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2007

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Recommended Citation

Colombo, F., Gentili, G., Sabadini, I., Struppa, D.C.: A functional calculus in a non commutative setting. *Electronic Research Announcements in Mathematical Sciences* 14, 60–68 (2007). Retrieved from http://www.aimsciences.org/journals/pdfsnews.jsp?paperID=2888&mode=full

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ELECTRONIC RESEARCH ANNOUNCEMENTS
IN MATHEMATICAL SCIENCES
Volume 14, Pages 60–68 (September 21, 2007)
S 1935-9179 AIMS (2007)

A FUNCTIONAL CALCULUS IN A NONCOMMUTATIVE SETTING

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(Communicated by Guido Weiss)

Abstract. In this paper we announce the development of a functional calculus for operators defined on quaternionic Banach spaces. The definition is based on a new notion of slice regularity, see [6], and the key tools are a new resolvent operator and a new eigenvalue problem. This approach allows us to deal both with bounded and unbounded operators.

1. Introduction

Let V be a Banach space over a (possibly skew) field $\mathbb{F}$ and let E be the Banach space of linear operators acting on it. If $T \in E$, then the standard eigenvalue problem seeks the values $\lambda \in \mathbb{F}$ for which $\lambda\mathcal{I} - T$ or $\mathcal{I}\lambda - T$ are not invertible. The classical case in which $\mathbb{F} = \mathbb{C}$ is well-known and leads to what is known as functional calculus. Indeed, in the complex case, the two eigenvalue problems coincide and one defines the spectrum of an operator as the set of values $\lambda \in \mathbb{C}$ for which either operator is not invertible. The key observation for the development of the functional calculus is the fact that the inverse of $\lambda\mathcal{I} - T$ formally coincides with a Cauchy kernel which allows an integral representation for holomorphic functions. As a consequence, one is able to consider for any function f holomorphic on the spectrum of T its value $f(T)$ which is formally defined as

$$f(T) = \frac{1}{2\pi i}\int_\Gamma (\lambda\mathcal{I} - T)^{-1} f(\lambda)d\lambda,$$

where Γ is a closed curve surrounding the spectrum of T, and, in this way, one converts complex functions defined on open subsets of $\mathbb{C}$ to E-valued functions defined on suitable subsets of E. The important fact is that such a definition coincides with the obvious meaning when f is a polynomial and that this definition behaves well with respect to linear combinations, product and composition of functions, i.e., $(af + bg)(T) = af(T) + bg(T)$, $a, b \in \mathbb{C}$, $(fg)(T) = f(T)g(T)$, $(f \cdot g)(T) = f(g(T))$. For the basic theory in the complex case a classical reference is [5].

Received by the editors June 11, 2007 .
2000 *Mathematics Subject Classification.* Primary 47A10, 47A60; Secondary 30G35.
Key words and phrases. Functional calculus, spectral theory, bounded and unbounded operators.
The first and third authors are grateful to Chapman University for the hospitality during the period in which this paper was written. They are also indebted to G.N.S.A.G.A. of INdAM and the Politecnico di Milano for partially supporting their visit.

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Important applications of this theory to specific operators are now an important field of investigation (see, for example, the references in [8]). The importance in mathematics and in physics is that the functional calculus can be used to write functions of operators, such as the exponential of a closed operator, for which the corresponding power series expansion is not convergent and thus not suitable to define it. In particular, in quantum mechanics, the exponential function of an operator defines the evolution operator associated to Schrödinger equation. As it is well-known, quantum mechanics can be formulated in the real, complex and quaternionic setting (see [1]), and for this reason it is important to introduce a quaternionic version of the functional calculus to allow the study of exponentials for quaternionic operators. Several difficulties arise when dealing with quaternionic linear operators. First of all, when working in a noncommutative setting, it is necessary to specify that the operators are linear, for example, on the right, i.e. $T(v\alpha+w\beta)=T(v)\alpha+T(w)\beta$, $\alpha,\beta\in\mathbb{H}$. As explained before, there are two different eigenvalue problems. The so called right eigenvalue problem, i.e. the search for nonzero vectors satisfying $T(v)=v\lambda$, is widely studied by physicists. The crucial fact is that whenever there is a non real eigenvalue λ then all quaternions belonging to the sphere $r^{-1}\lambda r$, $r\in\mathbb{H}\setminus\{0\}$, are eigenvalues. This fact allows to choose the phase and to work, for example, with complex eigenvalues. Note, however, that the operator of multiplication on the right is not a right linear operator, and so the operator $\mathcal{I}\lambda-T$ is not linear. On the other hand, the operator $\lambda\mathcal{I}-T$ is right linear but, in general, one cannot choose the phase as the eigenvalues do not necessarily lie on a sphere. Even more importantly, in the complex setting the inverse $(\lambda\mathcal{I}-T)^{-1}$ is related to a Cauchy kernel useful in the notion of a Cauchy integral. In the quaternionic setting it is not even clear which type of generalized holomorphy must be used. The regularity in the sense of Fueter (see e.g. [4]) does not seem to give a good notion of exponential function and does not allow to introduce polynomials (or even powers) of operators. A more recent notion, the so called *slice regularity*, see [6], where it is called Cullen-regularity, allows to show that power series in the quaternionic variable are regular, so this new notion seems to be the correct setting in which a functional calculus can be introduced and developed. The Cauchy kernel series $\sum_{n\geq 0} q^n s^{-n-1}$, $q,s\in\mathbb{H}$, which is used to write a Cauchy formula for slice-regular functions does not coincide, in general, with $(s-q)^{-1}$, thus the linear operator $S^{-1}(s,T)=\sum_{n\geq 0} T^n s^{-n-1}$ is not, in general, the inverse of $(s\mathcal{I}-T)$. The key idea of [2] and [3], whose results we announce in this note, is to identify the operator whose inverse is $\sum_{n\geq 0} T^n s^{-n-1}$. This new operator will give us a new notion of spectrum (which we call S-spectrum) which will allow to introduce a functional calculus As we will see, this spectrum, as the right-spectrum, will allow the choice of the phase, and this functional calculus can be introduced both for bounded and for unbounded operators. We close this introduction by pointing out that the spectral theory can be extended to the case of n-tuples of operators. For the complex case, the reader is referred to [9] (for the commuting case) and to [8] for the case when the operators do not commute. We plan to come back to this issue in a subsequent paper and show how our theory can be applied to this situation as well.

2. Slice regular functions

In this section we summarize the basic definitions from [6], [7], that we need in the sequel.
Let $\mathbb{H}$ be the real associative algebra of quaternions with respect to the basis $\{1, i, j, k\}$ satisfying the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \qquad ki = -ik = j.$$

We will denote a quaternion q as $q = x_0 + ix_1 + jx_2 + kx_3$, $x_i \in \mathbb{R}$, its conjugate as $\bar{q} = x_0 - ix_1 - jx_2 - kx_3$, and we will write $|q|^2 = q\overline{q}$.
Let $\mathbb{S}$ be the sphere of purely imaginary unit quaternions, i.e.

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Definition 2.1. *Let $U \subseteq \mathbb{H}$ be an open set and let $f : U \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex line $L_I := \mathbb{R}+I\mathbb{R}$ passing through 1 and I. We say that f is a slice left (resp. right) regular function if for every $I \in \mathbb{S}$*

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0, \qquad \left(resp.\ \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}I\right) f_I(x+Iy) = 0\right).$$

Definition 2.2. *We say that a function f is slice regular at infinity if there exists an open neighborhood U of ∞ on which f is regular.*

Remark 2.3. Slice left regular functions on $U \subseteq \mathbb{H}$ form a right vector space $\mathcal{R}(U)$ and slice right regular functions on $U \subseteq \mathbb{H}$ form a left vector space. It is not true, in general, that the product of two regular functions is regular.

Every slice regular function can be represented as a power series, [7]:

Theorem 2.4. *If $B(0, R)$ is a ball centered in the origin with radius $R > 0$ and $f : B \to \mathbb{H}$ is a slice left (resp. right) regular function, then f has a series expansion of the form*

$$f(q) = \sum_{n=0}^{+\infty} q^n \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(0), \qquad \left(resp.\ \ f(q) = \sum_{n=0}^{+\infty} \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(0) q^n\right)$$

converging on B.

Definition 2.5. *Let E be a bilateral quaternionic Banach space. A function $f : \mathbb{H} \to E$ is said to be slice left (resp. right) regular if there exists an open ball $B(0, r)$ and a sequence $\{a_n\}$ of elements of E such that, for every point $q \in B(0, r)$ the function $f(q)$ can be represented by the following series*

$$f(q) = \sum_{n=0}^{+\infty} q^n a_n, \qquad \left(resp. f(q) = \sum_{n=0}^{+\infty} a_n q^n\right), \qquad a_n \in E, \tag{1}$$

converging in the norm of E for any q such that $|q| < r$.

From now on we will not specify whether we are considering left or right regular functions, since the context will clarify it. Also we will simply say regular instead of slice regular.

3. The S-spectral problem

Definition 3.1. *Let V be a right vector space over $\mathbb{H}$. A map $T : V \to V$ is said to be right linear if*

$$T(u + v) = T(u) + T(v),$$
$$T(us) = T(u)s,$$

for all $s \in \mathbb{H}$ and all $u, v \in V$.

Remark 3.2. Note that the set of right linear maps is not a quaternionic left or right vector space. Only if V is both left and right vector space, then the set $\mathrm{End}(V)$ of right linear maps on V is both, a left and a right vector space over $\mathbb{H}$, since in that case we can define $(aT)(v) := aT(v)$ and $(Ta)(v) := T(av)$. The composition of operators can be defined in the usual way: for any two operators $T, S \in \mathrm{End}(V)$ we have

$$(TS)(u) = T(S(u)), \qquad \forall u \in V.$$

In particular, we have the identity operator $\mathcal{I}(u) = u$, for all $u \in V$ and setting $T^0 = \mathcal{I}$ we can define powers of a given operator $T \in \mathrm{End}(V)$: $T^n = T(T^{n-1})$ for any $n \in \mathbb{N}$. An operator T is said to be invertible if there exists an S such that $TS = ST = \mathcal{I}$ and, in this case, we will write $S = T^{-1}$.

Definition 3.3. *Let V be a bilateral quaternionic Banach space. We will denote by $\mathcal{B}(V)$ the vector space of all right linear bounded operators on V.*

It is easy to verify that $\mathcal{B}(V)$ is a Banach space endowed with its natural norm.

Definition 3.4. *An element $T \in \mathcal{B}(V)$ is said to be invertible if there exists a $T' \in \mathcal{B}(V)$ such that $TT' = T'T = \mathcal{I}$.*

It is obvious that the set of all invertible elements of $\mathcal{B}(V)$ is a group with respect to the composition defined in $\mathcal{B}(V)$.
Given a linear quaternionic operator T, there are two natural eigenvalue problems associated to T. The first, the left eigenvalue problem, consists in the solution of the equation $T(v) = \lambda v$, and the second, the right eigenvalue problem, consists in the solution of the equation $T(v) = v\lambda$.

The key observation is that none of them is useful to define a functional calculus. In this section we will identify the correct framework for the study of eigenvalues for quaternionic operators. We refer the reader to [2] for all the proofs of the results in this section.

Definition 3.5. *Let $T \in \mathcal{B}(V)$ and let $s \in \mathbb{H}$. We define the S-resolvent operator series as*

$$S^{-1}(s, T) := \sum_{n \geq 0} T^n s^{-1-n} \tag{2}$$

for $\|T\| < |s|$.

Theorem 3.6. *Let $T \in \mathcal{B}(V)$ and let $s \in \mathbb{H}$. Assume that $\overline{s}$ is such that $T - \overline{s}\mathcal{I}$ is invertible. Then*

$$S(s, T) := (T - \overline{s}\mathcal{I})^{-1}(sT - |s|^2\mathcal{I}) - T \tag{3}$$

is the inverse of

$$S^{-1}(s, T) = \sum_{n \geq 0} T^n s^{-1-n}.$$

Moreover, we have

$$S^{-1}(s,T) = \sum_{n\geq 0} T^n s^{-1-n} = -(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \tag{4}$$

for $\|T\| < |s|$.

Theorem 3.7. *Let* $T \in \mathcal{B}(V)$ *and let* $s \in \mathbb{H}$. *The operator*

$$\sum_{n\geq 0}(s^{-1}T)^n s^{-1}\mathcal{I}$$

is the right and left algebraic inverse of $s\mathcal{I} - T$. *Moreover, the series converges in the operator norm for* $\|T\| < |s|$.

Corollary 3.8. *When* $Ts\mathcal{I} = sT$, *the operator* $S^{-1}(s,T)$ *equals* $(s\mathcal{I} - T)^{-1}$ *when the series (2) converges.*

Proof. It follows immediately from Theorem 3.7. □

Definition 3.9. *(The S-resolvent operator) Let* $T \in \mathcal{B}(V)$ *and let* $s \in \mathbb{H}$. *We define the S-resolvent operator as*

$$S^{-1}(s,T) := -(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}). \tag{5}$$

Definition 3.10. *(The spectra of quaternionic operators) Let* $T : V \to V$ *be a linear quaternionic operator on the Banach space* V. *We denote by* $\sigma_L(T)$ *the left spectrum of* T *related to the resolvent* $(s\mathcal{I} - T)^{-1}$ *that is*

$$\sigma_L(T) = \{s \in \mathbb{H} \quad : \quad s\mathcal{I} - T \quad \textit{is not invertible}\}.$$

We define the S-spectrum $\sigma_S(T)$ *of* T *related to the S-resolvent operator (5) as:*

$$\sigma_S(T) = \{s \in \mathbb{H} \quad : \quad T^2 - 2\ Re[s]T + |s|^2\mathcal{I} \quad \textit{is not invertible}\}.$$

Theorem 3.11. *Let* $T \in \mathcal{B}(V)$ *and let* $s \in \mathbb{H}$. *Let* $S^{-1}(s,T)$ *be the S-resolvent operator defined in (2). Then the series converges for* $\|T\| < |s|$ *and* $S^{-1}(s,T)$ *satisfies the (S-resolvent) equation*

$$S^{-1}(s,T)s - TS^{-1}(s,T) = \mathcal{I}.$$

Theorem 3.12. *Let* $T \in \mathcal{B}(V)$. *Then* $\sigma_S(T)$ *and* $\sigma_L(T)$ *are contained in the set* $\{s \in \mathbb{H} : |s| \leq \|T\|\}$.

Theorem 3.13. *(Compactness of S-spectrum) Let* $T \in \mathcal{B}(V)$. *The S-spectrum* $\sigma_S(T)$ *is a compact nonempty set contained in* $\{\ s \in \mathbb{H}\ :\ |s| \leq \|T\|\ \}$.

Theorem 3.14. *(Structure of the S-spectrum) Let* $T \in \mathcal{B}(V)$ *and let* $p = p_0 + p_1 I \in p_0 + p_1\mathbb{S} \subset \mathbb{H}\setminus\mathbb{R}$ *be an S-eigenvalue of* T. *Then all the elements of the sphere* $p_0 + p_1\mathbb{S}$ *are S-eigenvalues of* T.

4. The main results for bounded operators

Definition 4.1. *A function* $f : \mathbb{H} \to \mathbb{H}$ *is said to be locally regular on the spectral set* $\sigma_S(T)$ *of an operator* $T \in \mathcal{B}(V)$ *if there exists a ball* $B(0,R)$ *containing* $\sigma_S(T)$ *on which* f *is regular. We will denote by* $\mathcal{R}_{\sigma_S(T)}$ *the set of locally regular functions on* $\sigma_S(T)$.

Theorem 4.2. *Let $T \in \mathcal{B}(V)$ and $f \in \mathcal{R}_{\sigma_S(T)}$. Choose $I \in \mathbb{S}$ and let L_I be the plane that contains the real line and the imaginary unit I. Let U be an open bounded set in L_I that contains $L_I \cap \sigma_S(T)$. If we set $ds_I = -Ids$, then the integral*

$$\frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\ ds_I\ f(s)$$

does not depend on the choice of the imaginary unit I and on the open set U.

Proof. By Theorem 3.14 the S-spectrum contains either real points or (entire) spheres of type $s_0 + r\mathbb{S}$, $(s_0, r \in \mathbb{R})$. Every plane $L_I = \mathbb{R} + I\mathbb{R}$ $(I \in \mathbb{S})$, contains all the real points of the S-spectrum. Moreover, L_I intersects each sphere $s_0 + r\mathbb{S}$ in the two (conjugate) points $s_0 + rI$ and $s_0 - rI$. Now we show that the integral in the statement does not depend on the plane L_I and on U. For any imaginary unit $I' \in \mathbb{S}$, $I' \neq I$, and any U' containing $L_{I'} \cap \sigma_S(T)$ we obtain:

$$\frac{1}{2\pi}\int_{\partial U}\sum_{n\geq 0} T^n s^{-1-n}\ ds_I f(s) = \frac{1}{2\pi}\int_{\partial U'}\sum_{n\geq 0} T^n s^{-1-n}\ ds_{I'} f(s)$$

thanks to the Cauchy theorem and to the fact that the points of the spectrum have that same coordinates on each "complex" plane L_I. □

We give a preliminary result that motivates the functional calculus, [2].

Theorem 4.3. *Let $q^m a$ be a monomial, $q, a \in \mathbb{H}$, $m \in \mathbb{N} \cup \{0\}$. Let $T \in \mathcal{B}(V)$ and let U be an open bounded set in L_I that contains $L_I \cap \sigma_S(T)$. If we set $ds_I = -Ids$, then*

$$T^m a = \frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\ ds_I\ s^m\ a. \tag{6}$$

The preceding discussion allows to give the following definition.

Definition 4.4. *Let $T \in \mathcal{B}(V)$ and $f \in \mathcal{R}_{\sigma_S(T)}$. We define*

$$f(T) = \frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\ ds_I\ f(s),$$

where U is an open bounded set that contains $L_I \cap \sigma_S(T)$.

This definition offers a new functional calculus, as the two following theorems show, [2].

Theorem 4.5. *Let $T \in \mathcal{B}(V)$ and let f and $g \in \mathcal{R}_{\sigma_S(T)}$. Then*

$$(f+g)(T) = f(T) + g(T), \qquad (f\lambda)(T) = f(T)\lambda \qquad \text{for all} \quad \lambda \in \mathbb{H}.$$

Moreover, if $\phi(s) = \sum_{n\geq 0} s^n a_n$ and $\psi(s) = \sum_{n\geq 0} s^n b_n$ are in $\mathcal{R}_{\sigma_S(T)}$ with a_n and $b_n \in \mathbb{R}$, then

$$(\phi\psi)(T) = \phi(T)\psi(T).$$

Theorem 4.6. *(Spectral decomposition of a quaternionic operator) Let $T \in \mathcal{B}(V)$. Let $L_I \cap \sigma_S(T) = \sigma_{1S}(T) \cup \sigma_{2S}(T)$, with $dist(\sigma_{1S}(T), \sigma_{2S}(T)) > 0$. Let U_1 and U_2 be two open sets such that $\sigma_{1S}(T) \subset U_1$ and $\sigma_{2S}(T) \subset U_2$, on L_I, with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Set*

$$P_j := \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)\, ds_I, \quad T_j^m := \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)\, ds_I\, s^m\,, \quad m \in \mathbb{N},\ \ j = 1, 2.$$

Then

(I) $P_j^2 = P_j$,
(II) $P_1 + P_2 = \mathcal{I}$,
(III) $TP_j = T_j$,
(IV) $T = T_1 + T_2$,
(V) $T^m = T_1^m + T_2^m, \ \ m \geq 2$.

Proof. We sketch the proofs of (I)–(III). Observe that $P_j = T_j^0$ and note that the resolvent equation for $m = 0$ is trivially $T_j^0 S^{-1}(s,T) = S^{-1}(s,T)s^0 = S^{-1}(s,T)1$. We have

$$\begin{aligned} P_j^2 =& P_j \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)\, ds_I \ = \frac{1}{2\pi}\int_{\partial U_j} P_j S^{-1}(s,T)\, ds_I \\ =& \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)1\, ds_I \ = P_j. \end{aligned}$$

This proves (I). To prove (II) we use the Cauchy integral theorem. If $\overline{U}_1 \cup \overline{U}_2 \subset U$, then

$$\frac{1}{2\pi}\int_{\partial U_1} S^{-1}(s,T)\, ds_I \ + \frac{1}{2\pi}\int_{\partial U_2} S^{-1}(s,T)\, ds_I \ = \frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\, ds_I.$$

This gives $P_1 + P_2 = \mathcal{I}$ since $\frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\, ds_I = \mathcal{I}$.
To prove (III) we recall the resolvent relation

$$TS^{-1}(s,T) = S^{-1}(s,T)s - \mathcal{I},$$

so

$$\begin{aligned} TP_j =& \frac{1}{2\pi}\int_{\partial U_j} TS^{-1}(s,T)\, ds_I = \frac{1}{2\pi}\int_{\partial U_j} [S^{-1}(s,T)s - \mathcal{I}]\, ds \\ =& \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)\, ds_I\, s \ = T_j. \end{aligned}$$

□

5. The main results for unbounded operators

In this section we announce the results from [3]. These results show to what extent one can push the theory developed in the previous section, when unbounded operators are considered. First of all, we note that if T is a closed operator, the series $\sum_{n\geq 0} T^n s^{-1-n}$ does not converge. However, the right hand side of formula (4) contains the continuous operator $(T^2 - 2TRe[s] + |s|^2\mathcal{I})^{-1}$. From an heuristical point of view, the composition $(T^2 - 2TRe[s] + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I})$ gives a bounded operator, for suitable s, as it will be proved in Theorem 5.5, thus the following definition makes sense.

Definition 5.1. *Let T be a linear closed quaternionic operator. The S-resolvent operator is (formally) defined by*

$$S^{-1}(s,T) = -(T^2 - 2TRe[s] + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}). \tag{7}$$

We now define the resolvent set, the spectrum will be defined as its complement.

Definition 5.2. *Let V be a quaternionic Banach space, and T be a closed linear quaternionic operator on V. We define the S-resolvent set*

(8)
$$\rho_S(T) = \{s \in \mathbb{H} : \overline{Range(S(s,T))} = V \text{ and } S^{-1}(s,T) \text{ exists and is bounded on } V\}$$

and the S- spectrum

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T). \tag{9}$$

Theorem 5.3. *Let V be a quaternionic Banach space, and T be a closed linear quaternionic operator on V. Let $s \in \rho_S(T)$. Then the S-resolvent operator defined in (7) satisfies the (S-resolvent) equation*

$$S^{-1}(s,T)s - TS^{-1}(s,T) = \mathcal{I}.$$

Remark 5.4. Let $D(T)$ be the domain of the operator T. If $T : D(T) \subseteq V \to V$ is a closed operator, then the S-resolvent set can be the empty set, the whole $\mathbb{H}$, a bounded or an unbounded set.

Theorem 5.5. *Let T be a closed operator, and let $s \in \rho_S(T)$. Then the S-resolvent operator can be represented by*

$$S^{-1}(s,T) = \sum_{n\geq 0}(Re[s]\mathcal{I} - T)^{-n-1}(Re[s] - s)^n \tag{10}$$

if and only if

$$|Im[s]| \ \|(Re[s]\mathcal{I} - T)^{-1}\| < 1. \tag{11}$$

Theorem 5.6. *Let T be a closed linear quaternionic operator. Then $\rho_S(T)$ is open.*

If $T = T_0 + iT_1 + jT_2 + kT_3$ is a closed operator, then necessarily at least one of the T_j's is closed and its resolvent is not defined at infinity. It is therefore natural to consider closed operators T for which the resolvent $S^{-1}(s,T)$ is not defined at infinity and to define the extended spectrum as

$$\overline{\sigma}_S(T) := \sigma_S(T) \cup \{\infty\}.$$

Theorem 5.7. *Let T be a closed quaternionic operator with a bounded inverse such that $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. Let V be an open set in L_I that contains $L_I \cap \overline{\sigma}_S(T)$ such that its boundary ∂V is positively oriented and consists of a finite number of regular curves. Let f be regular on $\overline{V} \cup \{\infty\}$. Then*

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial V} S^{-1}(s,T)ds_I f(s). \tag{12}$$

Theorem 5.8. *Let T be a closed quaternionic operator with a bounded inverse, and such that $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. Suppose that (11) holds. Let V be an open set in L_I that contains $L_I \cap \sigma_S(T)$ such that its boundary ∂V consists of the segment I_R of length $2R$ on the imaginary axis I symmetric with respect to the origin and of the semicircle γ_R with diameter I_R and $Re[s] > 0$. Let f be regular on $\overline{V} \cup \infty$ such that $f(\infty) = 0$ and suppose that $\int_{\gamma_R} S^{-1}(s,T)ds_I f(s) \to 0$ as $R \to +\infty$. Then*

$$f(T) = \sum_{n\geq 0} T^{-n-1}\mathcal{F}_n(f), \tag{13}$$

where

$$\mathcal{F}_n(f) = -\frac{1}{2\pi}\int_{I\mathbb{R}} (Im[s])^n ds_I f(s). \tag{14}$$

Proof. From the previous theorem, we immediately obtain

$$f(T) = \frac{1}{2\pi}\int_{I_R\cup\gamma_R} S^{-1}(s,T)ds_I f(s) \tag{15}$$

$$= \frac{1}{2\pi}\int_{I_R} S^{-1}(s,T)ds_I f(s) + \frac{1}{2\pi}\int_{\gamma_R} S^{-1}(s,T)ds_I f(s) := A_1(R,f) + A_2(R,f).$$

By the hypotheses on f we have that $\|A_2(R,f)\| \to 0$ as $R\to\infty$, and therefore $f(T) = A_1(\infty, f)$. In this case the S-resolvent representation (10) implies

$$S^{-1}(s,T) = \sum_{n\geq 0}(-1)^{n+1}T^{-n-1}(-1)^n(Im[s])^n = -\sum_{n\geq 0}T^{-n-1}(Im[s])^n.$$

The statement follows from the definition of $A_1(\infty, f)$ and $\mathcal{F}_n(f)$. □

Remark 5.9. The Cauchy theorem shows that any set V whose boundary consists of a finite number of regular curves can be assumed to satisfy the conditions in the statement.

References

[1] S. Adler, "Quaternionic Quantum Mechanics and Quantum Fields Theory," Oxford University Press, 1995. MR 1333599 (97d:81001)

[2] F. Colombo, G. Gentili, I. Sabadini and D. C. Struppa, *Non commutative functional calculus: bounded operators*, preprint, 2007.

[3] F. Colombo, G. Gentili, I. Sabadini and D. C. Struppa, *Non commutative functional calculus: unbounded operators*, preprint, 2007.

[4] F. Colombo, I. Sabadini, F. Sommen and D. C. Struppa, "Analysis of Dirac Systems and Computational Algebra," Progress in Mathematical Physics, **39**, Birkhäuser, Boston, 2004. MR 2089988 (2005m:30052)

[5] N. Dunford and J. Schwartz, "Linear Operators, Part I: General Theory," J. Wiley and Sons, 1988. MR 1009162 (90g:47001a)

[6] G. Gentili and D. C. Struppa, *A new approach to Cullen-regular functions of a quaternionic variable*, C.R. Acad. Sci. Paris, **342** (2006), 741–744. MR 2227751 (2006m:30095)

[7] G. Gentili and D. C. Struppa, *A new theory of regular functions of a quaternionic variable*, Adv. Math., **216** (2007), 279-301.

[8] B. Jefferies, "Spectral Properties of Noncommuting Operators," Lecture Notes in Mathematics, **1843**, Springer-Verlag, Berlin, 2004. MR 2069293 (2007a:47007)

[9] J. L. Taylor, *The analytic-functional calculus for several commuting operators*, Acta Math., **125** (1970), 1–38. MR 0271741 (42 6622)

[10] J. L. Taylor, *Functions of several noncommuting variables*, Bull. Amer. Math. Soc., **79** (1973), 1–34. MR 0315446 (47 3995)

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