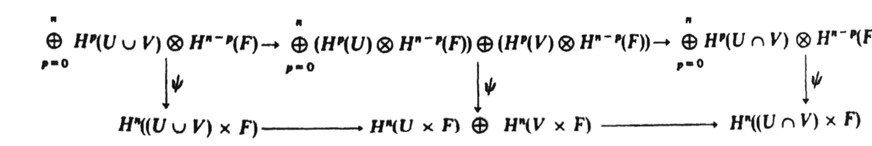

Figure 3: The commutative diagram given by the direct sums of the tensor product with the Mayer-Vietoris sequence

- If ψ is an isomorphism in the cases of U, V, and $U \cap V$, then it will also be true of $U \cup V$ by the Five Lemma.
- Because M has a finite good cover, if it is true of $U \cup V$, it will be true generally for M.

Applications and Examples

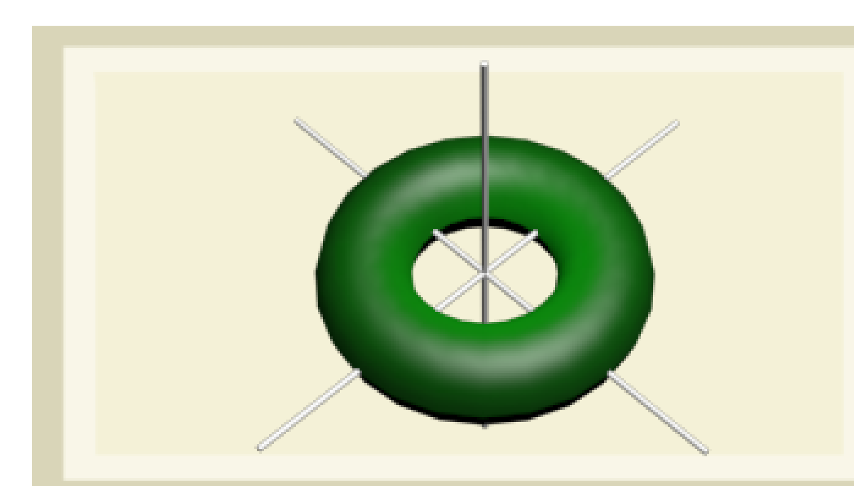

Figure 4: A representation of the torus in 3D space

The Cohomology of the Torus.
The torus, T, can be thought of as the product of two copies of the sphere. That is, $T = S^1 \times S^1$. By the Künneth Formula,

$$H^*(S^1 \times S^1) = H^*(S^1) \otimes H^*(S^1).$$

Using the Mayer-Vietoris sequence we obtain that

- $H^0(S^1) = \mathbb{R}$
- $H^1(S^1) = \mathbb{R}$

It follows that

- $H^0(T) = H^0(S^1) \otimes H^0(S^1) = \mathbb{R} \otimes \mathbb{R} = \mathbb{R}^2$
- $H^1(T) = (H^0(S^1) \otimes H^1(S^1)) \oplus (H^1(S^1) \otimes H^0(S^1)) = \mathbb{R}^2$
- $H^2(T) = H^1(S^1) \otimes H^1(S^1) = \mathbb{R}^2$

We confirm that

$$H^*(T) = \mathbb{R}^2,$$

as can be calculated through the Mayer-Vietoris sequence.

Applications (Continued)

The Cohomology of $(S^k)^n$.
We define the **k-dimensional unit sphere** $\mathbb{S}^k$ as follows:

$$\mathbb{S}^k := \{(x_1, ..., x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1^2 + x_2^2 + ... + x_{k+1}^2 = 1\}.$$

We can apply the Künneth Formula inductively to calculate the cohomology of n-products of the k-sphere. To do so, we first need the cohomology of the k-sphere.
Let $S^k = U \cup V$ where $U \cap V$ is diffeomorphic to $S^{k-1} \times \mathbb{R}$. Then, through the Mayer-Vietoris sequence,

$$H^*(S^k) = \begin{cases} \mathbb{R} & \text{in dimensions } 0, k \\ 0 & \text{otherwise.} \end{cases}$$

By the Künneth Formula,

$$H^*(S^k \times S^k) = H^*(S^k) \otimes H^*(S^k),$$

and since the tensor product is associative,

$$H^*((S^k)^n) = \bigotimes_{i=1}^{n} H^*(S^k).$$

The computation of the cohomology at any qth level follows similarly to the computation of the torus.

Proof of the Leray-Hirsch Theorem.
In the case of the trivial fiber bundle $E = M \times F$, the result follows immediately by the Künneth Formula.
In the more general case, by the definition of a fiber bundle, and since $\{e_1, ..., e_r\}$ restricted to F generate the cohomology of F, there exist fiber preserving isomorphisms corresponding to each set of an open cover of M, $\{U_\alpha\}$

$$\phi_\alpha : E\mid_{U_\alpha} \to U_\alpha \times F.$$

Then $H^*(E \mid U_\alpha) \simeq H^*(U_\alpha \times F) = H^*(U_\alpha) \otimes H^*(F)$.
Since M is of finite type, the result will hold for the union of all U_α. Thus,

$$H^*(E) = H^*(M) \otimes H^*(F).$$

The Künneth Formula for Compact Cohomology

Just as with the general de Rham cohomology, for any manifolds M and F having finite good cover,

$$H^*_c(M \times F) = H^*_c(M) \otimes H^*_c(F).$$

In the case that M and F are orientable, this follows directly from Poincaré Duality.
The more general case can be proven using the Mayer-Vietoris sequence for compact support, by a similar argument to the proof of Theorem 1.

Conclusion

The Künneth Formula relates the cohomology of a product to the cohomology of the individual spaces via the tensor product, providing a convenient tool by which to calculate the cohomologies of otherwise difficult spaces.

References

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