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Efficient Monetary Allocations and the Illiquidity of Bonds[1]

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Abstract

We construct a monetary economy with heterogeneity in discounting and consumption risk. Agents can insure against this risk with money and nominal government bonds, but all trades must be monetary. We demonstrate that a deflationary policy *à la* Friedman cannot sustain the constrained-efficient allocation as no-arbitrage imposes too stringent a bound on the return money can pay. The constrained-efficient allocation can be sustained when bonds have positive yields and, under certain conditions, only if they are illiquid. Illiquidity, meaning that bonds cannot be transformed into consumption as easily as cash, is necessary to eliminate arbitrage opportunities due to disparities in shadow interest rates.

Keywords: Money, Heterogeneity, Friedman Rule, Illiquidity
JEL codes: E4, E5

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1 Introduction

A considerable segment of theoretical monetary literature underscores the social desirability of setting nominal interest rates to zero, a policy known as the Friedman rule. The basic idea is that if impatient agents must do business with cash, then allocative efficiency is achieved by simply eliminating the opportunity cost of saving with money. In fact, zero-interest-rate policies are unusual, and in reality agents largely save with illiquid assets that are sold for cash as consumption needs arise. Among these assets there are highly illiquid government securities such as the U.S. Savings Bonds, of which about $200 billion were held in 2005.[2]

Our study is motivated by the desire to reconcile these observations. In particular, we focus on two issues. If agents must do business with cash, should interest rates be set to zero, and if not, what is the reason? And if interest rates should indeed be set above zero, are there reasons for the government to issue illiquid securities? Of course, to study these questions we need a theoretical framework where money has an explicit role. This in turn implies frictions must be introduced, thus generating market incompleteness. For this reason, we present a physical environment in the tradition of Townsend [21], where money facilitates spot exchange on spatially separated markets that preclude borrowing and lending. Specifically, random consumption needs motivate trade but the process of market interactions is subject to frictions. As a first fundamental friction, a meeting process is imposed that effectively renders trade partners complete strangers and severs any durable links among them, as in [2]. A second basic friction concerns commitment and enforcement limitations. Essentially, agents' actions must be compatible with individual incentives. Thus, trade must be *quid pro quo* and a sudden consumption shock generates an immediate need for cash.

Because idiosyncratic trading shocks can lead to complicated distributions of money balances (as in [7]), we draw from the preference and sequential market formulation proposed by Lagos and Wright in [13] to achieve degeneracy in asset holdings. However, two basic features set our model apart from such a monetary framework. First, agents need not rely exclusively on cash to insure against consumption shocks. They can also acquire government nominal bonds that cannot be directly exchanged for goods but can be liquidated for cash if need be. Second, the model accounts for the possibility of a natural

[2] The figures are from the U.S. Treasury Department. The illiquidity of U.S. Savings Bonds is striking. They are non-marketable registered securities, i.e., they are owned exclusively by the person(s) named on them, ownership is non-transferable, they cannot be used as collateral, there are purchase ceilings and minimum holding times, and there is a penalty for early redemption.

form of heterogeneity, as agents are assumed to differ in their rate of time preference and in their exposure to consumption risk.

We compare stationary monetary allocations in competitive equilibrium to the constrained-efficient allocation. The latter is stationary and corresponds to the selection of a planner bound by the physical and informational limitations that define the economy. Two results emerge.

First, if bonds pay no interest (equivalently, if money is the only asset available) the constrained-efficient allocation is unattainable. The reason is that when nominal interest rates are set to zero, agents save with cash. Equilibrium deflation is bounded by the lowest discount rate because, as in [4], giving cash a return exceeding the lowest shadow interest rate generates arbitrage opportunities. Hence, impatient agents under-save with cash, which impairs trading efficiency. Yet, because money in our model is essential to execute trades, everyone holds some cash in equilibrium, unlike [4] where the most patient agents hold the entire stock of assets (capital).

This first result complements a theoretical monetary literature that finds zero nominal interest rates are generally optimal unless factors such as income shocks, as in [1], redistributive issues, as in [5] or [10], distortionary taxes, as in [15] or trading externalities, as in [17] are taken into account. Our work points out that shadow interest rates disparities are sufficient to blunt the effectiveness of a zero-interest-rate policy. Since existence of such disparities is empirically well established (e.g., see [16]) our result suggests yet one more reason why allocative efficiency cannot be achieved by simply eliminating the opportunity cost of holding cash.

Second, we demonstrate that positive nominal interest rates can sustain the constrained-efficient allocation under certain conditions, but only if the government issues bonds that are sufficiently illiquid, i.e. bonds that cannot be turned into consumption as easily as cash. This friction takes the form of a proportional fee for early redemption. If the government prices bonds correctly, agents fully insure against consumption shocks by holding bonds that are sold for cash once consumption needs arise. In short, we need a friction (illiquidity) to cure an inefficiency.

When is illiquidity a necessary friction? In our model this occurs when the most patient agents are also those who trade more frequently. Illiquidity acts as a tax that lowers the bonds' expected return according to the anticipated incidence of consumption shocks. Thus, illiquidity affects the bonds' expected return unequally across agent types. By selecting bond yield and illiquidity appropriately, the policy maker can manipulate the expected rates of return to eliminate arbitrage opportunities while allowing every agent

to perfectly insure against consumption risk.

This second result complements recent theoretical research that rationalizes the social desirability of illiquid securities as tools to overcome social or individual commitment limitations. In the non-monetary economy described in [14] agents display dynamically inconsistent preferences and so illiquid assets help agents to beneficially constrain their own future choices. In the monetary economy laid out in [12], instead, agents value initial consumption differently but cannot commit to a redistributive plan. Illiquid bonds overcome this limitation by forcing their owners to postpone consumption. Finally, the study in [18] takes a different angle and characterizes illiquidity of bonds as an endogenous outcome linked to government restrictions on market trades. In our work bonds are not commitment tools that force agents to postpone consumption. Instead, they are assets designed to provide optimum consumption insurance when disparities in discounting and consumption risk lead to self-insurance problems that cannot be mended by simply eliminating the opportunity cost of holding cash.

2 The model

We describe a spatially separated economy in which money has an explicit medium of exchange function and there is no role for private credit. The model builds on [2], [11], [13] and [21]. Time is discrete, starts with date 1 and the horizon is infinite. There is a population $X = \mathbb{N}$ of heterogeneous infinitely-lived agents who want to consume perishable goods and discount only even to odd dates. Thus, as in [13], we work with trading cycles indexed by $t = 1, 2, ...$ each including an odd and an even date. As in [21] there are infinitely many spatially separated trading groups, each of which defines a competitive market. On each date, every market includes infinitely many anonymous agents who have never met before. Thus, in each trading cycle each agent visits two anonymous markets, denoted 'one' in the odd date and 'two' in the even.

2.1 Trading groups

Trading groups are formed by a matching process that repeatedly partitions the population into disjoint sets of agents, as formalized in [2] and [3]. On each date there is a correspondence $\mu : X \twoheadrightarrow X$ that creates a partition $X = \sqcup_{s \in S} X_s$, with S an index set. Here $\mu(x) = X_s \subset X$ identifies the trading group of agent $x \in X_s$, with $X_s \cap X_{s'} = \emptyset$ for $s \neq s'$. Since $X = \mathbb{N}$, it is possible to form infinitely many groups each with countably many agents and an identical proportion of agents types. The matching process can then be defined as an infinite deterministic sequence $(\mu_1, \mu_2, \ldots)$, where the set $\mu_\tau(x)$ denotes

the trading group or market to which agent x belongs on date $\tau = 1, 2, ...$

Frictions exist that rule out the possibility of private credit, as in [21]. Precisely, agents can trade only within their group (spatial separation) and can neither communicate nor observe events in other groups (limited communication). In addition, agents ignore the partition of the population and know neither identity, nor type, nor trading history of others (anonymity). Finally, the matching process is such that agents neither meet the same partners again nor meet agents with whom they have direct or indirect partners in common. Equivalently, markets are formed by 'complete strangers.' This is a central feature in several models of money, which we formalize following [11].

Denote by $G_\tau(x)$ the set of all direct and indirect partners of agent x up to date τ. It contains all of x's past and current partners, the past partners of x's current partners, the partners that x's partners in $\tau - 1$ met prior to that date, and so on. If we let $G_0(x) = \{x\}$ then for $\tau = 1, 2, \ldots$ define recursively

$$G_\tau(x) = G_{\tau-1}(x) \cup [\cup_{y \in \mu_\tau(x)} G_{\tau-1}(y)].$$

Traders are complete strangers when $(\mu_1, \mu_2, \ldots)$ is such that for all τ and all $y \in \mu_\tau(x)$:

$$G_{\tau-1}(x) \cap G_{\tau-1}(y) = \emptyset. \tag{1}$$

A technique to construct trading groups satisfying (1) is analyzed in [2] and [3].[3]

2.2 Preferences and technologies

Dates differ in terms of agents' preferences and economic activities, as in [13]. Odd dates are characterized by idiosyncratic trading risk as an arbitrary agent either works but does not wish to consume, or consumes but cannot work, or is idle, i.e., he neither wishes to consume nor is able to work. Everyone can work and consume on even dates.

Agents are heterogeneous. We assume two types of agents denoted by $j = H, L$ in proportion ρ and $1 - \rho$. These agents differ in two dimensions. First, the discount factors β_j satisfy $0 < \beta_L < \beta_H < 1$, so we refer to agents L as impatient and agents H as patient. Second, agents draw different i.i.d. trading shocks at the start of each odd date. Specifically, an agent of type j is idle with probability $1 - \alpha_j$, where $0 < \alpha_L < \alpha_H \leq 1$.

[3] The procedure consists of three steps. Start by partitioning the population into countably many sets of identical cardinality. Based on this partition, construct partitions recursively for each date. Finally, build trading groups out of these partitions insuring that no pair of agents is ever in the same group. Notice that the matching scheme in [21] does not satisfy (1). The same applies to [13], because the entire population regularly meets in the centralized market. See [3] for more details.

Those who are not idle either wish to consume or are able to produce, states that are assumed to be equally likely and mutually exclusive. Hence, on odd dates each agent faces idiosyncratic consumption risk, but patient agents are more active traders. On even dates everyone can produce and consume. Thus, while only $\rho\alpha_H + (1-\rho)\alpha_L$ of the population trades on odd dates, everyone trades on even dates and there is always an equal number of buyers and sellers in each market.

On each date, a single perishable good can be produced. Sellers can supply any positive amount of labor and can access a technology that transforms each unit of labor into one unit of consumption goods. As in [13], it is assumed that preferences on even dates are quasilinear $U(q_j) - x_j$ for every agent type j. The first term denotes the utility from $q_j \geq 0$ consumption and the second term is disutility from supplying $x_j \geq 0$ labor. Odd date preferences are as follows. A consumer of any type j derives utility $u(c_j)$ from consuming $c_j \geq 0$ of someone else's production. A producer of any type j suffers y_j disutility from supplying $y_j \geq 0$ labor to produce y_j goods. The functions u and U satisfy the standard Inada conditions and $u(0) = U(0) = 0$. Also, let c^* be the solution to $u'(c^*) = 1$ and let q^* be the solution to $U'(q^*) = 1$.

As is standard in monetary models, we assume limited enforcement and limited commitment. This simply means that agents have exclusive rights to their assets and endowments, and their actions cannot be subject to retribution, so that trading plans must be compatible with individual incentives. This together with the market frictions assumed above implies an essential role for money (see [8], [11]) since on odd dates trade is *quid pro quo* but consumers cannot produce. Thus, a consumption shock on odd dates corresponds to a need for currency.

2.3 Assets and policy tools

We assume a government exists that is the sole supplier of fiat currency, of which there is an initial stock $\bar{M} > 0$. We let the money stock evolve deterministically at gross rate π by means of lump-sum cash transfers at the beginning of even dates. The government also buys and sells nominal pure-discount bonds having two distinctive features similar to U.S. Savings Bonds. First, they are non-marketable claims to currency redeemable only by their owner.[4] To formalize it, we assume bonds are intangible and non-transferable, ownership of which is recorded by the government. The government knows neither identity, nor type, nor trading history of agents and can credibly commit to repayment since it can

[4] U.S. Savings Bonds can be traded only by issuing and paying agents authorized by the U.S. Treasury.

print currency.[5]

Second, bonds are illiquid in that early redemption may come at a cost and cannot involve fractions of the asset. As we will see later, this insures agents do not make a speculative use of bonds and use them to self-insure against consumption risk. Specifically, bonds are issued at the end of even dates at price $p_A \leq 1$ and mature the following cycle at the beginning of even dates paying off one unit of money. Unmatured bonds can be redeemed for $p_\ell \leq 1$ money by traders at the beginning of odd dates after individual shocks have been realized. Hence, p_ℓ naturally captures the notion of illiquidity as the cost of immediate execution of a trade: $1 - p_\ell$ is lost to convert a bond into cash at the start of a trading cycle.

3 Efficient allocations

We start by discussing the allocation selected by a benevolent planner who maximizes the agents' lifetime utilities treating agents identically. Because the physical environment displays a set of obstacles to economic interactions, two cases arise that differ in the constraints faced by the planner.

In the first case, the planner is unconstrained by the physical and informational limitations assumed to be in existence. In particular, this means that the planner can observe types and identities and commit to and enforce a consumption plan on the initial date. The solution to this unconstrained planning problem delivers the 'first-best' allocation and, because agents differ in their discount factors, implies non-stationary type-specific consumption paths. Indeed, impatient agents front-load consumption, while patient agents do the opposite (for a discussion, see [9]). It is obvious that such an allocation cannot be decentralized, as the frictions assumed prevent borrowing and lending.

In the second case, the planner is subject to the same physical and informational constraints faced by the agents and therefore can observe neither types nor identities. So, the planner can just propose a type-independent consumption plan in each trading cycle

[5] The government is an inanimate entity that interacts with agents in each date. Its role is to issue and redeem bonds and to implement lump-sum monetary transfers. This typical abstraction is used to introduce policy in monetary models with spatial separation (e.g., [21] or [12]), random matching and divisible money (e.g., [13], [17], [18]) as well as other models (e.g., [5], [10]). Note that we do not assume special powers for the government as it does not need to know agents' identities to sell and redeem bonds. Agents can select an arbitrary ID to buy bonds and use it to redeem them. The government's record-keeping technology can be as simple as making the agent sign his arbitrary ID on the money spent to buy the bond. Signatures are compared at redemption and then erased. This destroys all records and ensures the payoff can be claimed only by the agent who purchased the bond.

without having the ability to transfer resources across agents over time. Equivalently, the planner maximizes expected utility of the arbitrary agent on each date. The solution to such a sequence of static problems is called a constrained-efficient allocation. It corresponds to the outcome arising in each market if traders can coordinate and commit to a plan ex-ante, before realizing their individual shocks. Since agents have identical preferences ex-ante, then the constrained-efficient allocation maximizes trade surplus in each market. Precisely, since the marginal rate of transformation of labor into consumption is one, then marginal consumption utility must simply equal marginal production disutility in each market, i.e., on odd dates $u'(c_H) = u'(c_L) = 1$ and on even dates $U'(q_H) = U'(q_L) = 1$. Thus, the constrained-efficient consumption is stationary across trading cycles and is defined uniquely by $c_j = c^*$ and $q_j = q^*$ for each type $j = H, L$ (details in the Appendix). This allocation is the relevant benchmark for our purposes, and we simply call it efficient, when no confusion arises. Indeed, a sensible notion of efficiency must take into account existing physical and informational limitations.

4 Stationary monetary allocations

Now, we investigate if the constrained-efficient allocation can be decentralized by means of monetary exchange on competitive markets. Thus, we focus on stationary monetary outcomes such that consumption is unaffected by the trading cycle and nominal prices evolve so that the money stock has constant real value.[6]

Due to stationarity, we simplify notation omitting t subscripts and use a prime superscript to identify next-cycle variables, when necessary. Accordingly, we let p_1 and p_2 denote the nominal price of goods on odd and even dates of an arbitrary trading cycle t. In addition, we find it convenient to work with real variables normalizing all nominal variables by p_2, so that market one trades occur at real price $p = \frac{p_1}{p_2}$. In this manner, the timing of events during cycle t for an agent of type j can be discussed as follows.

The arbitrary agent of type j enters cycle t with portfolio $\omega_j = (m_j, a_j)$ listing real holdings of money $m_j \geq 0$ and bonds $a_j \geq 0$, carried over from the preceding cycle. Then, the idiosyncratic trading shock is realized and the agent can choose to liquidate bonds. Subsequently, trade occurs and after market one closes the agent enters market two on the even date with portfolio $\omega_{j,k} = (m_{j,k}, a_{j,k})$ where $k = n, s, b$ denotes the trading shock

[6] Any monetary outcome that is non-stationary must be constrained inefficient. Indeed, suppose it is not. Then, someone should be able to consume more or work less, on some date, without affecting anyone. But this is impossible since the constrained-efficient allocation satisfies aggregate feasibility constraints with equality. Thus, every non-stationary monetary allocation must be constrained inefficient.

experienced in market one. Here, n identifies an agent who was idle, while b and s identify a buyer and a seller, respectively.

To clarify how portfolios $\omega_{j,k}$ evolve, notice that $a_{j,k} \in \{0, a_j\}$ since by assumption early liquidation must involve the entire stock of bonds, where $a_{j,k} = 0$ corresponds to liquidation. We work under the conjecture that early liquidation occurs only if cash is needed to buy consumption, i.e., $a_{j,s} = a_{j,n} = a_j$ (the proof is provided later). Thus, if we let c_j denote consumption and y_j production of type j on an odd date, money holdings evolve within the cycle as follows:

$$\begin{aligned} m_{j,b} &= m_j + p_\ell(a_j - a_{j,b}) - pc_j \\ m_{j,s} &= m_j + py_j \\ m_{j,n} &= m_j. \end{aligned} \tag{2}$$

That is, buyers deplete balances by pc_j while sellers increase them by py_j. Cash left over is used to trade in market two, when the real price is one, q_j is consumption bought and $x_{j,k}$ is production sold by an agent who experienced shock k (the notation q_j is without loss in generality, as we later show). In market two, agents also choose their savings. Let $m'_j \geq 0$ and $a'_j \geq 0$ denote the real values of the agent's money and bond holdings at the start of next trading cycle (multiply by p'_2 or $\frac{p'_2}{p_2}$ to get the current nominal or real values).

In a stationary economy real asset holdings must be constant, i.e., $(m'_j, a'_j) = (m_j, a_j)$. If M is cash at the start of a cycle and $M' = \pi M$ is cash available in market two, then

$$\frac{p'_2}{p_2} = \frac{M'}{M} = \pi, \tag{3}$$

i.e., in a stationary economy aggregate real balances are constant so the inflation rate equals the rate of growth of money. This rate is controlled by means of per capita lump-sum transfers τ in market two, so the government budget constraint is

$$\begin{aligned} \tau + \rho a_H + (1-\rho)a_L = \quad & (1-p_\ell)[\rho\tfrac{\alpha_H(a_H - a_{H,b})}{2} + (1-\rho)\tfrac{\alpha_L(a_L - a_{L,b})}{2}] \\ & + p_A\pi[\rho a_H + (1-\rho)a_L] + [\rho m_H + (1-\rho)m_L](\pi - 1). \end{aligned} \tag{4}$$

The left hand side collects outflows of real balances due to transfers and bonds' redemption. The right hand side accounts for inflows due to early redemption fees and bond sales (spend $p_A\pi$ today to have one real bond in the next trading cycle).

Stationarity also implies that in each trading cycle

$$\begin{aligned} \tfrac{M\pi}{p_2} = \quad & \tau + \rho[m_H + a_H - \tfrac{\alpha_H}{2}(1-p_\ell)(a_H - a_{H,b})] \\ & + (1-\rho)[m_L + a_L - \tfrac{\alpha_L}{2}(1-p_\ell)(a_L - a_{L,b})], \end{aligned}$$

i.e., real balances available in market two must equal transfers τ, plus initial money holdings m_j, plus net money inflows from redemption of bonds $a_j - \frac{\alpha_j}{2}(1-p_\ell)(a_j - a_{j,b})$.

4.1 Even dates

Given the recursive nature of the problem, we use a dynamic programming approach to describe the problem faced by the representative agent of type j at any date. We let $V_j(\omega_j)$ be the expected lifetime utility of this agent when he starts the trading cycle with ω_j, before trading shocks are realized. We also let $W_j(\omega_{j,k})$ be the expected lifetime utility from entering an even date with $\omega_{j,k}$.[7]

The agent's problem at the start of an even date is:

$$\begin{aligned} W_j(\omega_{j,k}) = & \max_{q_j, x_{j,k}, \omega'_j \geq 0} \left[U(q_j) - x_{j,k} + \beta_j V_j(\omega'_j)\right] \\ \text{s.t.} \quad & x_{j,k} = q_j + \pi(m'_j + p_A a'_j) - (m_{j,k} + a_{j,k} + \tau) \end{aligned} \tag{5}$$

The resources available to the agent in market two partly depend on the realization of the trading shock k, as he has $m_{j,k}$ real balances carried over from market one and $a_{j,k}$ receipts from matured bonds. Other resources are $x_{j,k}$ receipts from current sales of goods and the lump-sum real balances transfer τ.[8] These resources can be used to finance current consumption q_j, to buy $\pi a'_j$ bonds at price p_A, or simply to carry $\pi m'_j$ real money balances into tomorrow's markets (short-selling is not allowed). The factor $\pi = \frac{p'_2}{p_2}$ multiplies a'_j and m'_j because the budget constraint lists current real values.

Notice that the composition of savings depends on expected rates of return on cash and bonds since agents can save only with money or bonds and cannot lend to each other. In particular, the most patient cannot lend to the less patient because the structure of the environment severs all future direct and indirect links among current trade partners.

Substituting $x_{j,k}$ from the real resource constraint, (5) is rearranged as:

$$W_j(\omega_{j,k}) = \max_{q_j, \omega'_j \geq 0} \left\{U(q_j) - q_j - \pi(m'_j + p_A a'_j) + m_{j,k} + a_{j,k} + \tau + \beta_j V_j(\omega'_j)\right\} \tag{6}$$

It follows that in a stationary monetary economy

$$\frac{\partial W_j(\omega_{j,k})}{\partial m_{j,k}} = \frac{\partial W_j(\omega_{j,k})}{\partial a_{j,k}} = 1 \quad \text{for } j = H, L. \tag{7}$$

The result hinges on the linearity of production disutility and the use of competitive pricing, linear in the quantity sold. It follows that the marginal value of any asset must

[7] It can be proved that these value functions exist and are unique using Banach's fixed point theorem.

[8] Notice that $x_{j,k} \geq 0$ so we must verify that this is true for all k in equilibrium.

simply reflect the price of real balances, which is one. The economic implication is the marginal valuations of real balances and bonds in market two are identical and do not hinge on the agent's type j, wealth $\omega_{j,k}$ or trade shock k.

The model allows us to disentangle the agents' portfolio choices from their trading histories since

$$W_j(\omega_{j,k}) = \quad W_j(0) + m_{j,k} + a_{j,k}, \tag{8}$$

i.e., the agent's expected value from having portfolio $\omega_{j,k}$ at the start of an even date is the expected value from having no wealth $W_j(0)$, letting $\omega_j = (0,0) \equiv 0$, plus the current real value of wealth $m_{j,k}+a_{j,k}$. This implies agents of identical type exit an even date with identical portfolios ω'_j, independent of their trading histories, much as in [13]. However, different types might choose different portfolios, as we demonstrate next.

Start by observing that by (6) we have

$$q_j = q^* \quad \text{for } j = H, L. \tag{9}$$

That is, everyone consumes the same amount q^* independent of his asset holdings. The reason is agents in market two can produce any amount at constant marginal cost. Thus goods market clearing on even dates requires

$$\begin{aligned} q^* = \quad & (1-\rho)[\tfrac{\alpha_L}{2}(x_{L,s} + x_{L,b}) + (1-\alpha_L)x_{L,n}] \\ & +\rho[\tfrac{\alpha_H}{2}(x_{H,s} + x_{H,b}) + (1-\alpha_H)x_{H,n}]. \end{aligned} \tag{10}$$

Given (9) we write

$$W_j(\omega_{j,k}) = U(q^*) - q^* + m_{j,k} + a_{j,k} + \tau + \max_{\omega'_j \geq 0} [-\pi(m'_j + p_A a'_j) + \beta_j V_j(\omega'_j)].$$

The central implication is the agents' lifetime utility and the efficiency of the decentralized monetary solution will hinge on the trades that take place in market one. Since these depend on the availability and the liquidity of financial resources, then we expect that efficiency will impinge on the agents' portfolio decisions ω'_j. This is studied next.

Let $\lambda^a_j \geq 0$ and $\lambda^m_j \geq 0$ denote the Kuhn-Tucker multipliers on the non-negativity constraint on bonds and money holdings. The first order conditions from the optimal portfolio choice are

$$\begin{aligned} \pi = \beta_j \frac{\partial V_j(\omega'_j)}{\partial m'_j} + \lambda^m_j \quad & \Rightarrow \quad 1 \geq \frac{\beta_j}{\pi} \times \frac{\partial V_j(\omega'_j)}{\partial m'_j} \quad (= \text{if } m'_j > 0) \\ p_A \pi = \beta_j \frac{\partial V_j(\omega'_j)}{\partial a'_j} + \lambda^a_j \quad & \Rightarrow \quad p_A \geq \frac{\beta_j}{\pi} \times \frac{\partial V_j(\omega'_j)}{\partial a'_j} \quad (= \text{if } a'_j > 0). \end{aligned} \tag{11}$$

Recalling that one unit of real balances buys one unit of consumption, the left hand sides of the expressions simply define the marginal cost of assets. The right hand sides define the expected marginal benefit from holding the asset, either money or bonds, discounted according to time preferences and inflation. The weak inequalities reflect the optimality requirement that the expected benefit from buying an asset cannot surpass its current cost. It is important to realize that the benefit from holding an asset in this model hinges not only on the asset's yield but also on its illiquidity, i.e., the loss from converting it into immediate cash. Indeed, since agents differ in their frequency of consumption shocks, it follows that the expected benefit of holding any asset will generally differ across types j. To see how, we must study trades on odd dates.

4.2 Odd dates

The problem faced by an arbitrary agent of type j who starts a trading cycle with ω_j is:

$$\begin{aligned} V_j(\omega_j) = \quad & \tfrac{\alpha_j}{2} \max_{c_j, a_{j,b}} [u(c_j) + W_j(\omega_{j,b})] + \tfrac{\alpha_j}{2} \max_{y_j, a_{j,s}} [W_j(\omega_{j,s}) - y_j] \\ & +(1-\alpha_j) \max_{a_{j,n}} W_j(\omega_{j,n}) \qquad (12) \\ \text{s.t.} \quad & pc_j \leq m_j + p_\ell(a_j - a_{j,b}) \ \text{ and } a_{j,k} \in \{0, a_j\} \end{aligned}$$

The agent maximizes his expected lifetime utility by choosing consumption $c_j \geq 0$ as a buyer and production $y_j \geq 0$ as a seller. Agents can also choose to liquidate their bonds early by solving a discrete choice problem. Note that such a choice is relevant only for buyers because $W_j(\omega_{j,k})|_{a_{j,k}=0} \leq W_j(\omega_{j,k})|_{a_{j,k}=a_j}$ for all k since $p_\ell \leq 1$ (see (8)). It follows that it is optimal to set:

$$a_{j,s} = a_{j,n} = a_j \qquad (13)$$

From (12) we see that consumption c_j hinges on the available cash m_j, the liquidation value $p_\ell a_j$ of bonds, and the relative price p. We start by discussing the latter. To do so, consider a seller's problem:

$$\max_{y_j} [\, W_j(\omega_{j,s}) - y_j]$$

Given (2), (8), and (13), the seller's problem is linear in y_j since

$$W_j(\omega_{j,s}) = W_j(0) + m_j + py_j + a_j.$$

Hence, the optimal choice y_j is a correspondence. Precisely, positive and finite work effort can arise only if prices in market one and two are identical, i.e.,

$$p = 1. \qquad (14)$$

The reason is that sellers have unit marginal disutility from production in any market. Income raised in market one at price p_1 can be spent in market two at price p_2. Thus, market one sellers work infinite amounts if $\frac{p_1}{p_2} > 1$, or not at all if $\frac{p_1}{p_2} < 1$. When $p = 1$ sellers are indifferent to supplying any amount. Thus, in a stationary monetary economy (14) must hold in which case, without loss in generality, we work under the conjecture that sellers serve an equal share of aggregate demand. Goods market clearing then implies:

$$y_j = y = \frac{\rho\alpha_H c_H + (1-\rho)\alpha_L c_L}{\rho\alpha_H + (1-\rho)\alpha_L} \quad \text{for} \quad j = H, L \tag{15}$$

Now we determine c_j. Given some choice $a_{j,b}$, a buyer's problem is:

$$\begin{aligned} \max_{c_j \geq 0} \quad & u(c_j) + W_j(\omega_{j,b}) \\ \text{s.t.} \quad & c_j \leq m_j + p_\ell(a_j - a_{j,b}) \end{aligned}$$

Let $\lambda_j \geq 0$ be the Kuhn-Tucker multiplier on the resource constraint. Since $u'(0) = \infty$ we have $c_j > 0$. Recall from (2) that $m_{j,b}$ depends on c_j. Hence, the first-order condition is

$$u'(c_j) + \frac{\partial W_j(\omega_{j,b})}{\partial m_{j,b}} \frac{\partial m_{j,b}}{\partial c_j} - \lambda_j = 0.$$

Using (7), $\frac{\partial m_{j,b}}{\partial c_j} = -p$ from (2), and (14), we have $u'(c_j) = 1 + \lambda_j$.

If $\lambda_j = 0$, then $c_j = c^*$ since $u'(c_j) = 1$. Otherwise, $c_j < c^*$. Thus we have:

$$c_j = \min(m_j + p_\ell(a_j - a_{j,b}), c^*) \tag{16}$$

If we define $m^* = c^*$, then liquidating bonds before maturity makes sense only if $m_j < m^*$. Hence, we say that a buyer of type j is cash constrained if $m_j + p_\ell(a_j - a_{j,b}) < m^*$. As for the liquidation choice, in what follows we focus on outcomes where $a_{j,b} = 0$ is optimal. To better understand when early liquidation is optimal for a buyer, we need to study savings decisions.

4.3 Savings decisions

To find the optimal portfolio of an agent we must calculate the expected marginal values of each asset, $\frac{\partial V_j(\omega_j)}{\partial m_j}$ and $\frac{\partial V_j(\omega_j)}{\partial a_j}$. To do so, use (2) and (8) in $V_j(\omega_j)$. Given a liquidation choice $a_{j,b} \in \{0, a_j\}$, we have

$$V_j(\omega_j) = m_j + a_j + \frac{\alpha_j}{2}[u(c_j) - c_j - (a_j - a_{j,b})(1 - p_\ell)] + W_j(0), \tag{17}$$

where c_j satisfies (16) and m_j satisfies (2).

Expression (17) tells us that the expected lifetime utility at the start of an arbitrary trading cycle depends on the agent's real wealth $m_j + a_j$ and two additional elements. First, the expected utility from trade in market one. With probability $\frac{\alpha_j}{2}$ the agent spends c_j of his wealth on consumption and gets net utility $u(c_j) - c_j$. If a buyer liquidates bonds, we have $a_{j,b} = 0$ and must account for the capital loss $a_j(1 - p_\ell)$. Second, there is the continuation payoff $W_j(0)$. Hence, we have

$$\frac{\partial V_j(\omega_j)}{\partial m_j} = 1 + \frac{\alpha_j}{2}[u'(c_j) - 1]\frac{\partial c_j}{\partial m_j},$$

where $\frac{\partial c_j}{\partial m_j} = 1$ if the agent is cash constrained and zero otherwise, from (16). It follows that:

$$\frac{\partial V_j(\omega_j)}{\partial m_j} = \begin{cases} 1 + \frac{\alpha_j}{2}[u'(c_j) - 1] & \text{if } m_j + p_\ell(a_j - a_{j,b}) < m^* \\ 1 & \text{otherwise.} \end{cases} \tag{18}$$

Furthermore,

$$\frac{\partial V_j(\omega_j)}{\partial a_j} = 1 + \frac{\alpha_j}{2}[u'(c_j) - 1]\frac{\partial c_j}{\partial a_j} - \frac{\alpha_j}{2}(1 - p_\ell)(1 - \frac{\partial a_{j,b}}{\partial a_j}),$$

so the bond's marginal value depends on its intended use. If the agent uses bonds to finance market one consumption, then $\frac{\partial c_j}{\partial a_j} = p_\ell(1 - \frac{\partial a_{j,b}}{\partial a_j})$, $\frac{\partial a_{j,b}}{\partial a_j} = 0$ and $\frac{\partial c_j}{\partial a_j} = p_\ell$. Instead, if bonds are never liquidated, i.e., $a_{j,b} = a_j$, then we have $\frac{\partial a_{j,b}}{\partial a_j} = 1$ and $\frac{\partial c_j}{\partial a_j} = 0$. It follows that:

$$\frac{\partial V_j(\omega_j)}{\partial a_j} = \begin{cases} 1 + \frac{\alpha_j}{2}[p_\ell u'(c_j) - 1] & \text{if} \quad a_{j,b} = 0 \quad \text{and} \quad m_j + p_\ell a_j < m^* \\ 1 + \frac{\alpha_j}{2}(p_\ell - 1) & \text{if} \quad a_{j,b} = 0 \quad \text{and} \quad m_j + p_\ell a_j \geq m^* \\ 1 & \text{if} \quad a_{j,b} = a_j \end{cases} \tag{19}$$

The bond's marginal value always reflects the price of real balances, which is equal to one. If bonds are liquidated to finance consumption (first line) this value is adjusted by $\frac{\alpha_j}{2}[p_\ell u'(c_j) - 1]$, i.e., the expected value from having p_ℓ additional cash ready to spend. This term is likely to be positive when cash constraints are severe since there is a large marginal benefit from buying extra consumption. Of course, if the agent is not cash constrained (second line), the early cashing of bonds generates a capital loss $1 - p_\ell$ and no benefit. This loss is absent if bonds are not liquidated (third line).

The central observation is that illiquid bonds are valued dissimilarly in the economy. Indeed, the heterogeneity in consumption risk, governed by α_j, induces heterogeneity in expected rates of return. To see why, observe that the gross nominal yield for money is equal to one and for bonds is

$$1 + i = \frac{1}{p_A}.$$

Now, consider nominal rates of return. The return on money is always the yield but the return on illiquid bonds is the yield only if they are held until maturity. Indeed, early redemption implies a capital loss and so the expected nominal rate of return is type dependent, as for a type j we have

$$\frac{1}{p_A}[1 - \frac{\alpha_j}{2}(1 - p_\ell)].$$

In short, $1 - p_\ell$ acts as a proportional tax on liquidation and so it affects expected returns dissimilarly across agents who have unequal consumption frequencies.

Of course, if assets finance consumption in market one we must also account for marginal consumption utility. Using (3), (11) and (18)-(19), the agents' optimal portfolio choices must satisfy:

$$\begin{aligned} 1 &\geq \tfrac{\beta_j}{\pi}\left\{1 + \tfrac{\alpha_j}{2}[u'(c_j) - 1]\right\} && (= \text{if } m_j > 0) \\ 1 &\geq \tfrac{\beta_j}{\pi p_A}\left\{1 + \tfrac{\alpha_j}{2}[p_\ell u'(c_j) - 1]\right\} && (= \text{if } a_j > 0 \text{ and } a_{j,b} = 0) \\ 1 &\geq \tfrac{\beta_j}{\pi p_A} && (= \text{if } a_j > 0 \text{ and } a_{j,b} = a_j) \end{aligned} \tag{20}$$

i.e., the marginal cost must be no less than the discounted expected marginal benefit.

The expressions in (20) indicate that the composition of portfolios depends on the real interest rate and on the bonds' illiquidity. For instance, bonds are not superior to money if $i = 0$. What is crucial, however, is that the policy parameters i and π affect everyone's returns identically, but the bonds' illiquidity does not. To see why this is the case, we look in detail at the different expressions in (20).

The first line refers to the choice of real balances, the second and the third lines refer to the choice of bonds under early liquidation or not. The first line tells us that in choosing real balances the agent evaluates three components. The first and the second are standard: the discount factor β_j and the real yield on cash $\frac{1}{\pi}$. The third component, which is non-standard, is $\frac{\alpha_j}{2}[u'(c_j) - 1]$, a non-negative value since $u'(c_j) \geq 1$ from (16). This can be interpreted as the expected liquidity premium from having cash available in market one and it arises because money is needed to trade in that market. This premium grows with the severity of the cash constraint and the likelihood of a consumption shock.

A similar interpretation applies to the choice of bonds, with two key differences. First, bonds have a possibly higher real yield $\frac{1}{\pi p_A}$. Second, if illiquid bonds are used to buy consumption (second line) they have a smaller liquidity premium $\frac{\alpha_j}{2}[p_\ell u'(c_j) - 1]$ relative to cash. Agents consider this trade-off between bonds' illiquidity and superior return in choosing their portfolios. The third line in (20) instead must be considered when buyers

hold bonds until maturity, which is relevant only if $m_j > 0$. Indeed, if $m_j = 0$, then setting $a_{j,b} = a_j$ violates the first line in (20) because $c_j = 0$ implies $u'(0) = \infty$. In any event, in what follows we concentrate on outcomes where $a_{j,b} = 0$, which can be optimal only if the agent is cash constrained.[9]

To conclude this section, we discuss money market clearing in a stationary outcome. Note that, unlike some models of money, our agents are not forced to insure against consumption shocks solely with money as they can also use bonds. Indeed, (5) indicates that real savings of type j are $\pi(m'_j + p_A a'_j)$ with $(m'_j, a'_j) = (m_j, a_j)$ in a stationary outcome. Thus, the money market clears if at the end of each trading cycle aggregate real savings equal the real money stock $\frac{M\pi}{p_2}$, i.e.,

$$\frac{M\pi}{p_2} = \rho\pi(m_H + p_A a_H) + (1-\rho)\pi(m_L + p_A a_L). \tag{21}$$

We can now provide a definition of equilibrium.

Definition 1 *Given an initial money stock $\bar{M} > 0$ and a government policy as specified by (π, τ, p_A, p_ℓ), a competitive stationary monetary equilibrium is a constant list of real quantities $(c_j, y_j, q_j, x_{j,k}, m_j, a_j, a_{j,k})$ and cycle-dependent prices $(p_{1,t}, p_{2,t})$ that solve the agents' problems (5) and (12), satisfy (14), the government budget constraint (4) and market clearing (10), (15), (21).*

Summing up, policy shapes economic outcomes in our model by affecting the expected returns of the different assets, which in turn influence agents' portfolio choices. These choices determine the cash available to each agent type in market one. Ultimately, this affects the agent's ability to consume and therefore the efficiency of the allocation.

5 The failure of the Friedman rule

A natural question, at this point, is whether the constrained optimum can be achieved simply by eliminating the opportunity cost of holding money. Indeed, as noted in the introduction, a common result in monetary theory is that setting $i = 0$, known as the

[9] If $m_j > 0$, use (17) to note that a constrained buyer's liquidation problem is $\max_{a_{j,b} \in \{0, a_j\}} [u(c_j) - c_j - (a_j - a_{j,b})(1 - p_\ell)]$. Thus, $a_{j,b} = 0$ is optimal if $u(m_j + p_\ell a_j) - u(m_j) > a_j$. It should be now clear why not permitting liquidation of fractions of bond holdings is important to achieve efficiency. If buyers can liquidate a fraction θ of bonds then in the problem above $a_{j,b} = (1-\theta)a_j$ and so $\theta < 1$ is optimal. Intuitively, keeping some bonds and buying slightly below c^* generates a loss in consumption utility that is smaller than the gain from avoiding liquidation fees, $u'(c^*) \leq \frac{1}{p_\ell}$. Prohibiting partial liquidation implies that in the optimum bonds are held to insure against consumption risk and not for speculative purposes.

Friedman rule, allows to achieve efficiency. To answer this question we move in steps and start by determining the highest return on money that is consistent with equilibrium.

Lemma 2 *In any stationary monetary equilibrium we must have* $\pi \geq \beta_H$.

Proof. By way of contradiction, suppose a monetary equilibrium exists with $\pi < \beta_H$. Consider $j = H$ in the first line of (20). We need $\pi \geq \beta_H + \beta_H \frac{\alpha_H}{2}[u'(c_H) - 1] \geq \beta_H$. This is in contradiction with $\pi < \beta_H$.■

The lesson here is that the rate of return on money $\frac{1}{\pi}$ cannot be excessive in a stationary monetary equilibrium. Precisely, the upper bound for the return on money corresponds to the lowest pure rate of time preference $\frac{1}{\beta_H}$, i.e., the shadow interest rate. Intuitively, if $\frac{1}{\pi} > \frac{1}{\beta_H}$ then cash pays such a good return that a patient agent would want to keep accumulating money, which cannot be a stationary equilibrium.[10]

The implication is policy makers are constrained in their ability to give cash a return that is sufficiently attractive for everyone. Thus, inefficiencies are to be expected when saving can only take the form of cash. To formalize this intuition we remove the incentives to save with bonds by setting $i = 0$, running the Friedman rule.[11] Thus, we now ask the question: is there any $\pi \geq \beta_H$ that sustains the constrained-efficient allocation when $i = 0$?

Lemma 3 *Consider* $i = 0$ *and* $\pi > \beta_H$. *A unique stationary monetary equilibrium exists and money holdings are heterogeneous,* $0 < m_L < m_H < m^*$. *As* $\pi \to \beta_H$ *we have* $c_H \to c^*$ *but* $c_L < c^*$. *The allocation is inefficient for all* $\pi \geq \beta_H$.

Proof. Let $p_A = 1$ so $i = 0$. From (20) we get:

$$\begin{aligned} \pi \geq \quad & \beta_j \left\{1 + \tfrac{\alpha_j}{2}[u'(c_j) - 1]\right\} \qquad (= \text{if } m_j > 0) \\ \pi \geq \quad & \beta_j \left\{1 + \tfrac{\alpha_j}{2}[u'(c_j)p_\ell - 1]\right\} \quad (= \text{if } a_j > 0 \text{ and } a_{j,b} = 0) \end{aligned} \tag{22}$$

Bonds and money are equivalent assets only if $p_\ell = 1$, and bonds are inferior otherwise. Thus, suppose $p_\ell = 1$ and simply consider money.

[10] This is in line with the finding in [4] where heterogeneous agents trade a fixed stock of capital, but there is no money. There, too, the steady-state equilibrium rate of return on capital cannot exceed the lowest rate of time preference.

[11] In a representative agent model setting $i = 0$ requires a deflation equal to the real interest rate, i.e. the unique discount factor. In our model we have more than one discount factor, but we have established that π cannot fall below β_H.

Note that $\pi \geq \beta_H$ is necessary from Lemma 2. From (22), if $m_H > 0$ then

$$\pi = \beta_H \left\{1 + \tfrac{\alpha_H}{2}[u'(c_H) - 1]\right\}.$$

If $\pi > \beta_H$ then $c_H < c^*$ and $m_H < m^*$. As $\pi \to^+ \beta_H$ then $c_H \to c^*$ and $m_H \to m^*$. Thus, suppose $\pi = \beta_H$. Now, $m_L > 0$ implies

$$\pi = \beta_L \left\{1 + \tfrac{\alpha_L}{2}[u'(c_L) - 1]\right\} = \beta_H.$$

Since $\beta_L < \beta_H$, it follows that $c_L < c^*$ and $m_L < m^*$. Hence, if $i = 0$ then a unique stationary monetary equilibrium exists in which $0 < m_L < m_H < m^*$ and $0 < c_L < c_H < c^*$. In equilibrium $\lim_{\pi \to^+ \beta_H} m_H = m^*$, so $\lim_{\pi \to^+ \beta_H} c_H = c^*$; also, $\frac{\partial c_L}{\partial \pi} < 0$. Thus, the Friedman rule cannot achieve the efficient allocation. Existence easily follows from inspection of the individual optimality and market clearing conditions.■

What is the intuition? When $i = 0$ effectively we have a model where agents insure against consumption shocks only with money. Due to discounting disparities, equilibrium returns must obey the optimality condition $\pi \geq \beta_H$, so the more impatient tend to under-insure. This leaves them cash constrained in market one, which creates an inefficiency. Of course, letting $\pi \to \beta_H$ allows the more patient agents to perfectly insure, since $m_H \to m^*$ and $c_H \to c^*$.

This result seems quite robust. The Friedman rule should fail to sustain perfect consumption insurance when money has an explicit transactions role and agents price future consumption unequally. In fact, lowering the return on bonds to that of money by setting $i = 0$ seems to be the source of the problem. It eliminates the opportunity cost of holding money, which is good, but it fails to provide adequate incentives for everyone to save enough, which is bad, since $\pi \geq \beta_H > \beta_L$. Thus, we next consider a policy where $i > 0$. Before doing so, several remarks are in order.

First, we emphasize that the Friedman rule does not fail to sustain the constrained-efficient allocation just because bonds are illiquid. Setting $p_\ell = 1$ and $i = 0$ simply makes money and bonds indistinguishable financial instruments. Second, the result does not hinge on the mere existence of some arbitrary heterogeneity element that gives different agents incentives to hold unequal money balances. In fact, the Friedman rule can be quite effective in eliminating equilibrium heterogeneity in real balances when agents differ in aspects other than time preferences.

To see why, consider for example an economy in which $\beta_H = \beta_L = \beta$. However, retain the assumption of disparities in trade shocks, $\alpha_H > \alpha_L$. Now, set $i = 0$ so from Lemma

3 a unique monetary equilibrium exists for $\pi > \beta$. Specifically, we have

$$\pi = \beta\left\{1 + \tfrac{\alpha_H}{2}[u'(c_H) - 1]\right\} = \beta\left\{1 + \tfrac{\alpha_L}{2}[u'(c_L) - 1]\right\}.$$

Here, balances and consumption are heterogeneous, $c_L < c_H < c^*$ and $m_L < m_H < m^*$. Types L under-insure as they do not need cash as frequently as types H. The opposite occurs if $\alpha_H < \alpha_L$. However, as $\pi \to^+ \beta$ all real balances converge to m^* because agents become indifferent between having a dollar today or one tomorrow.[12] In this case, trade-frequency considerations do not enter saving decisions (see also [6]).

6 Using bonds to finance consumption

We now want to demonstrate that the efficient allocation can be sustained when the bonds' yield is positive. To simplify our task, we start by proving that such an allocation is inconsistent with agents holding money in their portfolios.

Lemma 4 *Consider a stationary monetary equilibrium in which $i > 0$. If $m_j > 0$ and $a_j > 0$, then $c_j < c^*$.*

Proof. Let $p_A < 1$ so $i > 0$. We want to show that an agent who holds bonds and money in equilibrium must be cash constrained, i.e., $c_j < c^*$. By means of contradiction, suppose $m_j > 0$ and $a_j > 0$ but $c_j = c^*$. There are two cases to consider: $m_j \geq m^*$ and $0 < m_j < m^*$.

Suppose $m_j \geq m^*$. Here bonds are not liquidated since the agent is not cash constrained. From (20), we have $\pi = \beta_j$, since $m_j > 0$. This implies $\pi < \frac{\beta_j}{p_A}$. But this is inconsistent with equilibrium (see the third line in (20)).

Now suppose $m_j \in (0, m^*)$. Using (20) and our hypothesis $c_j = c^*$, we must have the following in equilibrium:

$$\pi \begin{cases} = \beta_j \\ \geq \frac{\beta_j}{p_A}\left\{1 - \frac{\alpha_j}{2}(1 - p_\ell\right\} \\ \geq \frac{\beta_j}{p_A} \end{cases} \tag{23}$$

The first line in (23) follows from $m_j > 0$. We readily derive a contradiction since

$$\pi = \beta_j < \frac{\beta_j}{p_A} \tag{24}$$

[12] For $\pi = \beta$ a continuum of monetary equilibria exists. The reason is price indeterminacy, as any sequence of nominal prices which is consistent with $\pi = \beta$ is a monetary equilibrium.

whenever $i > 0$. This violates the third line in (23). The reason is that, given the return on money $\frac{1}{\beta_j}$, the agent would prefer to shift some money into bonds. That is, the agent would definitely prefer to consume slightly less than c^* in market one and hold some bonds until maturity. The trade-off is favorable since the bonds' yield is greater than one (that of money), while decreasing consumption marginally in market one has a small (second order) effect as the marginal utility is one at c^*. We conclude that if $m_j > 0$ and $a_j > 0$ in a stationary monetary equilibrium with $i > 0$, then $c_j < c^*$. This is true independent of whether bonds are held until maturity or not.■

When the yield on bonds is positive, agents who hold both money and bonds must be cash constrained. Again, this is an arbitrage argument. In fact, suppose the agent is not cash constrained in market one but is holding both money and bonds. Then, since bonds pay positive interest, the agent could achieve the same consumption level and accumulate wealth by holding more bonds.

This result suggests that perhaps the optimal policy should encourage agents to save only with bonds and not with money. The government could make cash an unattractive asset for saving purposes by selecting a sufficiently high π. Then, agents would possibly fully insure against consumption shocks using bonds and liquidate them when needed. In the words of Tobin, "Why not hold transactions balances in assets with higher yields than cash, shifting into cash only at the time an outlay must be made?" ([19], p. 241).

The problem is this might induce the most patient agents to buy infinite amounts of bonds. To see why, consider for a moment an economy in which $i > 0$ and agents save only with bonds. Suppose also that $c_j = c^*$. Then, from (20) we have that $m_j = 0$ for all j if $\pi > \beta_H$ and $a_j > a_{j,b} = 0$ if:

$$\pi = \frac{1}{p_A}\beta_j \left[1 - \frac{\alpha_j}{2}(1 - p_\ell)\right]$$

As $p_\ell \to 1$ we have that if $\pi = \frac{1}{p_A}\beta_L > \beta_H$ then $\pi < \frac{1}{p_A}\beta_H$. That is, the patient agents would want to buy infinite amounts of bonds, which is not an equilibrium. Our next objective is to prove that, in certain economies, such arbitrage opportunities can be avoided in a simple way: by making bonds sufficiently illiquid.

6.1 The optimal illiquidity of bonds

We start by presenting a condition that we need in proving that the constrained-efficient allocation can be sustained as an equilibrium:

$$\frac{\beta_L}{\beta_H} > \frac{2 - \alpha_H}{2 - \alpha_L} \tag{25}$$

Since $\alpha_H > \alpha_L$ then $\frac{2-\alpha_H}{2-\alpha_L} < 1$. Thus (25) simply limits the extent of disparities in individual discount factors. We now proceed to demonstrate that, under this condition, the efficient allocation can be achieved if bonds are sufficiently illiquid.

We start by reminding the reader that in such an allocation every buyer consumes c^* in market one, q^* in market two, and agents save only with bonds. Precisely, everyone enters market one with real portfolio $(m, a) = (0, a^*)$, where $a^* = \frac{c^*}{p_\ell}$. Then, buyers liquidate all of their bonds to purchase c^* goods, sellers produce $y = c^*$ goods earning $m^* = c^*$ real balances, and the inactive agents do nothing. Thus, at the start of market two buyers have neither cash nor bonds, whereas idle agents and sellers have, respectively, a^* and $m^* + a^*$ real balances available (as bonds mature). In market two, everyone consumes q^*, receives the real balance transfer τ and purchases πa^* bonds at price p_A. To accomplish this, agents who bought in market one produce $x_b = q^* + p_A \pi a^* - \tau$, those idle in market one produce $x_n = x_b - a^*$, while market one sellers now produce $x_s = x_b - (m^* + a^*) \geq 0$.

Proposition 5 *Consider economies in which (25) is satisfied. If*

$$\begin{aligned} &\pi > \beta_H \\ &p_\ell = 1 - \tfrac{2(\beta_H - \beta_L)}{\alpha_H \beta_H - \alpha_L \beta_L} \\ &p_A = \tfrac{\beta_H}{\pi}\left[1 - \tfrac{\alpha_H}{2}(1 - p_\ell)\right], \end{aligned} \tag{26}$$

then $c_j = c^*$ for all j is a stationary monetary equilibrium. Here $p_A, p_\ell \in (0, 1)$.

Proof. We use a constructive proof. First we conjecture that the allocation is efficient, and then we prove that the expressions in (26) support existence of an equilibrium in which $c_j = c^*$ for all j.

Conjecture $c_j = c^*$ for some $i > 0$. Applying Lemma 4 we must insure that agents' savings consist only of bonds. Thus, we need $m_j = 0$ for all j, which requires $\pi > \beta_H$ from (20). Thus, let $\pi > \beta_H$. Since bonds must be liquidated to finance c^* consumption we also need $a_j > a_{j,b} = 0$ for all j, which requires

$$\pi = \tfrac{\beta_j}{p_A}\left[1 - \tfrac{\alpha_j}{2}(1 - p_\ell)\right], \tag{27}$$

from the second line in (20).

Consider $j = H$. Then (27) holds if

$$p_A = \tfrac{1}{\pi}\beta_H\left[1 - \tfrac{\alpha_H}{2}(1 - p_\ell)\right] \quad \equiv \tfrac{1}{\pi}h(p_\ell), \tag{28}$$

which defines uniquely p_A as a function of π. Since we are assuming $\pi > \beta_H$ then $p_A < 1 - \frac{\alpha_H}{2}(1 - p_\ell)$, i.e., $i > 0$.

Now consider $j = L$. Equation (27) holds when, using p_A from (28), the following equality is satisfied:

$$\beta_H \left[1 - \tfrac{\alpha_H}{2}(1 - p_\ell)\right] = \beta_L \left[1 - \tfrac{\alpha_L}{2}(1 - p_\ell)\right] \quad \Rightarrow \quad \tfrac{\beta_L}{\beta_H} = \tfrac{2-\alpha_H(1-p_\ell)}{2-\alpha_L(1-p_\ell)}$$

This can be rewritten as

$$p_\ell = 1 - \frac{2(\beta_H - \beta_L)}{\alpha_H \beta_H - \alpha_L \beta_L}, \tag{29}$$

which gives $p_\ell > 0$ only if (25) holds. Thus assume (25). Since $\beta_H > \beta_L$ and $\alpha_H > \alpha_L$, then $p_\ell < 1$ and $p_\ell = 1$ if $\beta_H = \beta_L$. Note also that $p_\ell > p_A$ if π is large. Thus, assuming $\pi > \beta_H$ and (25), if p_ℓ satisfies (29) and p_A satisfies (28) then (27) holds for all j.

We note that in this case (27) implies $\pi < \frac{\beta_j}{p_A}$, since $p_\ell < 1$. This does not mean that agents would buy and hold infinite amounts of bonds without liquidating them. Indeed, fractions of bonds cannot be liquidated by assumption. Thus, since $m_j = 0$ buying bonds without liquidating them is not an equilibrium, as the marginal utility of consumption in market one would be infinite. Hence, $a_j = a = c^*/p_\ell$ for $j = L, H$, as indicated by (16).

Money market clearing (21) requires $p_A a = \bar{m}$ and stationarity $a' = a$. Thus, from (4) the government sets

$$\tau = \bar{m}\pi - \frac{\bar{m}}{p_A}\left[1 - (1 - p_\ell)\left(\rho\frac{\alpha_H}{2} + (1-\rho)\frac{\alpha_L}{2}\right)\right],$$

i.e., τ equals real balances at the end of the cycle $\bar{m}\pi$, minus the payments to bond holders $\frac{\bar{m}}{p_A}$ net of liquidation fees $\bar{m}\frac{1-p_\ell}{p_A}\left(\rho\frac{\alpha_H}{2} + (1-\rho)\frac{\alpha_L}{2}\right)$. Finally, it can be proved that $x_{j,k} \geq 0$ if $U'(x)$ is sufficiently larger than $u'(x)$ for $x \in \mathbb{R}_+$ (see the Appendix).■

In short, when the most patient agents are also those who are more often in need of liquidity because of consumption shocks, then two elements are necessary to sustain the efficient allocation: savings with bonds must be encouraged by setting $i > 0$ and setting $\pi > \beta_H$, and bonds must be illiquid, i.e., $p_\ell < 1$. What is the intuition? First, we know that deflation cannot be too pronounced in a monetary equilibrium and therefore the impatient agents under-insure by using cash. Consequently, we must give bonds a return superior to cash by setting $i > 0$.

However, the patient agents would demand infinite quantities of bonds if they were fully liquid. Thus, we need to lower the expected return on bonds for these agents. As long as types H need cash more frequently than agents L, this can be done by making the bonds illiquid setting $p_\ell < 1$. When (25) holds, a unique $p_\ell \in (0, 1)$ exists that equates the present values of returns across agent types:

$$\beta_H \left[1 - \frac{\alpha_H}{2}(1 - p_\ell)\right] = \beta_L \left[1 - \frac{\alpha_L}{2}(1 - p_\ell)\right].$$

The necessary degree of illiquidity p_ℓ falls as discounting disparities increase, which is why heterogeneity in discounting cannot be too extreme, i.e., why (25) must hold.

Finally, once we have calculated the optimal degree of illiquidity, we back out the nominal interest rate that sustains the efficient equilibrium by setting the bond's price p_A equal to the deflated present value of bonds' returns. This gives us

$$i = \frac{\pi}{\beta_H}\theta - 1 \quad \text{where} \quad \theta = \frac{\alpha_H \frac{\beta_H}{\beta_L} - \alpha_L}{\alpha_H - \alpha_L} \geq 1. \tag{30}$$

Nominal interest rates are a function of a weighted measure θ of the agents' discount factors, with weights given by the frequencies of consumption shocks.

The analysis is consistent with the notion of a Fisher effect. Indeed, i fully accounts for inflationary pressure, rising or falling, but the allocation is unaffected. So money is superneutral when agents save only with correctly priced bonds.[13] In particular, bonds dominate cash in rate of return, which is why no one saves with cash. Bond yields also include a liquidity premium captured by θ, since an increase in the bonds' illiquidity lessens their attractiveness. In environments where the efficient equilibrium is associated to a lower p_ℓ, hence a higher θ, we see that the bonds' yield must be higher. As discounting disparities vanish, so does the need for illiquidity and

$$\lim_{\beta_H, \beta_L \to \beta} i = \frac{\pi}{\beta} - 1,$$

i.e., the real yield converges to the common rate of time preference.

6.2 Other considerations

In this section we make a few more considerations on the finding emerged from Proposition 5. First, we note that the result holds in economies in which inflation can be substantial. To build intuition consider the case $\beta_H = \beta_L = \beta$, so that the first best allocation satisfies $c_j = c^*$ for all j. This outcome can be sustained in two manners. A first possibility is to induce agents to save with cash that guarantees the return $\frac{1}{\beta}$. This is achieved by lowering the yield on bonds to that of money, setting $i = 0$ and running a deflation at rate $\pi = \beta$, since cash cannot pay interest. Here, we are at the Friedman rule and money and bonds are perfect substitutes if $p_\ell = 1$.

[13] Superneutrality is generally taken to mean that, in long-run equilibrium, the magnitudes of real macroeconomic variables are unaffected by the inflation rate and therefore by the rate of growth of the money stock ([20], p. 98-99). In our model, the money growth rate π is the inflation rate in stationary equilibrium. There are three key real variables: real interes rate, consumption, and real asset holdings. Proposition 5 indicates that in an optimum the magnitudes of these variables are invariant to π, for $\pi > \beta_H$, since the nominal interest rate is fully adjusted (see (30)).

Alternatively, if for some reason $\pi > \beta$ must be selected, the government can sell liquid bonds at price $p_A = \frac{\beta}{\pi}$, standing ready to redeem them costlessly. Here, agents save only with bonds that pay real return $\frac{1}{\beta}$ for any given π and the allocation is efficient. The lesson is that a deflation is unnecessary for efficiency as long as some asset exists that offers a real yield $\frac{1}{\beta}$ and that can be easily transformed into consumption. If bonds are illiquid, instead, the efficient allocation can be sustained when the interest rate is raised by setting

$$p_A = \frac{\beta}{\pi}[1 - \frac{1 - p_\ell}{2} \max(\alpha_H, \alpha_L)],$$

and agents are rationed in their purchases of bonds. This is reminiscent of the market for U.S. Savings Bonds, the purchases of which cannot exceed a fixed nominal amount (currently \$60,000 for the EE series). We note that this same rationing strategy would sustain an efficient allocation when $\beta_H > \beta_L$ but $\alpha_H \leq \alpha_L$. Thus, illiquid bonds can be useful under different assumptions on the relationship between time discounting and trading risk.

It is also interesting to consider what happens when the model is generalized to more than two types. To this end, relabel $L = 0$ and $H = 1$ and consider an economy in which the set of types is $[0, 1]$ instead of $\{0, 1\}$. That is, we have a continuum of types j each of which is characterized by a pair $(\beta_j, \alpha_j) \in (0, 1)^2$. We can prove the following Corollary to Proposition 5.

Corollary 6 *Consider an economy with types j indexed by $[0, 1]$. There exists a family of types $(\beta_j, \alpha_j)_{j\in[0,1]}$ on $(0, 1)^2$ increasing in j such that if (25) holds and government policy satisfies (26), then $c_j = c^*$ for all j is a stationary monetary equilibrium.*

Proof. We want to show that, given $\{\tau, \pi, p_A, p_\ell\}$, the Euler equation (27) holds for all $j \in [0, 1]$ given some $(\beta_j, \alpha_j)_{j\in[0,1]} \subset (0, 1)^2$. That is, for each type j we have

$$\beta_j \left[1 - \tfrac{\alpha_j}{2}(1 - p_\ell)\right] = p_A \pi.$$

Proceed as follows. Without loss in generality, index the agent types so that a higher j is associated to a higher discount factor β_j. To do so, fix two values $\beta_0 < \beta_1$ in $(0, 1)$ and then express each β_j as the convex combination of β_0 and β_1, i.e.,

$$\beta_j = \beta_0 + j(\beta_1 - \beta_0) \text{ for } j \in [0, 1].$$

Now define the continuous function $\theta : [0, 1] \times (0, 1) \to \mathbb{R}$ with

$$\theta(j, \alpha) = \beta_j \left[1 - \frac{\alpha}{2}(1 - p_\ell)\right] - p_A \pi.$$

Fix two values of α, called $\alpha_0, \alpha_1 \in (0,1)$, that satisfy $\alpha_0 < \alpha_1$ and (25) with the obvious relabeling. Therefore, $p_A \pi$ is a constant. Now impose (26).

Note that $\theta(0, \alpha_0) = \theta(1, \alpha_1) = 0$, from Proposition 5. Also, $\theta(j, \alpha_1) < 0 < \theta(j, \alpha_0)$ for all $j \in (0,1)$, since β_j monotonically increases in j. Finally, the partial derivatives of θ are such that $\theta_\alpha(j, \alpha) < 0 < \theta_j(j, \alpha)$ for all j and α. Since θ is continuous, the Intermediate Value Theorem guarantees there is a unique $\alpha = \alpha(j) \in (\alpha_0, \alpha_1) \subset (0,1)$ such that $\theta(j, \alpha) = 0$. Let $\alpha_j \equiv \alpha(j)$, and notice that the implicit function theorem assures that α_j varies continuously with j and is increasing in j.

Therefore, we can define a continuous function $f : [0,1] \to (0,1)^2$ from the set of types to the sets of discount factors β and trading probabilities α, such that $f(j) = (\beta_j, \alpha_j)$ is monotonically increasing in j. The family $(\beta_j, \alpha_j)_{j \in [0,1]}$ satisfies (25) and supports the efficient allocation when inflation and debt structure are as in (26).■

The central finding is that, if we retain the government debt structure previously considered, then the efficient allocation is sustainable even when we allow for a continuum of types. Once again, the general parameter requirement is that those agents who are more patient also face a greater probability of trading in market one.

7 Final remarks

Our study offers two basic lessons. First, heterogeneity in preferences over future consumption blunts the effectiveness of the Friedman rule. Under zero interest rates, agents essentially must rely on the available stock of fiat money as a means to insure against consumption risk. A simple arbitrage argument indicates that cash cannot promise a return greater than the discount factor of the most patient agents, much as it happens for the return on capital in [4]. Hence, the more impatient will under-insure, which is detrimental to efficiency. Under-insurance implies that in equilibrium agents hold different amounts of the available stock of money. However, unlike [4], everyone holds some cash. These findings should be obtained in any environment with similar heterogeneity, where money is essential to execute trades.

A second lesson is that nominal interest rates should be positive in order to sustain the constrained-efficient allocation. Under certain conditions, an additional friction is needed. Specifically, bonds should be illiquid, i.e., they should be convertible into immediate consumption less efficiently than cash. In the model, this necessity stems from differences in discounting and consumption needs. Illiquidity is a friction that removes arbitrage opportunities if the individuals who have the lowest discount rate are also those who trade

more frequently. Although this result is less general, it suggests one more reason as to why illiquid government bonds might be socially desirable financial instruments.

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Appendix

The constrained-efficient allocation

When the social planner is subject to the same spatial and informational frictions of agents, the planning problem corresponds to a sequence of static maximization problems subject to the technological constraints.

Recall that we are assuming that the planner weighs each agent identically and that the planner cannot recognize agents' types. On each date agents have identical preferences ex-ante. Also, on each date there is an identical proportion of buyers and sellers, so consumption of each buyer must correspond to production of some seller. Thus, on each odd date the planner maximizes expected utility of an arbitrary active agent, subject to technological feasibility. Since on odd dates agents that are active can produce or consume with equal probability, then the planner's problem is:

$$\begin{array}{ll} \max_{c,y} & \frac{1}{2}[u(c) - y] \\ \text{s.t.} & c = y \end{array}$$

On each even date the problem to be solved is similar:

$$\begin{array}{ll} \max_{q,x} & U(q) - x \\ \text{s.t.} & q = x \end{array}$$

Hence, the constrained-efficient allocation is stationary across trading cycles, i.e., $c_j = y_j = c^*$ and $q_j = x_j = q^*$ for each type j in each cycle t. Recalling that to each buyer corresponds a seller in each market, then the constrained-efficient allocation maximizes the trade surplus $u(c) - c$ in market one and $U(q) - q$ in market two.

Conditions for $x_{j,k} \geq 0$

We now want to provide conditions that guarantee $x_{j,k} \geq 0$ in the constrained-efficient equilibrium described in Proposition 5. We know that $q_j = q^*$ for all j. These results and the budget constraint in (5) imply

$$x_{j,k} = q^* + \pi(m'_j + p_A a'_j) - (m_{j,k} + a_{j,k} + \tau)$$

In the stationary constrained-efficient equilibrium agents save only with bonds, i.e., $m'_j = 0$ and $a_j = a = a'$ for all j. Let $\bar{m} = \frac{M}{p_2}$ denote the real stock of money at the start of each trading cycle. Since $p_A a = \bar{m}$, then:

$$x_{j,k} = q^* + \bar{m}\pi - (m_{j,k} + a_{j,k} + \tau)$$

From now on we focus on the seller's case, since $x_{j,b} > x_{j,s}$. Since $p = 1$, we have $m_{j,s} = c^* = \frac{\bar{m}p_\ell}{p_A}$ and $a_{j,s} = a = \frac{\bar{m}}{p_A}$. Therefore, the constrained-efficient allocation production in market two is type independent:

$$\begin{aligned} x_s &= q^* + \bar{m}\pi - c^* - \frac{\bar{m}}{p_A} - \tau \\ &= q^* + \frac{c^* p_A \pi}{p_\ell} - c^* - \frac{c^*}{p_\ell} - c^* \frac{p_A}{\bar{m}p_\ell}\tau \\ &= q^* - c^* + \frac{c^* p_A \pi}{p_\ell} - \frac{c^* p_A}{p_\ell} - c^* \frac{1-p_\ell}{p_\ell}\left(\rho\frac{\alpha_H}{2} + (1-\rho)\frac{\alpha_L}{2}\right) \end{aligned}$$

since

$$\tau = \pi\bar{m} - \frac{\bar{m}}{p_A} + \bar{m}\frac{1-p_\ell}{p_A}\left[\rho\frac{\alpha_H}{2} + (1-\rho)\frac{\alpha_L}{2}\right].$$

Therefore

$$x_s = q^* - c^*[1 + \frac{1-p_\ell}{p_\ell}(\rho\frac{\alpha_H}{2} + (1-\rho)\frac{\alpha_L}{2})]. \tag{31}$$

Since the term multiplying c^* is greater than one, then q^* must be sufficiently larger than c^* in order to have $x_s > 0$. In the efficient equilibrium $U'(q^*) = u'(c^*) = 1$, so that (31) implies we need preferences that satisfy $U'(x) > u'(x)$ for any $x \in \mathbb{R}_+$.